Dimer Models, Integrable Systems and Gauge Theory

Great Lakes String Conference

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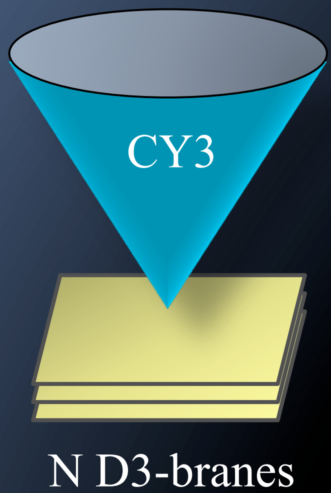
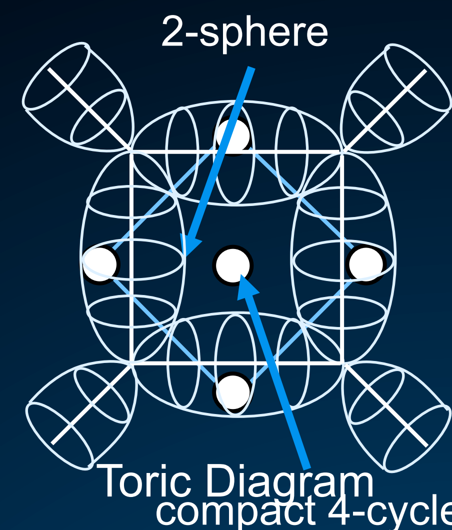
D-branes over Toric Singularities



Toric Geometry

- Torus fibrations over base spaces
- Described by specifying shrinking cycles and their relations
- Encoded by web or toric diagrams



The Quiver Gauge Theory

- On the worldvolume of D3-branes, N=1 superconformal field theory with:
  - Gauge group: Π U(N)
  - F-terms: monomial = monomial

Every field appears exactly twice in W with opposite signs
(Toric Condition)

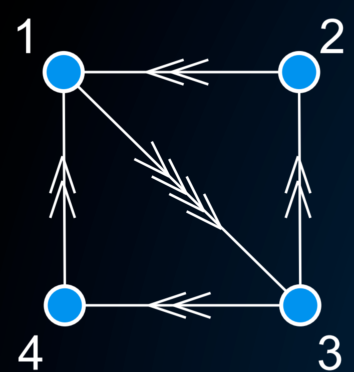
Brane Dimers

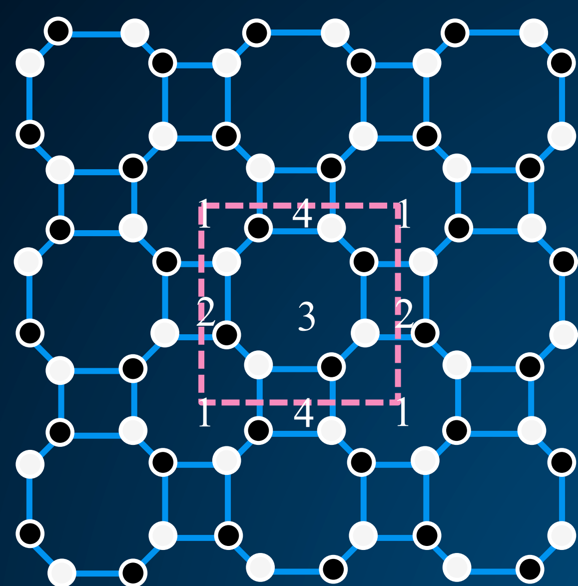
Franco, Hanany, Kennaway, Vegh, Wecht

- All the information defining the gauge theory can be encoded in a dimer model on T^2

Gauge Theory	Dimer
U(N) gauge group	face
bifundamental (or adjoint)	edge
superpotential term	node

