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IMPROVED STATE FEEDBACK CONTROLLER SYNTHESIS FOR PIECEWISE-LINEAR SYSTEMS

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Abstract. *We propose a new technique for the design of state feedback controller for piecewise-linear systems, such that, the closed-loop systems are well-posed and asymptotically stable. First, a new criterion for the avoidance of sliding motion on the boundaries is presented. Then, the piecewise affine controller is constructed in a way such that the resulting closed-loop system satisfies the proposed criterion and a piecewise quadratic Lyapunov function can be used to establish the asymptotical stability. In this way, the control design problem is formulated as a numerical optimization problem under Linear Matrix Inequality constraints. The results are illustrated by application to control an aerobatic helicopter and balance an inverted pendulum on a cart, respectively, which demonstrates the efficacy and advantage of the proposed approach.*
Keywords: Piecewise-linear systems, Piecewise quadratic Lyapunov function, Continuous controller, Sliding motion, Linear matrix inequalities

1. Introduction.

Piecewise-linear systems {PLS} have been a subject of research in the systems and control community for some time; see for example, [3-17]. PLS constitute a special class of switched systems [1,2] that often arise in practice when piecewise-linear components are encountered. These components include dead-zone, saturation, relays and hysteresis. In addition, many other types of nonlinear systems can also be approximated by PLS [3,4]. Thus, PLS provide a useful framework for the analysis and synthesis of a large class of nonlinear systems.

A number of significant results have been obtained on controller design for PLS in the last few years [5-17]. Johansson and Hassibi [5,6] first discussed stability analysis and controller design for PLS via piecewise quadratic Lyapunov function {PQLF}. It has been shown that the piecewise-affine {PWA} controller design using such Lyapunov functions can be cast as optimization problems via Bilinear Matrix Inequalities {BMIs} [7]. Furthermore, Ding and Yang [8] showed how BMIs can be avoided by presenting a new framework for the synthesis of PLS based on piecewise quadratic Lyapunov functions and the Reciprocal Projection Lemma, resulting in a convex optimization problem via Bilinear Matrix Inequalities {LMIs}. Recently, motivated by the PQLF theory in [5-7], Feng et al. [9-13] extended the stability analysis method in [3] to uncertain PLS, and then developed new design methods that yield LMIs. Zhang and Tang [14] considered the output feedback control of uncertain PLS, assuming that the uncertainty is norm-bounded and enters all system matrices. By constructing piecewise quadratic Lyapunov function for the closed-loop augmented system, the H_∞ output feedback controller design procedure is formulated as solving a set of BMIs. Moreover, Zhang [15] extended the piecewise quadratic Lyapunov functions technique to design the output feedback optimal guaranteed cost controller for uncertain PLS, and the existence of the guaranteed cost controller for closed-loop system is still cast as the feasibility of a set of BMIs. More recently, the adaptive control techniques are extended to the PLS area to deal with the parameter uncertainty in system model, and the key tool remains to be PQLF, which yields the optimization problem under LMIs or BMIs. (See [16,17])

To guarantee the uniqueness and convergence of the state trajectories, it must be shown the trajectories do not slide along the boundaries between the polytopic regions. However, none of the work in [3-17] has explicitly addressed this condition. Although Rodrigues [18] considered this condition via adding the equation constraint to guarantee the continuity of the vectors field for avoidance of sliding motion on the boundaries of PLS, which brings much conservatism, and the resulting controller can be discontinuous with attendant practical problems. On the other hand, the avoidance of sliding motion for PLS can also be treated as checking well-posedness problem [19-21] in some sense. However, the existing references for checking well-posedness are hard to be applied for the controller design of PLS because the expressions of criterions are always not convex problems. Motivated by these observations, we will propose a less conservative criterion for the avoidance of sliding motion, which can be formulated as LMIs. Incorporating the new criterion with PQLF, the controller design problem is cast as an convex optimization problem under LMIs.

The rest of the paper is organized as follows. First, we present as background in Section 2 a brief outline of the design method proposed in [18]. In particular, we show the main limitation of the method. In Section 3, we present in Proposition 3.2 a less conservative condition for avoidance of sliding motion on the boundaries between the polytopic regions. This condition is translated to LMI constraints, which in turn yields a controller design method in Theorem 3.1. Finally, we apply the proposed methods to two practical engineering designs in Section 4 and conclude the paper in Section 5.

2. **Problem Statement and Preliminaries.**

2.1. **System description.** This paper considers the PLS as below,

$$\dot{x}(t) = f(x,u) = A_i x(t) + B_i u(t) + b_i, \quad x \in R_i \tag{1}$$

Here, $\{R_i\}_{i\in I} \subseteq \mathbb{R}^n$ is a partition of the state space into a number of closed polyhedral regions with pairwise disjoint interior. The index of the region is denoted as I. Let I_0 be the index set for regions that contain origin. It is assumed that $b_i = 0$ such that origin is an equilibrium of autonomous PLS (1).

