Scaling dimensions at small spin

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March 2, 2012
Great Lakes String Conference,
Purdue University

arXiv:1109.3154

Spectral problem and AdS/CFT correspondence

Spectral problem

- Starting point: $\mathcal{N} = 4$ Super-Yang-Mills theory is a conformal field theory (CFT)
- It depends on two dimensionless parameters: the 't Hooft coupling $\lambda \equiv g_{YM}^2 N_c$ and the number of colors N_c
- Important observables: spectrum of scaling dimensions Δ of (local) conformal operators $\mathcal{O}$

AdS/CFT correspondence

(Planar) $\mathcal{N} = 4$ SYM theory is equivalent to (free) type IIB superstring on $\mathrm{AdS}_5 \times \mathrm{S}^5$ background

$$\text{string tension} = \sqrt{\lambda}/2\pi \qquad \text{string coupling} \sim 1/N_c$$

Dictionary

Spectrum of (planar) scaling dimensions = spectrum of energies of (free) string

Spectral problem and integrability

Main difficulty: How to confront the gauge and string theory?

- Gauge theory is tractable at weak coupling: $\lambda \ll 1$
- String theory is tractable at strong coupling: $\lambda \gg 1$

In most cases, to test the correspondence we need control on the weak/strong coupling interpolation $\rightarrow$ need non-perturbative methods

Important recent progress: Discovery of integrable structures (in the planar limit)

[Minahan,Zarembo'02],[Beisert,Staudacher'03'05]
[Lipatov'98],[Braun,Derkachov,Korchemsky,Manashov'98],[Belitsky'99]
[Bena,Polchinski,Roiban'03],[Kazakov,Marshakov,Minahan,Zarembo'04]
[Gromov,Kazakov,Vieira'09],[Gromov,Kazakov,Kozak,Vieira'09],
[Bombardelli,Fioravanti,Tateo'09],[Arutyunov,Frolov'09]

$\rightarrow$ Complete solution to spectral problem in the planar limit

Motivations:

- Solving the four-dimensional gauge theory (at least in the planar limit)
- Quantizing the string theory on the curved background
- Testing the AdS/CFT correspondence

Probing the correspondence

Probe: consider (local) operators in the so-called $\mathfrak{sl}(2)$ sector

$$\mathcal{O} = \mathrm{tr}\, D^S Z^J + \text{mixing}$$

with

- Z a complex scalar field in the adjoint representation of the gauge group
- $D \equiv n^\mu D_\mu$ a light-cone covariant derivative $n^2 = 0$

They carry spin S and twist J

Spectrum of scaling dimensions

$$\Delta \equiv \Delta_{S,J}(\lambda)$$

from Bethe ansatz (TBA/Y-system) equations (for any coupling λ)

Comment: computing Δ in the short string regime, i.e., with $S, J \sim 1$ and $\lambda \gg 1$, is difficult, even with help of integrability

Small spin expansion

- Consider scaling dimension Δ of operator

$$\mathcal{O} \sim \mathrm{tr}\, D^S Z^J + \text{mixing}$$

- Δ is defined for physical operator (integer spin)

$$\Delta \equiv \Delta_J(S)$$

  as a function of spin S, twist J, and 't Hooft coupling λ

- Perform 'analytical continuation' in the spin S and expand around $S = 0$ (BPS point)

$$\Delta = J + \alpha_J(\lambda) S + O(S^2)$$

- The slope $\alpha_J(\lambda)$ is a function of J and λ only, computable at weak and strong coupling