- Example: complex cone over F_0

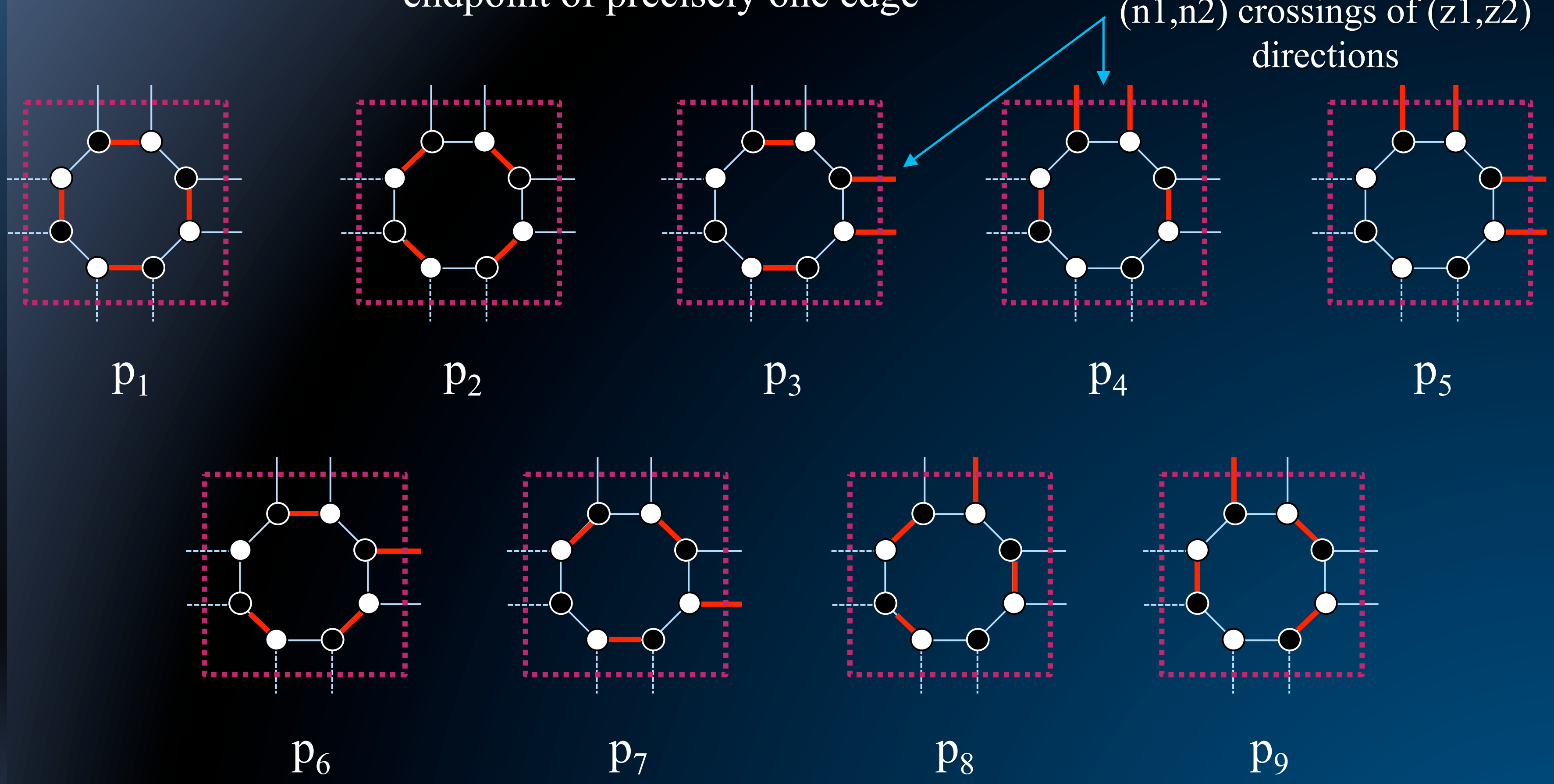




The dimer model is a physical configuration of NS5 and D5-branes

Perfect Matchings and Geometry

- Perfect matching: configurations of edges such that every vertex in the graph is an endpoint of precisely one edge



- Moduli Space: perfect matchings are the natural variables solving F-term equations

Franco, Hanany, Kennaway, Vegh, Wecht

Franco, Vegh

Perfect Matchings and Geometry

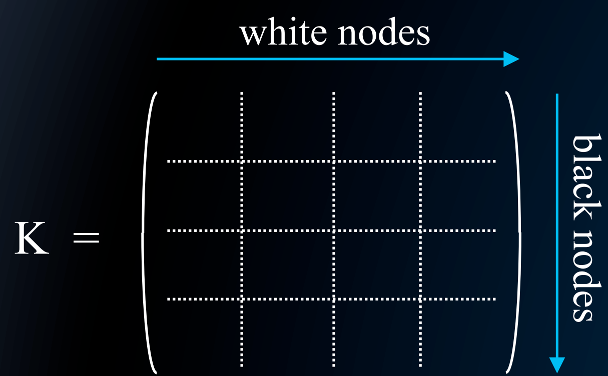
- There is a one to one correspondence between perfect matchings and GLSM fields describing the toric singularity (points in the toric diagram)

  Franco, Hanany, Kennaway, Vegh, Wecht

  Franco, Vegh

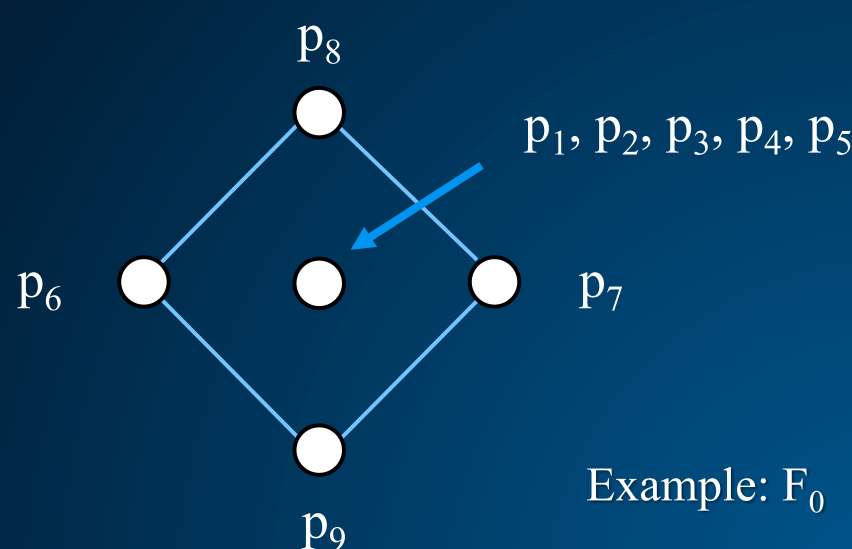
- This correspondence trivialized formerly complicated problems such as the computation of the moduli space of the SCFT, which reduces to calculating the determinant of an adjacency matrix of the dimer model (Kasteleyn matrix)

Kasteleyn Matrix



$$\det K = P(z_1,z_2) = \Sigma\, n_{ij}\, z_1^{\,i}\, z_2^{\,j}$$

Toric Diagram



Example: F_0

Multiple Applications

Mathematics ⟷ **Physics**

- BPS invariants of CYs (e.g. DT)
  Eager, SF
- Mirror symmetry
- Toric/Seiberg duality
  SF, Hanany, Kennaway, Vegh, Wecht
- D-brane instantons
  SF, Hanany, Krefl, Park, Uranga
- AdS/CFT correspondence in 3+1 and 2+1 dimensions
  SF, Hanany, Martelli, Sparks, Vegh, Wecht
  SF, Hanany, Park, Rodriguez-Gomez
  SF, Klebanov, Rodriguez-Gomez
- Dynamical SUSY breaking
  SF, Uranga
- Local constructions of MSSM + CKM

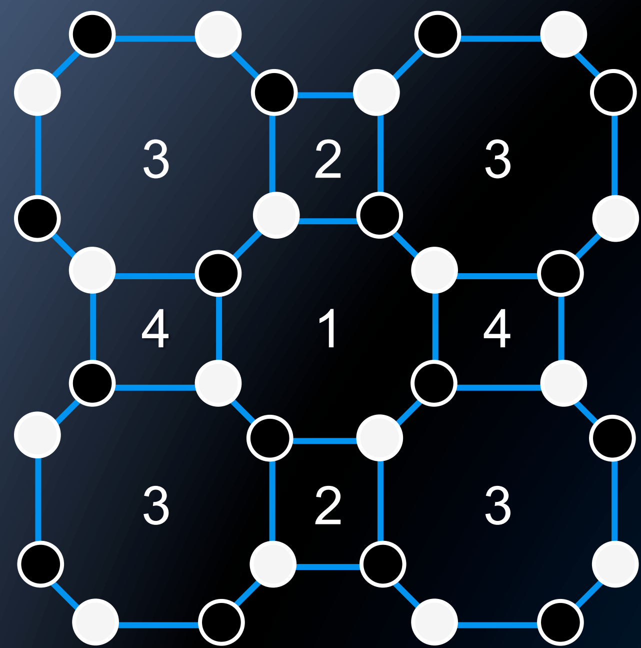
- The power of dimer models:
  - Define an infinite class of interesting objects: largest classification of 4d, N=1 SCFTs
  - Make previous complicated calculations trivial: determination of their moduli space

Can they do it again? ✓ YES!