For convenient notation, we introduce

$$\hat{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad \widehat{A}_i = \begin{bmatrix} A_i & b_i \\ 0 & 0 \end{bmatrix}, \quad \widehat{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix} \tag{2}$$

Using this notation, the systems model (1) can be expressed as

$$\dot{\hat{x}} = \widehat{A}_i \hat{x} + \widehat{B}_i u, \quad x \in R_i \tag{3}$$

Since the cells are polyhedra, we can construct matrices E_i and F_i such that

$$\begin{aligned} E_i \bar{x} &\geq 0, \quad && x \in R_i \\ F_i \bar{x} &= F_j \bar{x}, && x \in L_{ij} \end{aligned} \tag{4}$$

where L_{ij} denotes the common boundary between R_i and R_j, i.e., $L_{ij} = R_i \cap R_j$, and the vector inequality $z \geq 0$ means that each entry of z is nonnegative.

The following reasonable and practical assumptions are made:

Assumption 2.1. *For the regions containing origin, i.e., R_i, $i \in I_0$, the origin is also one of their vertexes.*

It is noted that, this assumption can always be satisfied by further refining the state-space partition.

Definition 2.1. *The PLS (1) is well-posed, if every solution is well-posed in the sense of Carathéodory [19]. This means that the following integral form:*

$$x(t; x_0) = x_0 + \int_0^t f(x(\tau), u(\tau))d\tau \tag{5}$$

has a unique solution without any sliding modes for each initial state x_0.

Our objective is to design PWA state feedback controller

$$u = K_i \bar{x} = k_i x + m_i, \quad x \in R_i, \tag{6}$$

such that the closed-loop system is well-posed and asymptotically stable.

2.2. **Research motivation.** Among the existing controller design methods for PLS (1), only Rodrigues in [18] considered the avoidance of sliding motion on the boundary, i.e., well-posedness problem (see the following theorem).

Theorem 2.1. *[18] Consider the PLS described by (1). If there exists PWA controller $u = k_i x + m_i, x \in R_i$ and a PQLF $V_i(x)$, such that the closed-loop system satisfies the conditions (7)-(10) described below.*

$$V_i(x) > 0, \quad x \in R_i \tag{7}$$

$$\tfrac{dV_i}{dt} < 0, \quad x \in R_i \tag{8}$$

$$V_i(x) = V_j(x), \quad x \in L_{ij} \tag{9}$$

$$\bar{A}_i x + \bar{b}_i = \bar{A}_j x + \bar{b}_j, \quad x \in L_{ij}. \tag{10}$$

Then the closed-loop system is well-posed and asymptotically stable, where $\bar{A}_i = A_i + B_i k_i$, $\bar{b}_i = b_i + B_i m_i$.

The role of constraint (10) is to guarantee the avoidance of sliding motion on the boundaries between the polytopic regions. However, it is an equality constraint, which will bring much conservatism. This is equal to reduce freedoms of the controller, sometimes yielding no solutions of the proposed optimization problem (see Example 4.1). Moreover, when using Theorem 2.1 to design the continuous controller, in some cases the constraint (10) will be inconsistent with the condition for continuous controller, and the details are shown as the following remark (also see Example 4.2).

Remark 2.1. *Consider design continuous PWA controller $u = k_i x + m_i$ to stabilize the PLS by Theorem 2.1. Suppose that the polytopic regions arise from the linearization of a nonlinear system $\dot{x} = f(x) + g(x)u$ around several operating points:*

$$A_i x + b_i = A_j x + b_j = f(x), \quad x \in L_{ij} \tag{11}$$

$$B_i = g(\widetilde{x}_i), \quad x \in R_i; \quad B_j = g(\widetilde{x}_j), \quad x \in R_j \tag{12}$$

where $\widetilde{x}_i$, $\widetilde{x}_j$ are the geometric centers of R_i and R_j.

That is to say, the original nonlinear system is approximatively expressed as $\dot{x} = A_i x + B_i u + b_i$, where A_i, B_i, b_i satisfy Equations (11) and (12).

Then, if (10) holds, Equations (11) and (12) yield

$$B_i(k_i x + m_i) = B_j(k_j x + m_j) \quad x \in L_{ij}. \tag{13}$$

If $B_i \neq B_i$, i.e., $g(\widetilde{x}_i) = g(\widetilde{x}_j)$, which is often the case, it can be shown that condition (10) requires

$$k_i x + m_i \neq k_j x + m_j, \quad x \in S_l, \tag{14}$$

which is inconsistent with the continuity of the controller. That is to say, for the PLS satisfying $A_i x + b_i = A_j x + b_j$ and $B_i \neq B_j$, which is often the case when approximating nonlinear system by PLS, Theorem 2.1 is not effective to design continuous controller.

