Chaos in String Theory

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Great Lakes Strings Conference
Purdue, March 3, 2012

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[arXiv:1103.4101, 1105.2540, 1201.5634, and ongoing work.]

Plan of the talk

Integrability, Classical and Quantum Chaos

Integrability in String Theory

- Motivation and summary of our programme
- Explicit demonstration of Chaos in Maldacena-Nuñez
- Analytical nonintegrability in confining backgrounds

What is Integrability?

In classical mechanics, a system is integrable when there are the same number of conserved quantities as the pairs of canonical coordinates.

One can transform to a coordinate system where these conserved quantities are the conserved momenta. In these action-angle coordinates, the conserved momenta are the action variables I_i and the corresponding coordinates are the angle variables θ_i.

$$\theta_i = I_i t \,, \qquad I_i = \mathrm{constant} \,.$$

The theory is thus exactly solvable.

For a non-integrable theory one can not write down a closed form analytic solution in general.

What is Chaos?

In many of these non-integrable systems, the behavior is not very predictable. The system is exponentially sensitive to initial conditions – infinitesimally close trajectories separate exponentially.

The Lyapunov index characterizes the sensitivity to initial conditions:

$$\lambda = \lim_{\tau \to \infty} \lim_{\Delta X_0 \to 0} \frac{1}{\tau} \ln \frac{\Delta X(X_0, \tau)}{\Delta X(X_0, 0)}$$

A positive Lyapunov index can be taken to be a definition of chaos.

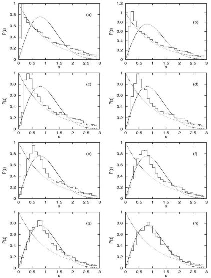
Most chaotic systems also show other typical features of chaos or transition to chaos – bifurcation diagrams, strange attactors in phase space, KAM behavior in Poincaré sections.

Quantum Chaos

Are there any characteristic quantum properties of a system that is classically chaotic?

Gaps between nearest energy levels for a non-chaotic system have a Poisson distribution, $P(s) = e^{-s}$.

For a chaotic system, the NND (Nearest Neighbour Distribution) is often approximated by a Wigner distributuion, $P(s) = \frac{\pi}{2} s e^{-\pi s^2/4}$.



NND of Rydberg atom

with increasing quantum defect.

Poisson $\rightarrow$ Wigner distr.

(Figure from Stöckmann, 'Quantum Chaos – An Introduction')

Motivation: String Integrability and AdS/CFT

For a recent review: [Beisert *et. al.* (2011)]

String theory on $AdS_5 \times S^5$ is integrable.

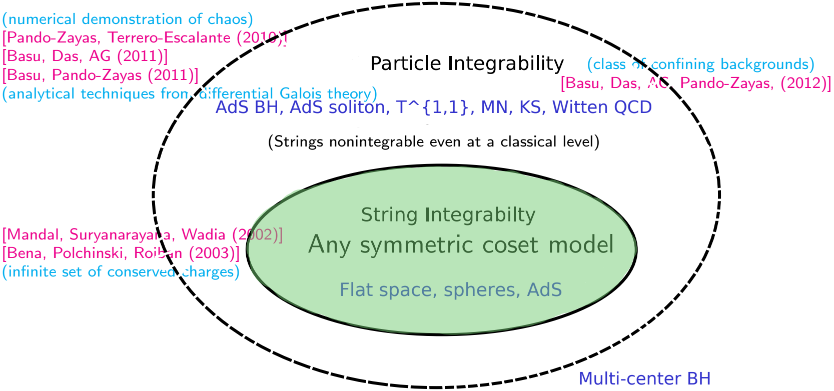
The dual field theory is similar to realistic gauge theories like QCD.

Solvability in the string theory would lead to closed form analytic expressions for hadron masses and excitations in the dual theory.

However the integrable $\mathcal{N} = 4$ SYM theory dual to $AdS_5 \times S^5$ is not confining and does not have a mass-gap.

It is thus interesting to look at integrability in spacetimes which have more QCD-like duals.

String Integrability and our programme



Confining backgrounds

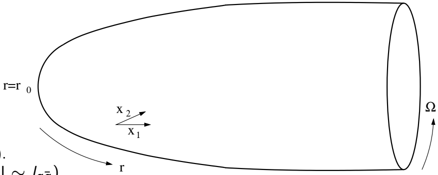
[Sonnenschein (2000)]

A generic confining geometry has a metric of the form:

$$ds^2 = a^2(r)dx_\mu dx^\mu + b^2(r)dr^2 + c^2(r)d\Omega_d^2\,,$$

$a(r)$ has a nonzero minimum at $r = r_0$ (confining wall):

$$\partial_r(a^2(r))|_{r=r_0} = 0, \qquad a^2(r_0) \neq 0$$



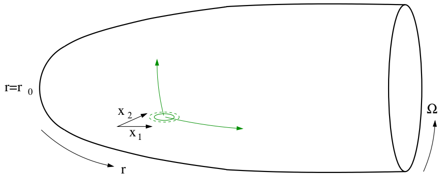
Mass gap (r_0 sets a scale).
Confinement ($q\bar{q}$ potential $\sim l_{q\bar{q}}$).

Klebanov-Strassler, Maldacena-Nunez and AdS soliton geometries, all fall in this category.