  - Define an infinite class of quantum integrable systems
  - Constructing all conserved charges becomes straightforward

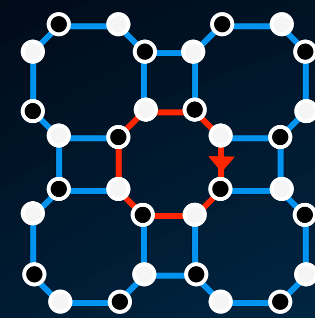
Dimers Models and Integrable Systems

From a Dimer to Phase Space

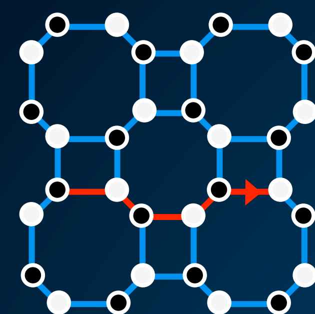


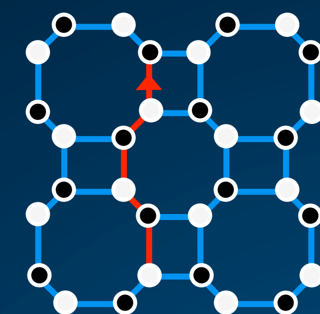
Example: F_0

- One w_i variable per gauge group:



- Two 2-torus directions:





Poisson Structure

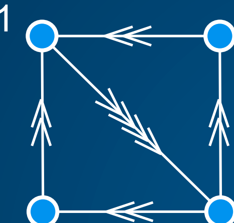
- $\{w_i, w_j\} = I_{ij}\, w_i\, w_j$

I_{ij}: intersection matrix

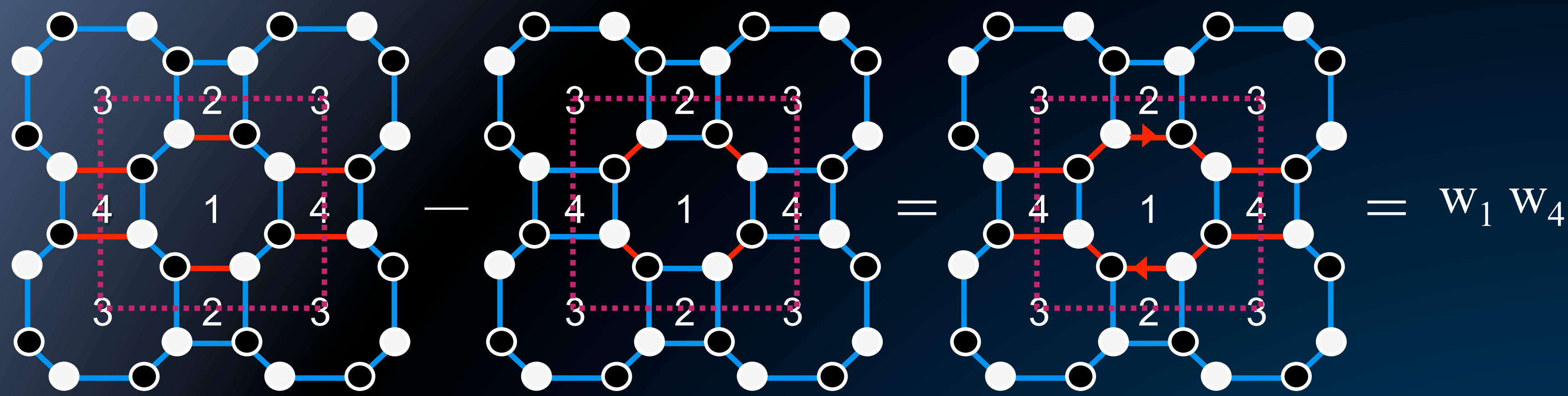
➢ exponential in p and q

- Idem for $\{w_i, z_j\}$ and $\{z_1, z_2\}$

e.g: $\{w_1, w_3\} = 4\, w_1\, w_3$ $\{w_1, w_2\} = -2\, w_1\, w_2$



- Every perfect matching defines a closed path on the tiling by taking the difference with respect to a reference perfect matching



- Every perfect matching can be expressed in term of loops variables

The Integrable System

Goncharov, Kenyon

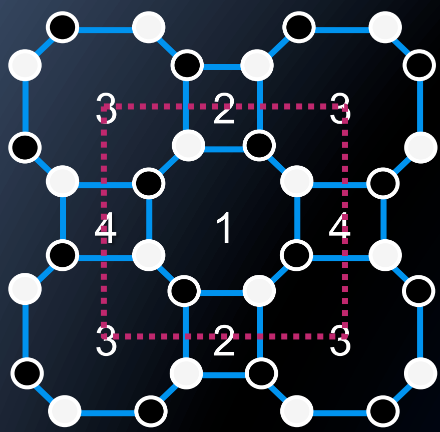
Eager, Franco, Schaeffer

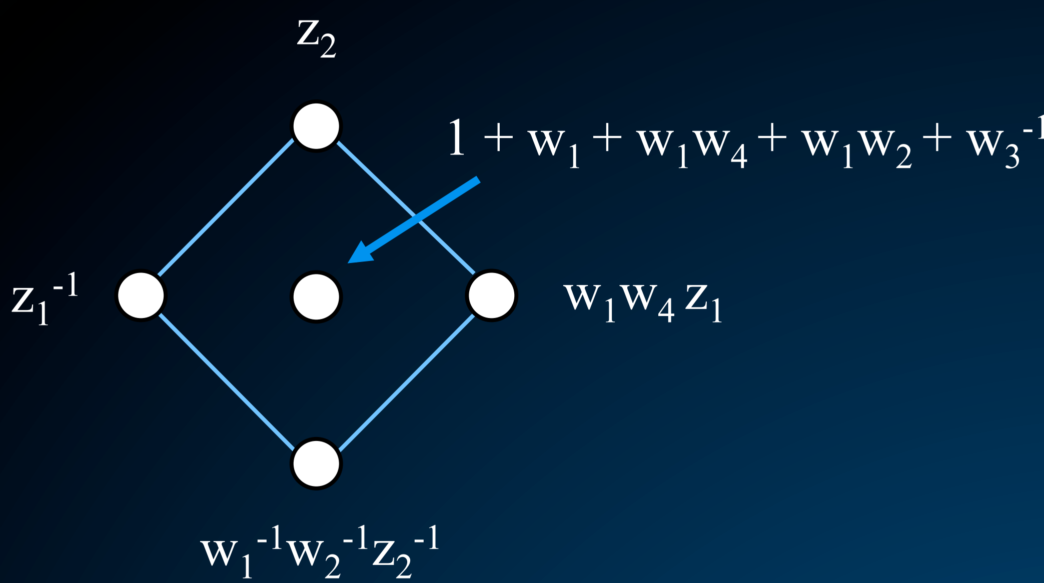
The commutators define a 0+1d quantum integrable system of dimension 2 + 2 Area (toric diagram), with symplectic leaves of dimension 2 $N_{interior}$

- Casimirs: ratios of boundary points (commute with everything)
- Hamiltonians: internal points (commute with each other)