Therefore, the key problem is to derive a new computable condition for avoidance of sliding motion at the boundary, which is the main contribution of this paper.

3. Main Results.

3.1. New sufficient condition for avoidance of sliding motion.

We begin by analyzing the state trajectory motion near the surface. The following proposition, taken from [22], gives a criterion for the state trajectory to cross a surface.

Proposition 3.1. *Let $S = \{x \mid s(x) = 0\}$ be a Lipschitz surface in X. Let $S^+ = \{x \mid 0 < s(x) < a\}$ and $S^- = \{x \mid b < s(x) < 0\}$ be two regions in X. Consider the system $\dot{x} = f(x, t)$, where for every bounded region $D \subseteq X$, there exists a bounded function $A(t)$ such that*

$$\|f(x, t)\| \leq A(t) \quad a.e. \;\; x \in D. \tag{15}$$

Let $K_F[f](x, t)$ denote the Filippov differential inclusion of system:

$$K_F[f](x, t) = \bigcap_{\delta > 0} \bigcap_{\mu(N) = 0} \overline{co} f(B_\delta(x) \setminus N, t), \tag{16}$$

where $\mu(N)$ denotes the measure of set N, $B_\delta(x)$ denotes the open ball with the center x and the radius δ.

Let $IntT^B_{S \bigcup S_+}(x)$ denote the interior of contingent cone for $S \bigcup S_+$:

$$T^B_\Omega(x) = \left\{ y \in \mathbb{R}_n : \; \liminf_{h \to 0} \frac{d_\Omega(x + hy)}{h} = 0 \right\}. \tag{17}$$

Indeed, $T^B_\Omega(x)$ is the set of all vectors that point to the interior of Ω from the boundary point x.

Suppose that

$$K_F[f](x, t) \subseteq Int\left(T^B_{S \bigcup S_+}(x)\right), \quad x \in S, \tag{18}$$

where $Int(X)$ denotes the interior of set X.

Then for all $x(t_0) = x \in S$, there exists $t_1 > t_0$ such that

$$x(t) \in S^+, \quad t \in (t_0, t_1). \tag{19}$$

When this proposition is applied to a PLS, we obtain the following proposition.

Lemma 3.1. *Consider the autonomous PLS (1). Let C_{ij} be the normal to the plane L_{ij} that separates regions A^+_{ij} and A^-_{ij} (see Figure 1), where $A^+_{ij} = \{x \mid C_{ij}x + d_{ij} > 0\}$, $A^-_{ij} = \{x \mid C_{ij}x + d_{ij} < 0\}$.*

If the conditions

$$C_{ij}(A_i x + b_i) > 0, \quad C_{ij}(A_j x + b_j) > 0 \quad x \in S_l \tag{20}$$

or

$$C_{ij}(A_i x + b_i) < 0, \quad C_{ij}(A_j x + b_j) < 0 \quad x \in S_l \tag{21}$$

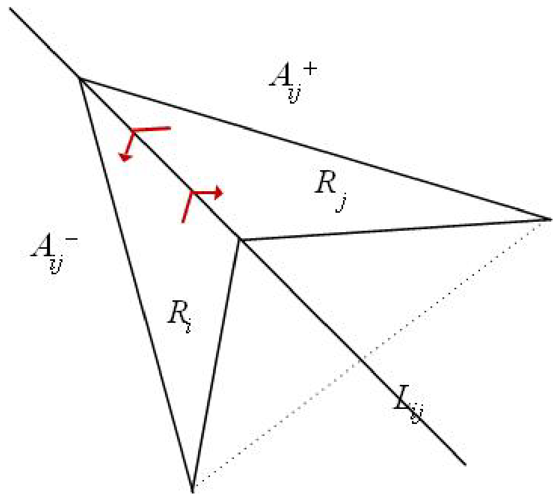

FIGURE 1. Solution motion of PLS

hold, then for $x(t_0) = x \in L_{ij}$*, there exists* $t_1 > t_0$ *such that*

$$x(t) \in A_{ij}^+(A_{ij}^-), \quad t \in (t_0, t_1), \tag{22}$$

i.e., the sliding motion on L_{ij} *is avoided.*

Proof: For the autonomous PLS $\dot{x}(t) = A_i x(t) + b_i \ x \in R_i$, note that the requirement of the right hand side to be bounded is evidently satisfied.