Illustration

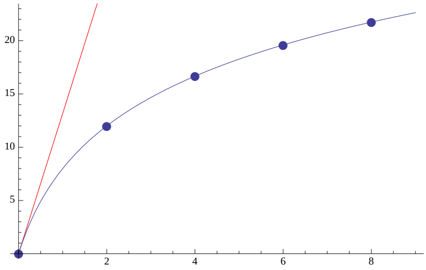
- Consider twist-two operator $(J = 2)$

$$\mathcal{O} = \mathrm{tr}\, D^S Z^2 + \text{mixing}$$

- Its scaling dimension is given up to one loop as

$$\Delta_{\text{twist-two}} = 2 + S + \frac{\lambda}{2\pi^2}(\psi(S+1) - \psi(1)) + O(\lambda^2)$$

with ψ the logarithmic derivative of Euler Gamma function



- Straigthforward expansion at small spin yields [Kotikov,Lipatov,Onishchenko,Velizhanin]

$$\alpha_{J=2}(\lambda) = \frac{d\Delta_{\text{twist-two}}}{dS}\Big|_{S=0} = 1 + \frac{\lambda}{12} - \frac{\lambda^2}{576} + \frac{\lambda^3}{17280} + O(\lambda^4)$$

Outline

- Small spin expansion using integrability
- Exact formula for the slope
- Application to short string energies

Tool : Integrability

Kinematics

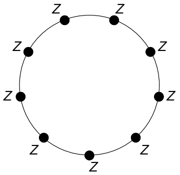
Operators

$$\mathcal{O}_{\{k_m\}} = \mathrm{tr}\, D^{k_1} Z ... D^{k_J} Z$$

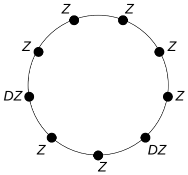
- tr $Z ... Z ... Z \to$ vacuum state of the spin chain
- tr $Z ... DZ ... Z \to$ one-particle state of the spin chain (magnon)

Quantum numbers

- Twist $J \to$ spin chain length
- Lorentz spin $S = k_1 + ... + k_J \to$ number of excitations (magnons) over the vacuum



Spin Chain (Ferromagnetic) Vacuum



Two−Magnon State

Tool : Integrability

Kinematics

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- Twist $J \rightarrow$ spin chain length
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Dynamics

Callan-Symanzik equation

$$\mu \frac{\partial}{\partial \mu} \mathcal{O}_{\{k_m\}} = -\delta \mathbb{D} \cdot \mathcal{O}_{\{k_m\}}$$

- Dilatation operator $\delta \mathbb{D} \rightarrow$ Hamiltonian of the spin chain
- Spectrum of anomalous dimensions $\delta \Delta \rightarrow$ spectrum of energies of the spin chain

One-loop example

Mapping with $\mathfrak{sl}(2)$ integrable Heisenberg spin chains

[Lipatov'97],[Braun,Belitsky,Derkachov,Korchemsky,Manashov'98]
[Minahan,Zarembo'02],[Beisert,Staudacher'03]

Kinematics : spin-chain Hilbert space $\mathcal{H} = V_{1/2}^{\otimes J}$

Dynamics : $\delta\mathbb{D}$ = Hamiltonian of $\mathrm{XXX}_{1/2}$ $\mathfrak{sl}(2)$ Heisenberg spin chain

Integrability

- System with J degrees of freedom... and J commuting conserved charges

  Liouville definition of a completely integrable system

- The complete family of conserved charges can be diagonalized simultaneously with $\delta\mathbb{D}$ by means of the algebraic Bethe ansatz

Bethe ansatz solution

Solution to mixing problem

- Bethe ansatz equations

$$\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right)^J = \prod_{j \neq k}^{S} \frac{u_k - u_j - i}{u_k - u_j + i}$$

- S magnons $\leftrightarrow$ S rapidities u_k

- One-loop scaling dimension

$$\Delta = J + S + \frac{\lambda}{8\pi^2} \sum_{k=1}^{S} \frac{1}{u_k^2 + 1/4} + O(\lambda^2)$$

Problem: how to go away from integer spin values?