An explicit example: F0

- This theory has 9 perfect matchings





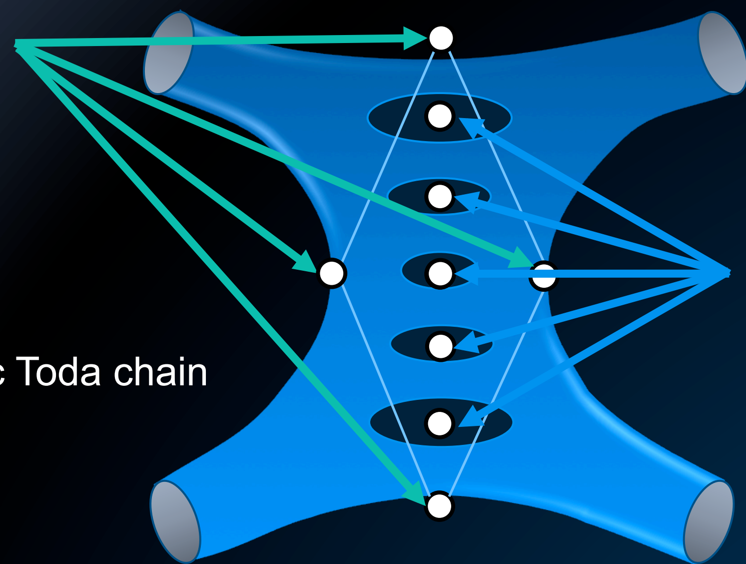
- Casimirs:
  $C_1 = z_1 z_2$
  $C_2 = w_1 w_2 z_2 / z_1$
  $C_3 = 1/(w_1^2 w_2^2 z_1 z_2)$

- Hamiltonian: $H = 1 + w_1 + w_1 w_4 + w_1 w_2 + w_3^{-1}$

The Integrable System

Franco

Eager, Franco, Schaeffer



e.g.: relativistic periodic Toda chain
(Conifold/Z_n)

Brane configuration for:
5d, N=1, pure SU(n)
gauge theory on S^1

- Characteristic polynomial: $P(z_1,z_2)$ → coeficients and their ratios give Hamiltonians and Casimirs

Spectral curve Σ		Mirror manifold
$P(z_1,z_2) = 0$	⇒	$P(z_1 z_2) = W \qquad u\,v = W$

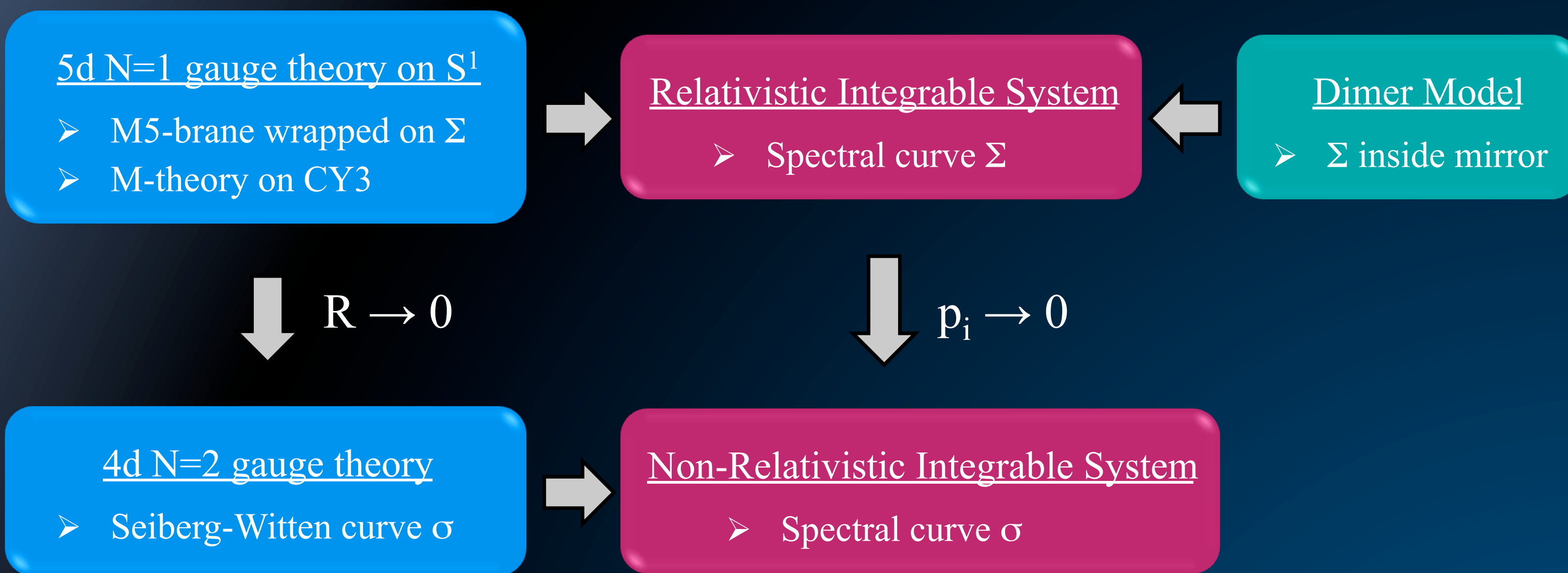
- Fully constructive prescription for building an integrable system given a spectral curve

  Feng, He, Kennaway, Vafa

  integrable system ⇕ quiver/dimer model; spectral curve ⇕ mirror manifold

Connection to 4d and 5d Gauge Theory

- Multiple avatars of the Riemann surface Σ Eager, Franco, Schaeffer



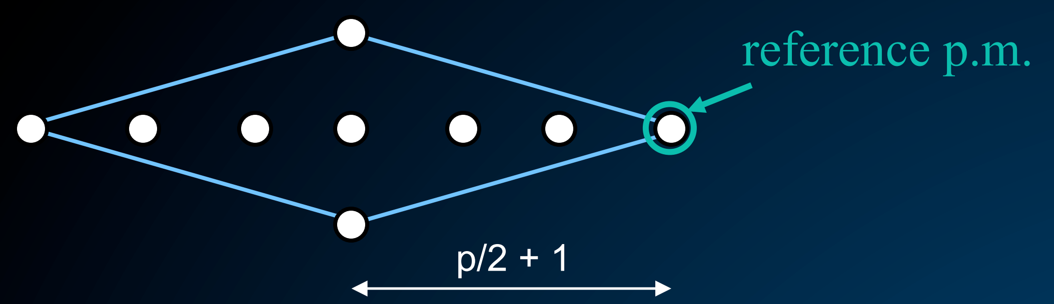
- Among other things, we systematically answer the question: what is the integrable system associated to an arbitrary 4d N=2 gauge theory? (spectral curve as SW curve)