Then, from Proposition 3.1, we need to prove the following requirement:

$$K_F[f](x) \subseteq \mathrm{Int}\left(T^B_{L_{ij}\cup A_{ij}^+(A_{ij}^-)}(x)\right) \tag{23}$$

We first assume the condition (20) holds. For the PLS (1) in regions R_i and R_j,

$$f(x) = \begin{cases} A_i x + b_i & x \in R_i \\ A_j x + b_j & x \in R_j \end{cases} \tag{24}$$

so the Filippov differential inclusion is

$$K_F[f](x) = \overline{\mathrm{co}}\{A_i x + b_i, A_j x + b_j\} \tag{25}$$

where $\overline{\mathrm{co}}S$ stands for the convex hull of a point set S.

On the other hand, for all $x \in L_{ij}$, the contingent cone of surface and regions can be expressed as

$$T^B_{L_{ij}\cup A_{ij}^+}(x) = \{Z \mid C_{ij}Z \geq 0\}, \quad \mathrm{Int}T^B_{L_{ij}\cup A_{ij}^+}(x) = \{Z \mid C_{ij}Z > 0\} \tag{26}$$

For all $f(x) \in K_F[f](x)$, there exists $0 \leq \lambda(x) \leq 1$, such that,

$$f(x) = \lambda(x)(A_i x + b_i) + (1 - \lambda(x))(A_j x + b_j) \tag{27}$$

Therefore,

$$C_{ij}f(x) = C_{ij}\lambda(x)(A_i x + b_i) + C_{ij}(1 - \lambda(x))(A_j x + b_j) > 0 \tag{28}$$

and hence,

$$f(x) \in \mathrm{Int}\left(T^B_{L_{ij}\cup A_{ij}^+}(x)\right) \tag{29}$$

Thus,

$$K_F[f](x) \subseteq \mathrm{Int}\left(T^B_{L_{ij}\cup A_{ij}^+}(x)\right) \tag{30}$$

A similar argument applies when condition (21) holds, yielding

$$K_F[f](x) \subseteq \mathrm{Int}\left(T^B_{L_{ij}\cup A_{ij}^-}(x)\right). \tag{31}$$

This completes the proof.

The following proposition enables the numerical verification of conditions (20) and (21) under the assumption that each polytopic region R_i is bounded.

Proposition 3.2. *Suppose that every R_i is bounded, then conditions (20) and (21) in Lemma 3.1 are equivalent to either set of LMIs:*

$$\begin{array}{ll} C_{ij}(A_i x_{l1} + b_i) > 0, & C_{ij}(A_j x_{l1} + b_j) > 0, \\ C_{ij}(A_i x_{l2} + b_i) > 0, & C_{ij}(A_j x_{l2} + b_j) > 0, \\ & \vdots \\ C_{ij}(A_i x_{lp} + b_i) > 0, & C_{ij}(A_j x_{lp} + b_j) > 0, \end{array} \tag{32}$$

or

$$\begin{array}{ll} C_{ij}(A_i x_{l1} + b_i) < 0, & C_{ij}(A_j x_{l1} + b_j) < 0, \\ C_{ij}(A_i x_{l2} + b_i) < 0, & C_{ij}(A_j x_{l2} + b_j) < 0, \\ & \vdots \\ C_{ij}(A_i x_{lp} + b_i) < 0, & C_{ij}(A_j x_{lp} + b_j) < 0, \end{array} \tag{33}$$

where $x_{l1}, x_{l2} \cdots x_{lp}$ are the nonzero vertices of boundary L_{ij}.

Proof: Necessity is obvious. Here we will prove sufficiency.

As R_i is a polytopic region, any $x \in L_{ij}$ can be expressed as

$$x = \sum_{k=1}^{p} a_k x_{lk}, \quad a_i \geq 0, \quad a_1 + a_2 + \cdots + a_p = 1.$$

It can be shown that

$$C_{ij}(A_i x + b_i) = \sum_{k=1}^{p} a_k C_{ij}(A_i x_{lk} + b_i), \quad x \in L_{ij}.$$

Using (32) and (33), we obtain

$$C_{ij}(A_i x + b_i) > 0 \ (< 0),$$

completing the proof.

3.2. **Controller design for PLS.** Incorporating Lemma 3.1 and Proposition 3.2 with PQLF theory, we now present a new controller synthesis method for PLS (1).