τ Embedding

The string has the following embedding:

$$\begin{aligned} t &= t(\tau), \qquad r = r(\tau), \\ x_1 &= R(\tau)\sin\alpha\sigma, \qquad x_2 = R(\tau)\cos\alpha\sigma. \end{aligned}$$

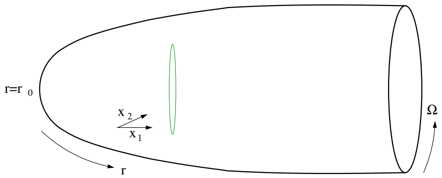


String wrapped around two of the x directions as a circle of radius R.
Allowed to move in the r direction and to change its radius R.

σ Embedding

The string has the following embedding (similar to GKP):

$$\begin{aligned} t &= t(\tau), \qquad r = r(\sigma), \\ x_1 &= f_1(\tau) g_1(\sigma), \qquad x_2 = f_2(\tau) g_2(\sigma). \end{aligned}$$



EoM are analogous and are related by a simple change of variables.

Spinning strings at $r = r_0$ give the Regge trajectories with $J \propto E^2$.

Demonstration of Chaos – Lyapunov Index for MN

We numerically solve the equations of motion obtained after plugging in the τ embedding in the Polyakov action.

Algorithm by: [Sprott (2003)]

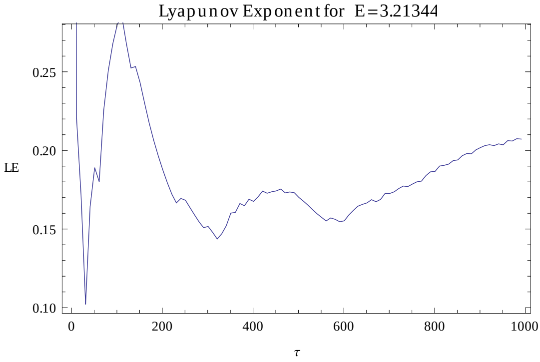
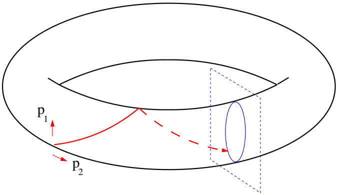


Figure: Positive LE indicating chaos.

Transition to chaos – Poincaré Sections

Consider an integrable system with 2 degrees of freedom (like a pair of coupled harmonic oscillators). The system decouples.

The trajectory in the 4 dimensional phase space is a 2-torus. A 2-dimensional section called Poincaré section is a circle.

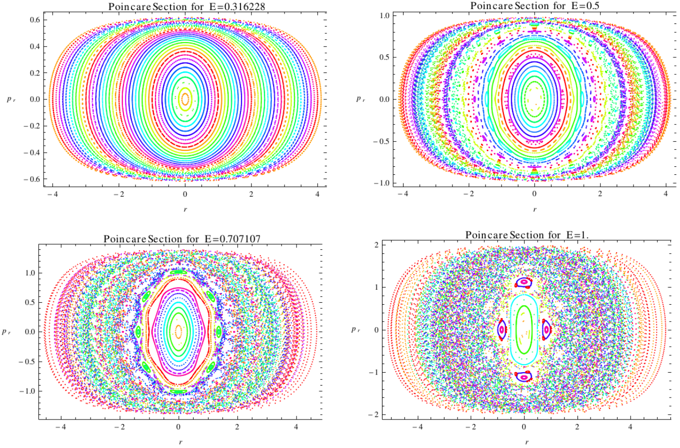


However if some nonintegrability (like an anharmonic coupling) is introduced, the trajectory is no longer confined to the torus, but moves out to a 3-dimensional space (since the energy is still conserved).

Poincaré Sections for MN

KAM Theorem:
Resonant tori start breaking
Elliptic and hyperbolic points

Tori are intact



More and more tori break
KAM tori survive

Colours get mixed
Trajectories roam around (Arnold diffusion)

Poincaré Sections with increasing values of the nonintegrable term. Each color represents a different initial condition.

Analytical Nonintegrability

[Ziglin (1982-83); Ramis, Ruiz, Simó (1994-2007); Kovacic (1986)]

Consider a system of second order differential equations in (p_1, q_1, p_2, q_2) with a conserved Hamiltonian H.

Suppose we can consistently set $p_2 = 0, q_2 =$ constant, and can analytically solve for $p_1^S(t), q_1^S(t)$. This solution is known as the straight line solution.

The linearized equation p_2, q_2 with the above solution plugged in is known as the Normal Variational Equation (NVE).

If there are two independent conserved quantities H and Q, the NVE has a first integral that can be solved using quadrature.

If the NVE can not be solved using quadrature, the Ramis-Ruiz theorem implies nonintegrability of the system of equations.

Kovacic Algorithm

The Kovacic algorithm provides a systematic procedure for calculating the first integral of the NVE if it exists.

The algorithm fails if the NVE does not have a first integral.

The algorithm is straightforward to follow step by step. It is also implemented in computer algebra software like MAPLE.

Kovacic algorithm fails.

Ramis-Ruiz theorem implies that the system is nonintegrable.

Was first used in context of string theory in [Basu, Pando-Zayas, arXiv: 1105.2540].

Kovacic algorithm goes through for pure AdS.

Conclusions and Outlook

We show that dynamics of classical strings in a class of confining backgrounds is nonintegrable. We moreover show that the dynamics is chaotic.

The Regge trajectories are special integrable islands. Are there other integrable islands?

Are there signatures of quantum chaos in the full quantum theory?

In a time-dependent (non-Hamiltonian) system do we see typical features like strange attractors?