An example: Relativistic Periodic Toda Chain

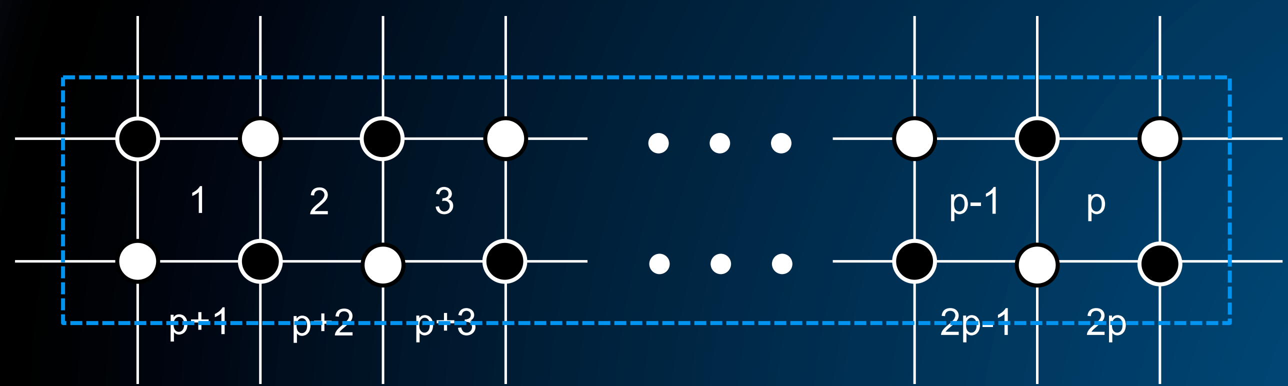
- Spectral curve Σ → 5d, N=1, pure SU(p) gauge theory on S^1

Nekrasov

- It corresponds to $Y^{p,0}$ (Z_p orbifold of the conifold)



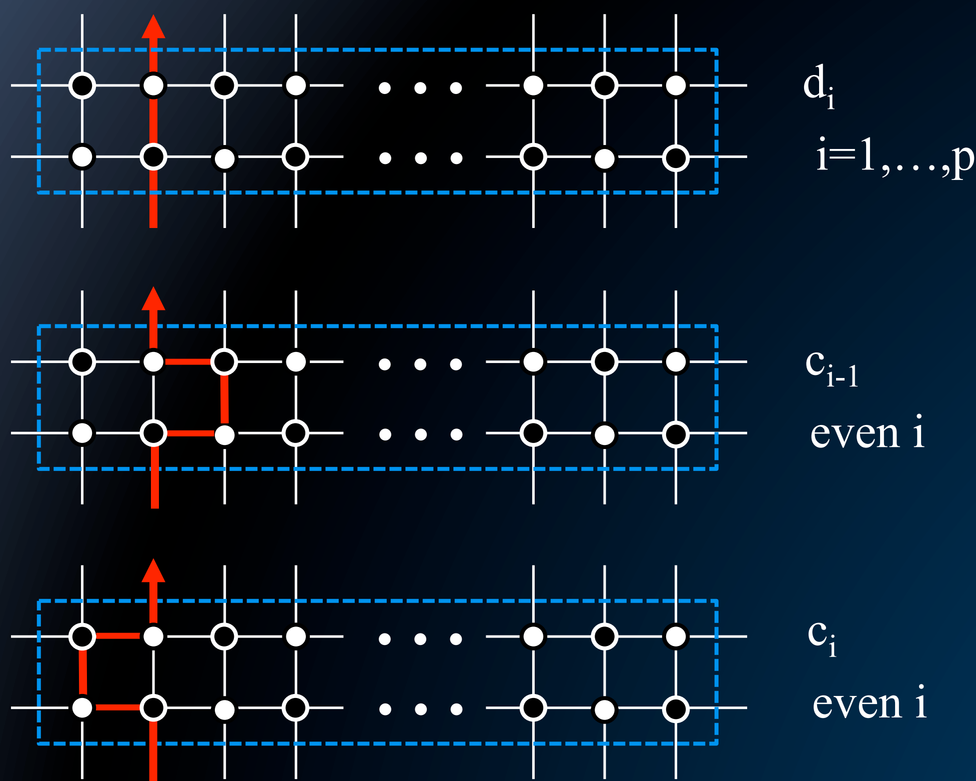
- Dimer model:



Relativistic Toda Chain: The Integrable System

Eager, Franco, Schaeffer

- Basic cyles: w_i ($i = 1, \ldots, 2p$), z_1 and z_2
- A more convenient basis:



$$\{c_k, d_k\} = c_k\, d_k$$
$$\{c_k, d_{k+1}\} = c_k\, d_{k+1}$$
$$\{c_k, c_{k+1}\} = -\, c_k\, c_{k+1}$$

- Two additional cycles fixed by Casimirs
- Hamiltonians in terms of non-intersecting paths:

$$H_k = \Sigma \underbrace{\Pi\, c_i\, d_j}_{\text{k factors}}$$

Bruschi, Ragnisco

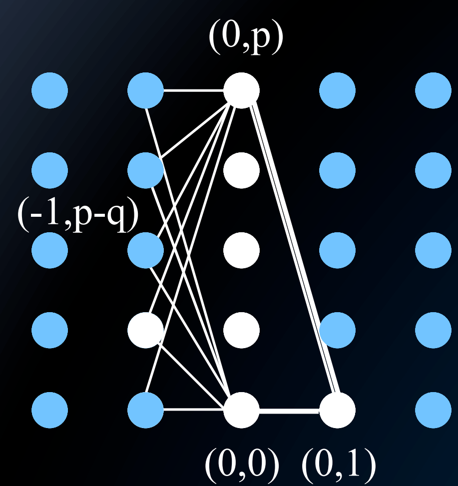
- Lax operator → Kasteleyn matrix

$$H_1 = \Sigma\, (c_i + d_i)$$

Quiver Impurities = Spin Chain Impurities

Eager, Franco, Schaeffer

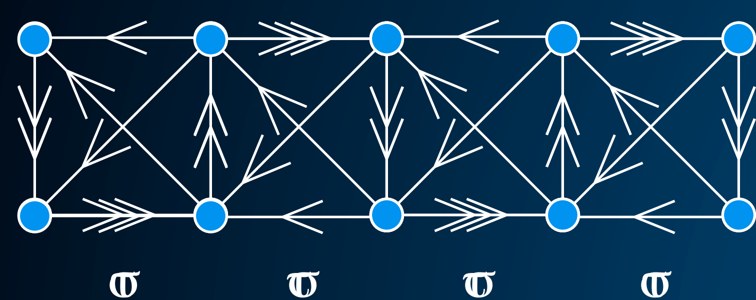
- Relativistic, periodic Toda chain →
  - 5d, N=1, pure SU(p) on S^1
  - Quantized cubic coupling in prepotential: $c_{cl} = 0, \ldots, p$ (disappears in 4d limit)