Theorem 3.1. *Consider the PLS described by (1), suppose that every region R_i is bounded. If there is a solution for the optimization problem below, the closed-loop system is well-posed and asymptotically stable.*

The optimization problem:

$$\max(\min_i \alpha_i) \ \text{ s.t. } (34) - (37)$$

Variables:

$$k_i, \ m_i; \quad T > 0; \quad U_i, \ W_i \succ 0; \quad \alpha_i > 0.$$

Constraints:

$$m_i = 0, \quad i \in I_0 \tag{34}$$

$$P_i - E_i^T U_i E_i > 0, \quad i \in I \tag{35}$$

$$\widehat{A}_i^T P_i + P_i \widehat{A}_i + \alpha_i P_i + E_i^T W_i E_i < 0, \quad i \in I \tag{36}$$

$$\begin{array}{l} C_{ij}(\bar{A}_i x_{ijq} + \bar{b}_i) > 0, \quad C_{ij}(\bar{A}_j x_{ijq} + \bar{b}_j) > 0 \\ \text{or} \\ C_{ij}(\bar{A}_i x_{ijq} + \bar{b}_i) < 0, \quad C_{ij}(\bar{A}_j x_{ijq} + \bar{b}_j) < 0 \end{array} \tag{37}$$

where $P_i = F_i^T T F_i$, $\bar{A}_i = A_i + B_i k_i$, $\bar{b}_i = b_i + B_i m_i$, and the matrix $Z \succeq 0$ stands for the matrix Z has nonnegative entries.

Moreover, the controller gain for each local subsystem is given by

$$K_i = [k_i, m_i] = -\widehat{B}_i^T \widehat{P}_i \tag{38}$$

Proof: First, it is noted from (34) and (38) that the origin is an equilibrium of resulting closed-loop system. Therefore, we choose the PQLF as below,

$$V(x) = \sum_i \beta_i V_i(x), \quad V_i(x) = \bar{x}^T P_i \bar{x}, \quad x \in R_i \tag{39}$$

where

$$\beta_i = \begin{cases} 1 & x \in R_i \\ 0 & \text{others} \end{cases} \tag{40}$$

It can be shown by region description (4), for all matrix U_i, W_i with compatible dimensions and nonnegative entries, we have,

$$x \in R_i \implies \hat{x}^T E_i^T U_i E_i \hat{x} \geq 0, \quad \hat{x}^T E_i^T W_i E_i \hat{x} \geq 0 \tag{41}$$

with this inequality and condition (35), we have for all $x \in R_i \backslash 0$,

$$V(x) = \hat{x}^T \hat{P}_i \hat{x} > \hat{x}^T E_i^T U_i E_i \hat{x} \geq 0 \tag{42}$$

Moreover, with the help of condition (36) and controller expression (38), we have for all $x \in R_i$,

$$\begin{aligned} \frac{dV}{dt} &= \hat{x}^T \left[\left(\widehat{A}_i + \widehat{B}_i K_i \right)^T \widehat{P}_i + \widehat{P}_i \left(\widehat{A}_i + \widehat{B}_i K_i \right) \right] \hat{x} \\ &= \hat{x}^T \left[\widehat{A}_i^T \widehat{P}_i + \widehat{P}_i \widehat{A}_i - 2 \cdot \widehat{P}_i \widehat{B}_i^T \widehat{B}_i \widehat{P}_i \right] \hat{x} \\ &< \hat{x}^T \left[-\alpha_i \widehat{P}_i - E_{i2} - 2 \cdot \widehat{P}_i \widehat{B}_i^T \widehat{B}_i \widehat{P}_i \right] \hat{x} \\ &< -\hat{x}^T E_i^T W_i E_i \hat{x} < 0 \end{aligned} \tag{43}$$

Furthermore, from the boundary expression (4) we know, the selected PQLF $V(x)$ is continuous when the state crosses from region R_i to region R_j, because for all $x \in L_{ij}$,

$$V_i(x) - V_j(x) = \hat{x}^T \left(F_i^T T F_i - F_j^T T F_j \right) \hat{x} = 0 \tag{44}$$

Therefore, the selected Lyapunov function $V(x)$ will decrease within every region R_i and keep invariant when state trajectory crossing the boundary L_{ij}. However, to complete the proof, we still need to consider the possibility of sliding motion on the boundary. It is noted from Lemma 3.1 and Proposition 3.2, the condition (37) guarantees that sliding motion cannot occur at the boundaries, i.e., the closed-loop system is well-posed.

To sum up, the closed-loop PLS is well-posed and asymptotically stable.

Remark 3.1. *It is noted that as long as the LMIs defined in (34)-(37) have solutions, then the stable state feedback control law can be constructively obtained via (38).*

Remark 3.2. *The continuity of the state feedback controllers on the boundary l can be achieved if we add the constraint $k_i x + m_i = k_j x + m_j$, $x \in L_{ij}$.*

Remark 3.3. *In comparison with Theorem 2.1, the limitation presented by Remark 2.1 is improved, and the inequality constraints are less conservative. However, we require that every R_i is bounded for the numerical computation, which is unnecessary for Theorem 2.1.*

Remark 3.4. *It is worth to note that, the proposed design method requires the stability of each open-loop subsystem (see (36)). Therefore, we use the following procedure [9]. First, we design local primary state feedback controller for each local model to make them become stable, then using Theorem 3.1 to find the secondary local controller for each local model so that the strong stability requirement is satisfied. Finally, we obtain total state feedback controller by combining the primary and the secondary controller together.*

4. **Numerical Example.** The purpose of this section is to show the efficacy and advantage of the proposed controller synthesis method through two practical engineering examples.