Alternative approach

Baxter polynomial

$$Q(u) = \prod_{k=1}^{S} (u - u_k)$$

Baxter equation

$$(u + i/2)^J Q(u + i) + (u - i/2)^J Q(u - i) = t_J(u) Q(u)$$

with $t_J(u)$ the so-called (eigenvalue of the) transfer matrix

Scaling dimension

$$\Delta = J + S + \frac{i\lambda}{8\pi^2} \left[\frac{Q'(i/2)}{Q(i/2)} - \frac{Q'(-i/2)}{Q(-i/2)} \right] + O(\lambda^2)$$

Interesting point

- We can look for non-polynomial solutions and perform the small spin expansion

$$Q(u) = 1 + S q(u) + O(S^2)$$

Illustration

Twist two solution

$$q(u) = \frac{1}{2}\left(\psi(\tfrac{1}{2}+iu)+\psi(\tfrac{1}{2}-iu)\right) - \frac{i}{4\pi}\sinh(2\pi u)\left(\psi_1(\tfrac{1}{2}+iu)-\psi_1(\tfrac{1}{2}-iu)\right)$$

Scaling dimension

$$\Delta = J + \alpha_{\text{twist-two}}(\lambda) S + O(S^2)$$

with

$$\alpha_{\text{twist-two}}(\lambda) = 1 + \frac{\lambda}{8\pi^2}\left(q'(i/2) - q'(-i/2)\right) + O(\lambda^2)$$

and thus

$$\alpha_{\text{twist-two}}(\lambda) = 1 + \frac{\lambda}{12} + O(\lambda^2)$$

Higher loops? Yes with higher-loop Baxter equation [Belitsky'09],[BB,Belitsky'11]

Exact slope

Exact slope in planar $\mathcal{N} = 4$ SYM theory [BB'11]

$$\alpha_J(\lambda) = \frac{\sqrt{\lambda}}{J} \frac{I_J'(\sqrt{\lambda})}{I_J(\sqrt{\lambda})} = 1 + \frac{\sqrt{\lambda}}{J} \frac{I_{J+1}(\sqrt{\lambda})}{I_J(\sqrt{\lambda})}$$

Expressed in terms of the modified Bessel's function $I_J(x)$ (and its derivative $I_J'(x) \equiv dI_J(x)/dx$)

Proposal: Formula is correct for any twist J and 't Hooft coupling λ

Immediate checks

- Weak coupling expansion

$$\alpha_J(\lambda) = 1 + \frac{\lambda}{2J(J+1)} - \frac{\lambda^2}{8J(J+1)^2(J+2)} + O(\lambda^3)$$

OK with previous twist-two expression for $J = 2$!

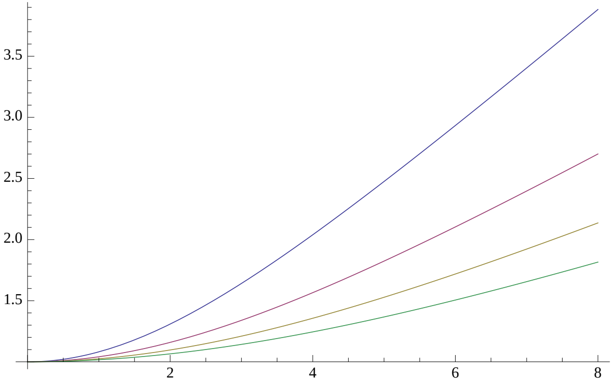
- At large J (and for any λ)

$$\alpha_J(\lambda) = 1 + \frac{\lambda}{2J^2} + O(1/J^2)$$

Correct BMN limit!

Numerical interpolation

Plot of the slope $\alpha_J(\lambda)$ as a function of the coupling $\sqrt{\lambda}$



for $J = 2$ (blue) to $J = 5$ (green)

Strong coupling expansion

Let us reformulate the proposal as

$$\Delta^2 = J^2 + \beta_J(\lambda)S + O(S^2)$$

where

$$\beta_J(\lambda) \equiv 2J\alpha_J(\lambda) = 2\sqrt{\lambda}\frac{I'_J(\lambda)}{I_J(\lambda)}$$

Motivation: remember the flat-space string theory result

$$\Delta^2 = J^2 + 2\sqrt{\lambda}S$$

Here we find that at strong coupling $\sqrt{\lambda}$ (i.e., large string tension)

$$\beta_J(\lambda) = 2\sqrt{\lambda} - 1 + \frac{J^2 - 1/4}{\sqrt{\lambda}} + \frac{J^2 - 1/4}{\lambda} + O(1/\lambda^{3/2})$$

- Correct flat-space limit!
- Correct one-loop correction!