- $Y^{p,0}$: conifold/Z_p
- $Y^{p,p}$: $C^3/(Z_2 \times Z_p)$
- $c_{cl} = q$

- These are the toric diagrams for $Y^{p,q}$ manifolds
- Quivers constructed iteratively starting for $Y^{p,p}$ and adding (p-q) impurities

Benvenuti, Franco, Hanany, Martelli, Sparks



- The quiver impurities are indeed impurities in XXZ spin chains

Conclusions

- Dimer models are brane configurations in String Theory connecting Calabi-Yau's and quantum field theories in various dimensions

- In addition, they define an infinite class of quantum integrable systems

- The computation of all conserved charges becomes straightforward

- These integrable systems are also associated to 5d N=1 and 4d N=2 gauge theories

    - Dimer models provide a systematic procedure for constructing the integrable system for an arbitrary gauge theory of this type

- Quantum Teichmüller Space: one-to-one correspondence between edges in dimer models and Fock coordinates in the Teichmüller space of Σ. The commutation relations required by integrability imply Chekhov-Fock quantization.

Future Directions

- Connection to quivers encoding the BPS spectrum of N=2 gauge theories, obtained from ideal triangulations of the SW curve.

  Alim, Cecotti, Cordova, Espahbodi, Rastogi, Vafa

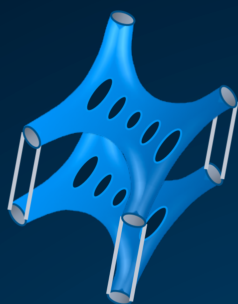
  Franco, He, *in progress*

- Applications to 3d-3d generalizations of the Alday-Gaiotto-Tachikawa (AGT) correspondence

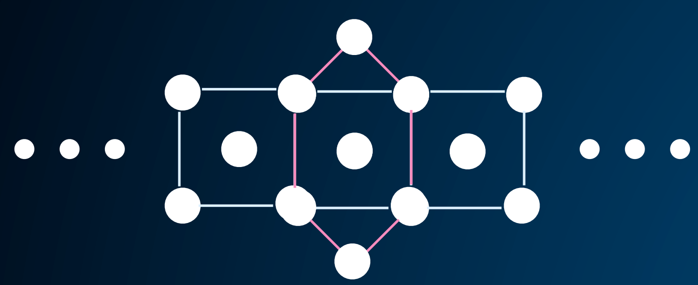
  Terashima, Yamazaki

$$Z_{3d\ SL(2,R)\ CS} = Z_{3d\ N=2\ theory}$$

$$M_3 = \Sigma \times I$$



- Study the continuous (1+1)-dimensional integrable field theory limit



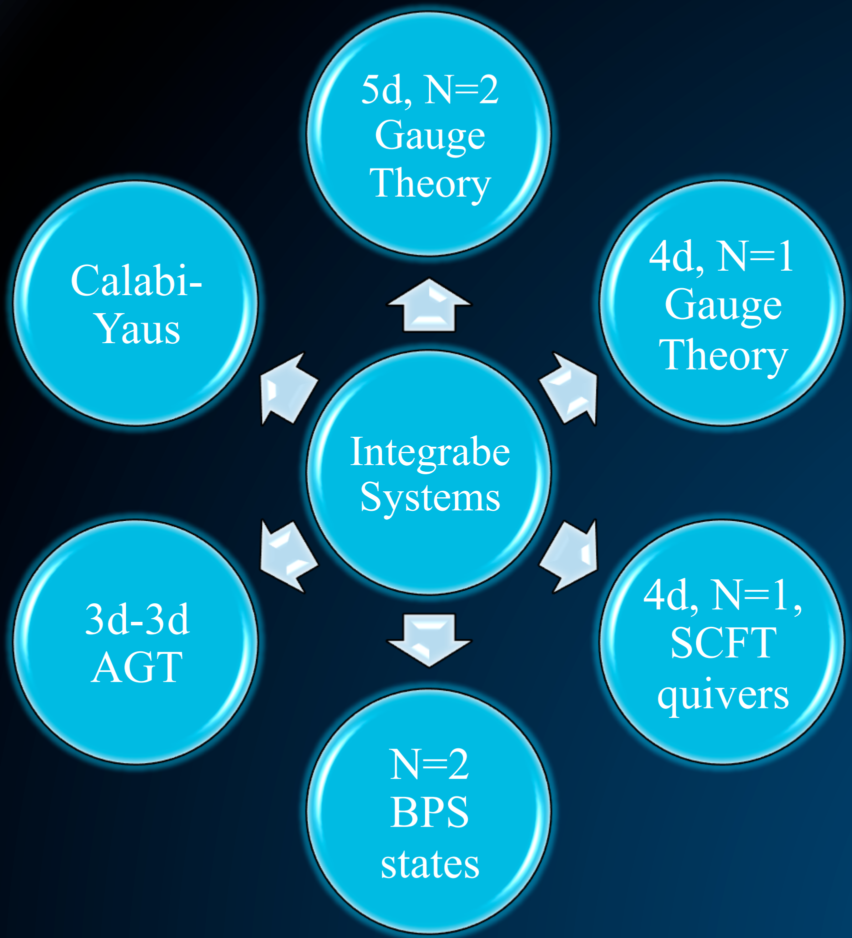
- Classification of possible integrable impurities and interfaces in integrable field theories

  Franco, Galloni, He, *in progress*

Thank you!

Multiple Connections

- Dimer models provide natural, systematic bridges connecting integrable systems to several physical systems.



Additional Slides

D-Branes on Singularities

- Bottom-up approach to String Phenomenology
  - Local constructions of SM-like theories — Aldázabal, Ibañez, Quevedo, Uranga; Verlinde, Wijnholt; Dolan, Krippendorf, Quevedo
  - Many features are independent of details of the compactification

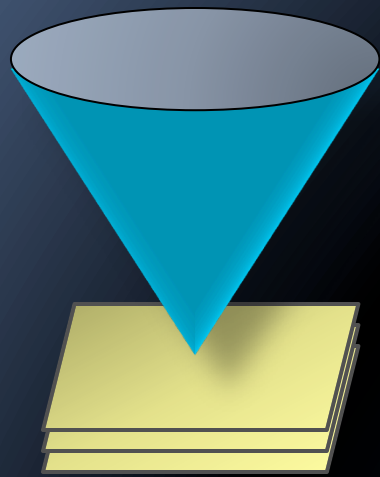
- Interesting strong gauge dynamics gets geometrized
  - Confinement — Klebanov, Strassler
  - SUSY breaking with runaway — Franco, Hanany, Saad, Uranga
  - Meta-stable SUSY breaking — Franco, Uranga