Example 4.1. *The objective of this example is to design a continuous PWA controller* $u = k_i x + m_i$, $x \in R_i$ *that forces an aerobatic helicopter follow a straight line trajectory in the lateral plane. The following three DOF helicopter model can be shown based on the six DOF model developed in [23].*

$$\begin{bmatrix} \dot{\psi} \\ \dot{r} \\ \dot{\nu} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{k_r}{I_{zz}} & 0 & 0 \\ 0 & -u_0 & -\frac{k_\nu}{m} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ r \\ \nu \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ Q_m \\ 0 \\ g(\psi, \nu) \end{bmatrix} + \begin{bmatrix} 0 \\ T(\nu, \delta_{ped}) \\ 0 \\ 0 \end{bmatrix}, \tag{45}$$

$$g(\psi, \nu) = u_0 \sin(\psi + \beta) + \nu \cos(\psi + \beta), \quad \beta = \arctan\left(\frac{\nu}{u_0}\right), \tag{46}$$

$$T(\nu, \delta_{ped}) = \frac{4\rho l_t \Omega_t R_t^3 \pi}{m} \nu \delta_{ped}, \tag{47}$$

where ψ, r, ν *and* y *are the heading angle, angular rate, translational velocity and longitudinal position, respectively. The control input is* δ_{ped}, *which denotes the tail rotor pedal control. All the parameter values and their physical meaning are illustrated in Table 1.*

Table 1. Simulation parameters

$m = 68.76$	Helicopter mass (Kg)
$\rho = 1.293$	Air density (Kg/m^3)
$u_0 = 0.7$	Helicopter forward velocity (m/s)
$I_{zz} = 0.01$	Moment of inertia (Kg·m^2)
$l_t = 0.5$	Tail rotor hub location behind c.g. (m)
$k_r = 0.1$	Yaw damping coefficient
$k_\nu = 1$	Slide damping coefficient
$Q_m = 1$	Main rotor torque (N · m)
$R_t = 1.853$	Tail rotor radius (ft)
$\Omega_t = 865.49$	Nominal tail rotor speed (rad/s)

Assume the helicopter can start from any possible initial angle in the range $\psi_0 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ *and any initial velocity in the range* $\nu_0 \in [-1, 1]$. *The nonlinear functions* $g(\psi, \nu)$ *and* $T(\nu, \delta_{ped})$ *can be approximated by PWA functions (see* [18]*) yielding a PLS with* 8 *regions* *as below.*

$$R_1 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[0, \frac{\pi}{2}\right], x_3 \in [-1, -0.5]\right\},$$
$$R_2 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[0, \frac{\pi}{2}\right], x_3 \in [-0.5, 0]\right\},$$
$$R_3 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[0, \frac{\pi}{2}\right], x_3 \in [0, 0.5]\right\},$$
$$R_4 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[0, \frac{\pi}{2}\right], x_3 \in [0.5, 1]\right\},$$
$$R_5 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[-\frac{\pi}{2}, 0\right], x_3 \in [-1, -0.5]\right\},$$
$$R_6 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[-\frac{\pi}{2}, 0\right], x_3 \in [-0.5, 0]\right\},$$
$$R_7 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[-\frac{\pi}{2}, 0\right], x_3 \in [0, 0.5]\right\},$$
$$R_8 = \left\{x|x \in \mathbb{R}^4|x_1 \in \left[-\frac{\pi}{2}, 0\right], x_3 \in [0.5, 1]\right\}.$$

The pole place method [24] is applied first to design a primary controller to stabilize each local system. Then, by solving the convex optimization presented in Theorem 3.1 using MATLAB, the following optimal solutions are obtained:

$$K_1 = K_8 = \begin{bmatrix}-31.2503 & 1.3230 & -16.6049 & 16.3479 & -41.5496\end{bmatrix}$$
$$K_2 = K_7 = \begin{bmatrix}-39.2600 & -9.6484 & -43.8773 & 22.5720 & -124.5330\end{bmatrix}$$
$$K_3 = K_6 = \begin{bmatrix}13.5607 & -1.0441 & -4.9824 & -10.4257 & 124.5330\end{bmatrix}$$
$$K_4 = K_5 = \begin{bmatrix}10.3747 & 1.2046 & -9.1763 & -16.3523 & 41.4252\end{bmatrix}$$

With the combined controller gains, simulations have been carried out with initial conditions $x(0) = \left[-\frac{\pi}{2}, -1, -1, 5\right]$, the results are shown in Figure 2. It can be observed in Figure 2 that the states converge to the origin, i.e., the controller makes the flight trajectory converge to the desired straight line, which demonstrates the efficacy of the proposed method. For comparison, we try to design continuous PWA controller using the Theorem 2.1, but there is no solution from the computation by MATLAB. This demonstrates Remark 2.1, that is, for the PLS satisfying Equation (12), Theorem 2.1 is not effective to design continuous PWA controller for PLS. This result shows the advantage of the synthesis method presented in this paper.