[Gromov,Serban,Shenderovitch,Volin'11], [Roiban,Tseytlin'11],[Vallilo,Mazzucato'11]

Further check: consider the semiclassical string regime where $\mathcal{J} \equiv J/\sqrt{\lambda}$ is fixed, then

$$\beta_J(\lambda) = 2\sqrt{\lambda}\sqrt{1+\mathcal{J}^2} - \frac{1}{1+\mathcal{J}^2} + O(1/\sqrt{\lambda})$$

Comment: it is in perfect agreement with classical and one-loop string prediction

[Frolov,Tseytlin'02],[Gromov,Valatka'11]

Physical application I

Apply the formula

$$\Delta^2 = J^2 + \beta_J(\lambda)S + \gamma_J(\lambda)S^2 + \delta_J(\lambda)S^3 + O(S^4)$$

to physical operators (i.e., for finite spin) at strong coupling

Assumption: coefficients of higher spin powers are suppressed by higher powers of $1/\sqrt{\lambda}$, e.g.,

$$\beta_J(\lambda) = O(\sqrt{\lambda})\,, \qquad \gamma_J(\lambda) = O(1)\,, \qquad \delta_J(\lambda) = O(1/\sqrt{\lambda})\,, \qquad \cdots$$

Further assumption: coefficients of small spin expansion can be directly matched against those predicted by the semiclassical string computation

Comments:

- Non-trivial claim since the semiclassical analysis produces an expansion at small semiclassical spin $\mathcal{S} \equiv S/\sqrt{\lambda}$ (possible order of limit issue)
- So far these assumptions have been found to be in good agreement with exact (numerical) predictions from Y-system [Gromov,Serban,Shenderovitch,Volin'11],[Gromov,Valatka'11]

Physical application II

Under the battery of assumptions

$$\Delta^2 = J^2 + \beta_J(\lambda) S + \gamma_J(\lambda) S^2 + \delta_J(\lambda) S^3 + \ldots$$

applies to physical operators (i.e., for finite spin) at strong coupling, with (up to two loops)

- $\beta_J(\lambda) = 2\sqrt{\lambda} - 1 + (J^2 - 1/4)/\sqrt{\lambda}$
- $\gamma_J(\lambda) = 3/2 - b/\sqrt{\lambda}$
- $\delta_J(\lambda) = -3/(8\sqrt{\lambda})$

Missing piece: the one-loop semiclassical coefficient b, found recently as [Gromov,Valatka'11]

$$b = \frac{3}{8} - 3\zeta_3$$

→ complete two-loop prediction for (minimal) scaling dimension at strong coupling!

In particular: for the Konishi scaling dimension, i.e., for $S = J = 2$, ones find [Gromov,Valatka'11]

$$\Delta = 2\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{1/2 - 3\zeta_3}{\lambda^{3/4}} + O(1/\lambda^{5/4})$$

Some interesting features

The expression from the slope hints that

- Weak coupling expansion is convergent

  Radius of convergency is finite and fixed by the first non-trivial zero of Bessel's function $I_J(\sqrt{\lambda})$

- Strong coupling expansion is asymptotic and non-Borel summable

  Strong coupling series determines the exact expression up to exponentially small contributions $\sim \exp(-2\sqrt{\lambda})$ only

Similar to the situation for the cusp anomalous dimension (as predicted from the BES equation) [Beisert,Eden,Staudacher'06],[Basso,Korchemsky,Kotanski'07]

Summary and outlook

Main result

- Formula for the slope of minimal scaling dimension at any coupling and twist

Extensions

- Spectrum of short strings?.... small spin expansion for more generic states?
- Can we control higher terms in the small spin expansion?
- Relation to cusp anomalous dimension? Recent result by [Correa,Henn,Maldacena,Sever'12], [Fiol,Garolera,Lewkowycz'12]

$$\Gamma_{\text{cusp}}(\lambda, \phi) = -B(\lambda)\phi^2 + O(\phi^4)$$

with

$$B(\lambda) = \frac{\sqrt{\lambda}}{4\pi^2} \frac{I_2(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}$$

in striking similarity with the slope