- Extensions of the AdS/CFT correspondence to theories with reduced (super) symmetry
  - Infinite families of explicit AdS/CFT dual pairs ($Y^{p,q}$ and $L^{a,b,c}$)

    Benvenuti, Franco, Hanany, Martelli, Sparks

    Franco, Hanany, Martelli, Sparks, Vegh, Wecht

D-branes over Toric Singularities



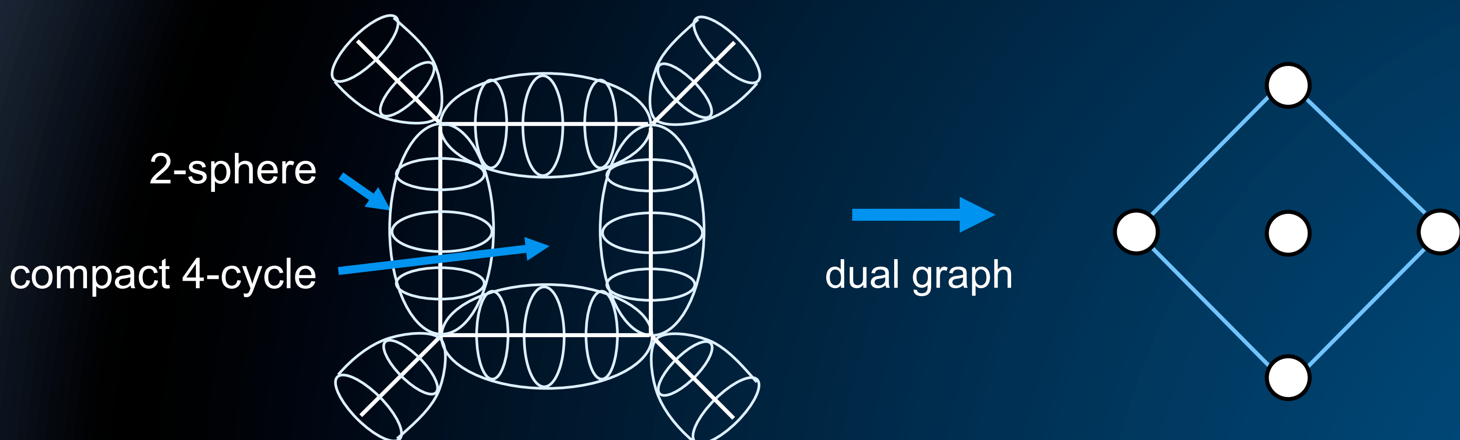
Toric CY3

N D3-branes

Toric Geometry

- Torus fibrations over base spaces
- Described by specifying shrinking cycles and their relations

Web Diagram — Toric Diagram



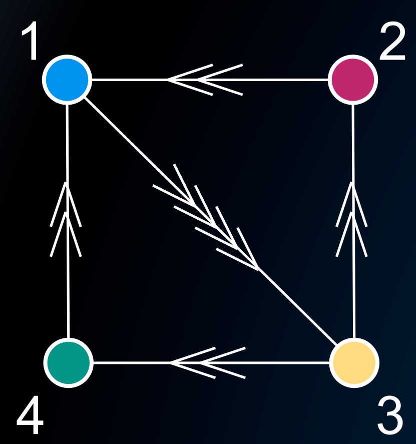
- Web and toric diagrams can be regarded as "pictures" of the geometry

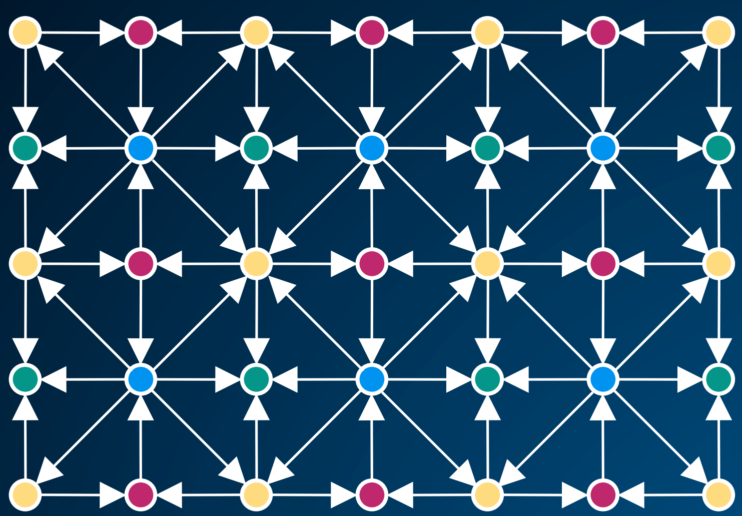
Combining Quiver and Superpotential Data: the Periodic Quiver

Franco, Hanany, Kennaway, Vegh, Wecht

Periodic Quiver: planar quiver drawn on the surface of a 2-torus such that every plaquette corresponds to a term in the superpotential

- Example: complex cone over F_0



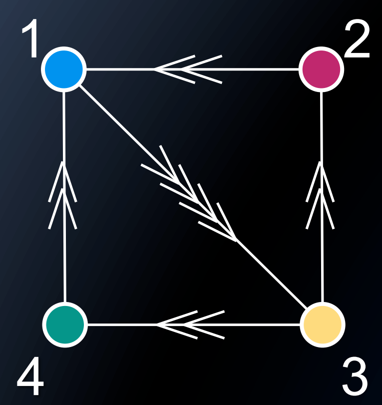


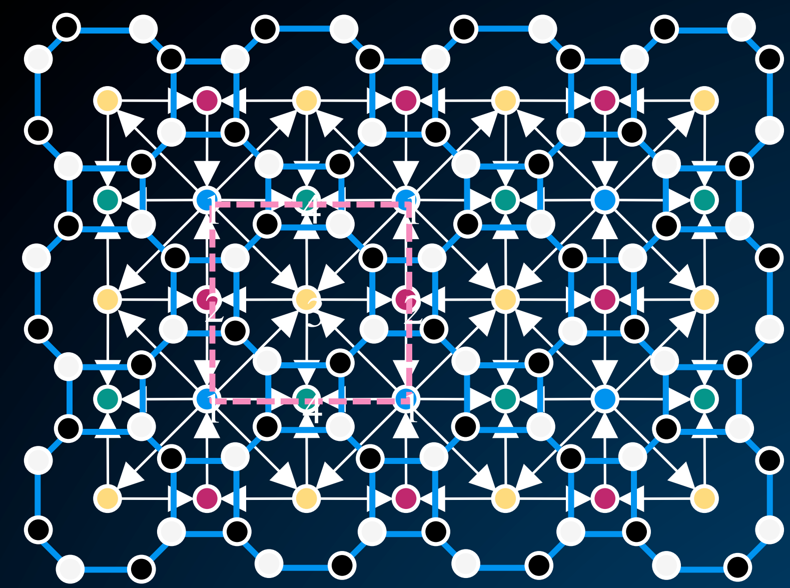
$$W = X^{11}_{13} X^{2}_{32} X^{2}_{21} - X^{12}_{13} X^{2}_{32} X^{1}_{21} - X^{21}_{13} X^{1}_{32} X^{2}_{21} + X^{22}_{13} X^{1}_{32} X^{1}_{21}$$
$$- X^{11}_{13} X^{2}_{34} X^{2}_{41} + X^{12}_{13} X^{2}_{34} X^{1}_{41} + X^{21}_{13} X^{1}_{34} X^{2}_{41} - X^{22}_{13} X^{1}_{34} X^{1}_{41}$$