Example 4.2. *This problem considers the controller design problem for the 2-D cart-pendulum system (Example 1 in [9]). The equations of motion of the pendulum are*

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{g\sin(x_1) - \frac{amlx_2^2\sin(2x_1)}{2} - a\cos(x_1)u}{\frac{4l}{3} - aml\cos^2(x_1)}$$

where $(x_1, x_2) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times [-5, 5]$ are the pendulum angle (with respect to the upright vertical) and its velocity, the input u is the acceleration of the cart. $g = 9.8m/s^2$ is the gravity constant, m is the mass of the pendulum, M is the mass of the cart, $2l$ is the length of the pendulum. In this simulation, the pendulum parameters are chosen as $m = 2kg$, $M = 8kg$ and $2l = 1.0m$. The objective is to design PWA controller $u = k_i x + m_i$ $x \in R_i$ to stabilize the cart-pendulum system, i.e., balance the inverted pendulum on its equilibrium.

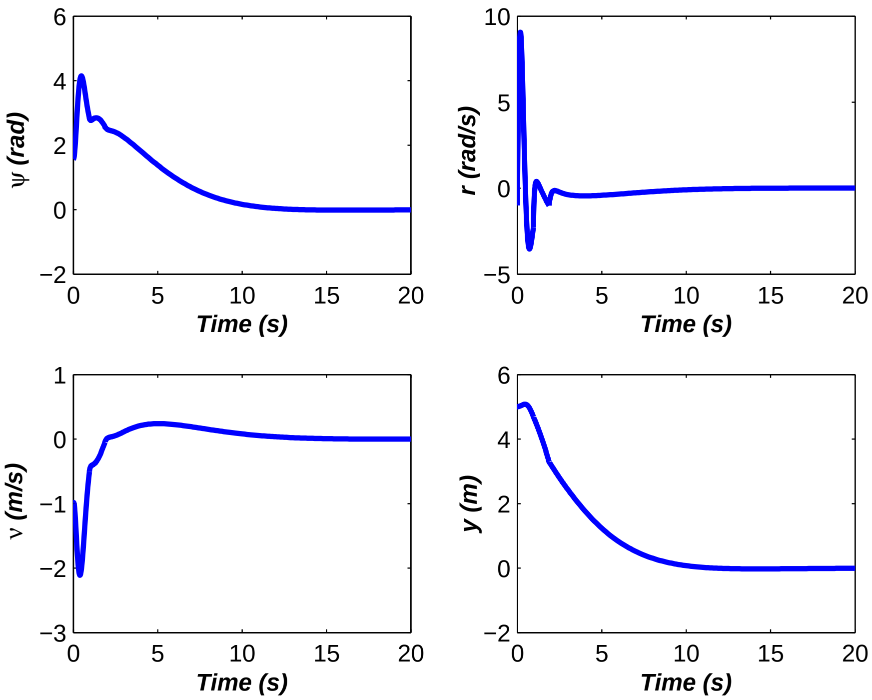

FIGURE 2. State trajectory with the obtained controller

With the following eight regions partitioned

$$R_1 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right], x_2 \in [0, 5]\right\},$$
$$R_2 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[-\frac{\pi}{4}, 0\right], x_2 \in [0, 5]\right\},$$
$$R_3 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[0, \frac{\pi}{4}\right], x_2 \in [0, 5]\right\},$$
$$R_4 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], x_2 \in [0, 5]\right\},$$
$$R_5 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right], x_2 \in [-5, 0]\right\},$$
$$R_6 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[-\frac{\pi}{4}, 0\right], x_2 \in [-5, 0]\right\},$$
$$R_7 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[0, \frac{\pi}{4}\right], x_2 \in [-5, 0]\right\},$$
$$R_8 = \left\{x|x \in \mathbb{R}^2|x_1 \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], x_2 \in [-5, 0]\right\},$$

the nonlinear functions $\sin(x_1)$*,* x_2^2*,* $\sin(2x_1)$*,* $\cos(x_1)$ *and* $\cos^2(x_1)$ *are linearized within each region. By substituting these nonlinear functions with the approximated PWA functions for each region, the approximated PLS is obtained.*

For the obtained PLS, by applying two syntheis approaches Theorem 3.1 and Theorem 2.1, the following optimization solutions are provided by MATLAB, respectively.