Brane Dimers

- Periodic Quiver
  - Take the dual graph
  - It is bipartite (chirality)

  → Dimer Model





The dimer model is a physical configuration of NS5 and D5-branes

Gauge Theory	Periodic Quiver	Dimer
U(N) gauge group	node	face
bifundamental (or adjoint)	arrow	edge
superpotential term	plaquette	node

- The Kasteleyn matrix is the adjacency matrix of the dimer

$$P(z_1,z_2) = \det K$$

It controls conserved quantities

- Rows: ○
- Columns: ●

$$K = \begin{pmatrix} -H_1-\widetilde{H}_1 z_1 & \widetilde{V}_1 & & & & V_p z_2 \\ V_1 & H_2+\widetilde{H}_2 z_1^{-1} & \widetilde{V}_2 & & & \\ & V_2 & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \widetilde{V}_{p-1} \\ \widetilde{V}_p z_2^{-1} & & & & V_{p-1} & H_p+\widetilde{H}_p z_1^{-1} \end{pmatrix}$$

$$V_i = \widetilde{V}_i \equiv e^{q_i - q_{i+1}} \qquad H_i = -\widetilde{H}_i \equiv e^{(-1)\ p_i/2} \qquad z_1 \equiv e^{-z} \qquad z_2 \equiv w$$

- Non-relativistic limit: linear orden in p_i and z and define

$$L(w) - z \equiv K$$

$$L(w) = \begin{pmatrix} p_1 & e^{q_1-q_2} & & & & e^{q_p-q_1} w \\ e^{q_1-q_2} & p_2 & e^{q_2-q_3} & & & \\ & e^{q_2-q_3} & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & e^{q_{p-1}-q_p} \\ e^{q_p-q_1} w^{-1} & & & & e^{q_2-q_3} & p_p \end{pmatrix}$$

- This is precisely the Lax operator of the non-relativistic periodic Toda chain!

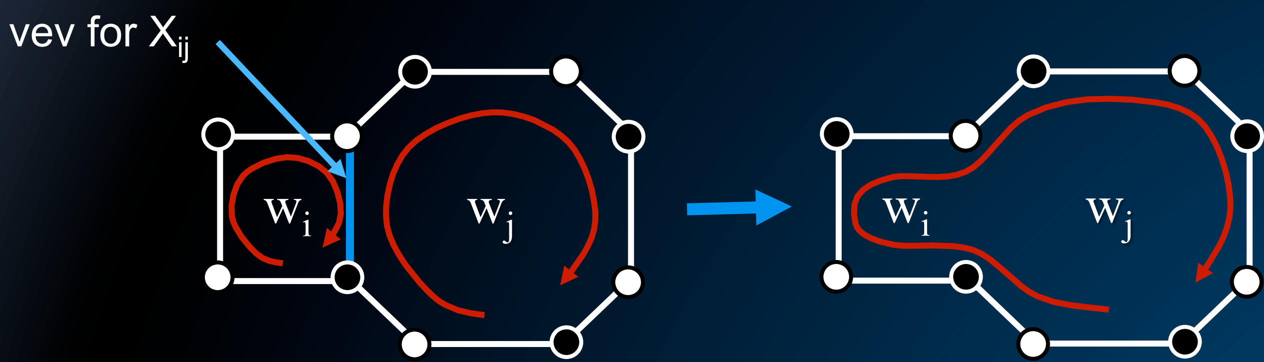
New Integrable Systems from Higgsing

Eager, Franco, Schaeffer

- Dimer models, and hence integrable systems, can be systematically constructed for any toric diagram

  Feng, He, Kennaway, Vafa
  Hanany, Vegh

- Another simple way to generate new integrable systems from existing ones is via the Higgs mechanism (geometrically, partial resolution)



- Start from the integrable system for the parent theory and turn on a vev for X_{ij}

  1) Remove loops containing an edge with a non-zero vev

  2) Re-express surviving loops with the replacement $(w_i\, w_j) \rightarrow w_{i/j}$

Dimers and Quantum Teichmüller Space

Franco

- Riemann surface Σ:

g: genus

n: # of punctures

- Teichmüller space: space of complex structure deformations

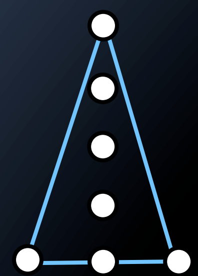
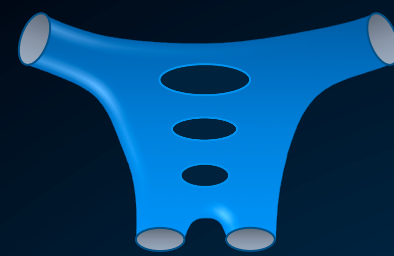
$$T_{g,n} = \frac{\text{Complex structure on } \Sigma}{\text{Diff}_0\, \Sigma}$$

- Coordinates in Teichmüller space:
  - Ideal triangulation (vertices at punctures)
  - 3m edges ($m = -\chi(\Sigma) = 2g - 2 + n$)

- Fock coordinates (shear coordinates): one real coordinate z_e for each edge in the triangulation

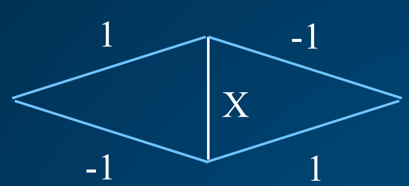
Dimers and Quantum Teichmüller Space

Franco

Dimer Model	Riemann Surface
	
Toric diagram	Σ
Dimer model on T^2	Ideal triangulation of Σ
Seiberg duality	Triangulation flip

- Weil-Petersson Poisson structure in Teichmüller space:

$$\{z_e, z_{e'}\} = n_{e,e'} \qquad n_{e,e'} \in \{-2,-1,0,1,2\}$$



- The Checkhov-Fock quantization of Teichmüller space promotes the Weil-Petersson Poisson brackets to commutators
- Integrable system commutators → Checkhov-Fock commutators