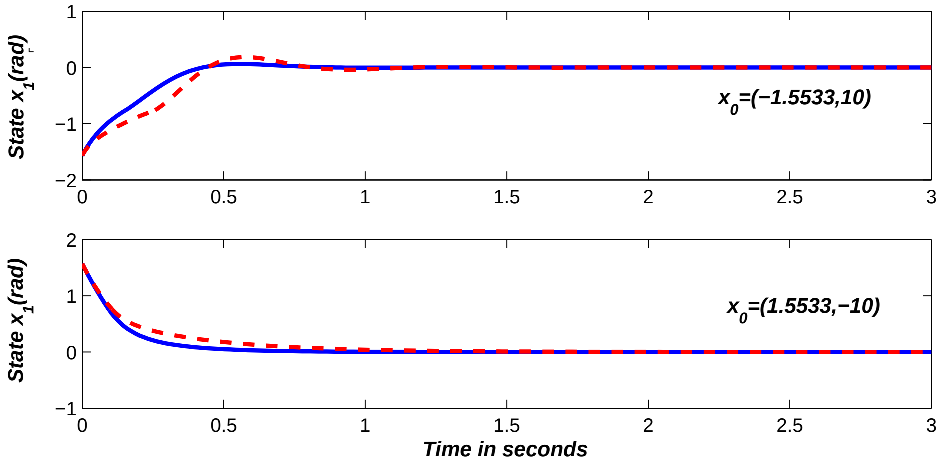

Figure 3. Comparison between the control performances of synthesis approaches Theorem 2.1 (dashed) and Theorem 3.1 (solid)

The solution obtained by Theorem 3.1:

$$\max(\min_i \alpha_i) = 0.3573$$
$$K_1 = \begin{bmatrix}2365.00 & 499.48 & -740.01\end{bmatrix}, \; K_2 = \begin{bmatrix}1071.76 & 106.88 & 0\end{bmatrix}$$
$$K_3 = \begin{bmatrix}4786.48 & 633.99 & 0\end{bmatrix}, \; K_4 = \begin{bmatrix}3280.91 & 314.40 & 732.32\end{bmatrix}$$
$$K_5 = \begin{bmatrix}1017.86 & 68.38 & -741.86\end{bmatrix}, \; K_6 = \begin{bmatrix}555.26 & 137.39 & 0\end{bmatrix}$$
$$K_7 = \begin{bmatrix}544.75 & 117.17 & 0\end{bmatrix}, \; K_8 = \begin{bmatrix}1144.23 & 140.31 & 731.98\end{bmatrix}$$

The solution obtained by Theorem 2.1:

$$\max(\min_i \alpha_i) = 0.1326$$
$$K_1 = \begin{bmatrix}3001.39 & 898.41 & -645.44\end{bmatrix}, \; K_2 = \begin{bmatrix}1184.90 & 236.73 & 0\end{bmatrix}$$
$$K_3 = \begin{bmatrix}2373.57 & 272.68 & 0\end{bmatrix}, \; K_4 = \begin{bmatrix}2693.39 & 267.80 & 714.58\end{bmatrix}$$
$$K_5 = \begin{bmatrix}964.23 & 6.45 & -645.44\end{bmatrix}, \; K_6 = \begin{bmatrix}533.06 & 58.42 & 0\end{bmatrix}$$
$$K_7 = \begin{bmatrix}527.37 & 67.84 & 0\end{bmatrix}, \; K_8 = \begin{bmatrix}1054.47 & 164.75 & 714.58\end{bmatrix}$$

It can be clearly seen from the above two solutions that, the decay rate of Lyapunov function obtained by the proposed approach (Theorem 3.1) is faster than the approach [18] (Theorem 2.1). The same result can also be observed from Figure 3, where the simulation is carried out with the initial condition $(-1.5533, 10)$ *and* $(1.5533, -10)$*, respectively. These results state that there actually exist some cases that the proposed synthesis method (Theorem 3.1) is less conservative than the existing synthesis method (Theorem 2.1).*

5. **Conclusions.** In this paper, a new method was developed to design state feedback controller for PLS. In comparison with existing methods, the key difference are a new criterion for avoidance of sliding motion on the boundary. Incorporating with PQLF theory, the stabilization problem is formulated as a convex optimization problem under LMI constraints. Two examples were illustrated to show the efficacy and advantage of the proposed method in practical engineering application. Example 4.1 shows that the new method can overcome the limitation of the existing method as we analysis in Section 2. Furthermore, for the Example 4.2, both synthesis approaches work for this example. By comparing the control performance of these two synthesis approaches, we learned that

the new method is less conservative than the existing method. However, the proposed new approach still has a limitation that every region R_i must be bounded, which will restrict its application. For the future work, we will focus on wiping out this assumption to achieve a much wider application.

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