(0, 2) Quantum Cohomology

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Outline

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Joint work with Ron Donagi, Josh Guffin, and Eric Sharpe

1110.3751, 1110.3752

Physical Model.

- Spacetime $M = \mathbf{R}^{3,1} \times X$, X 6 dimensional and compact

$$S = \int d^{10}x\sqrt{-g}\left(\partial_i x^\mu \partial^i x_\mu + \bar{\psi}_\mu \Gamma^\nu \partial_\nu \psi^\mu + \bar{\lambda}_a \Gamma^\nu \partial_\nu \lambda^a + \right.$$

$$\left. + R_{\mu\bar{\nu}a\bar{b}} \psi^\mu \bar{\psi}^\nu \lambda^a \bar{\lambda}^b + \ldots \right)$$

- ψ^μ right-handed fermions in TX; λ^a left-handed fermions living in gauge bundle E; a gauge index, ψ^μ superpartner of x^μ, $(0,2)$ SUSY
- In an E_6 heterotic, matter fits into 27, $\overline{27}$, singlet multiplets
- Yukawa couplings 27^3, $\overline{27}^3$
- If $E = TX$ (i.e. $(2,2)$ SUSY), 27^3 and $\overline{27}^3$ Yukawa couplings can be computed *exactly* by techniques of algebraic geometry (only tree level couplings)

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Quantum cohomology

- What makes this work so well: $(2, 2)$ theory can be twisted to become topological (A model)
- Fermions take values in simpler bundles (trivial and canonical)
- In topological sector, worldsheet vertex operators are identified with cohomology $H^*(X)$
- Operator products computed in terms of three point couplings, define quantum cohomology, and the algebra $H^*(X)$ still closes after quantum corrections
- $\mathcal{O}_a * \mathcal{O}_b = \langle \mathcal{O}_a, \mathcal{O}_b, \mathcal{O}_c \rangle \mathcal{O}^c$
- Three point couplings in twisted theory coincide with Yukawa couplings in physical theory
- We want to do this in $(0, 2)$ theory, but we can't

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Half-twisted $(0, 2)$ theories

- However, we can *half-twist* (right handed fermions only)
- Not topological, but have a finite-dimensional *quasi-topological* sector
- Three point couplings in half-twisted theory coincide with Yukawa couplings in the physical theory
- These theories are closely related to *gauged linear sigma models* (and sometimes coincide)
- Many other motivations from mathematics, especially algebraic geometry
- The rest of the talk will focus on $(0, 2)$ GLSMs

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$(0,2)$ GLSM

- The $(0,2)$ GLSM is a 2-dimensional gauge theory with $(0,2)$ supersymmetry *Witten*
  - Gauge group $G = U(1)^r$
  - Chiral fields $\Phi^1, \ldots, \Phi^n, \bar{\mathcal{D}}\Phi^i = 0$, lowest component scalar
  - Fermi superfields Υ^a, lowest component left-handed fermion
  - $\bar{\mathcal{D}}\Upsilon^a = E^a(\Phi),\ E^a$ holomorphic

Prior work.

- First studied in this context by *Adams, Basu, and Sethi*, who computed by mirror symmetry
- Direct computation of quantum cohomology ring done for a fixed $(0, 2)$ deformation by Guffin-K-Sharpe
- All $(0, 2)$ deformations of $\mathbf{P}^1 \times \mathbf{P}^1$ model computed by Guffin-K
- General form for linear deformations conjectured by McOrist-Melnikov
- We verify their conjecture and show that it holds verbatim for nonlinear deformations as well

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Vacuum Moduli Space

- The moduli space of vacua is a toric variety
- Q^i_α charge of Φ^i under α^{th} $U(1)$
- FI terms $\sum Q^i_\alpha |\Phi^i|^2 - r_\alpha = 0$
- Gauge equivalence classes give moduli space

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$\mathbf{P}^n$

- $G = U(1)$
- $n + 1$ charged chiral superfields $\Phi^0, \ldots, \Phi^n$, all charge 1
- $\sum |\Phi^i|^2 - r = 0$
- $M_{\mathrm{vac}} = \{\phi | \sum |\phi^i|^2 = r\}/(\phi^i \sim e^{i\theta}\phi^i)$
- This is complex projective n space $\mathbf{P}^n$

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Language of Algebraic Geometry

- Complex projective space $\mathbf{CP}^n = \mathbf{P}^n$:
- Described by homogeneous coordinates:

$$(x_0, \ldots x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$$

- Have holomorphic line bundles $\mathcal{O}(k)$ on $\mathbf{P}^n$ whose holomorphic sections are homogeneous degree k polynomials $f(x_0, \ldots, x_n)$

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Identification of Vacuum Moduli Space by Algebraic Geometry

- $M_{\mathrm{vac}} \simeq \mathbf{P}^n$
- $$(\phi^0, \ldots, \phi^n) \mapsto (\phi^0, \ldots, \phi^n)$$
- $$(\phi^0, \ldots, \phi^n) \mapsto \frac{1}{\sqrt{r}}(\phi^0, \ldots, \phi^n)$$

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Instanton Contributions

- Fix the worldsheet as $S^2 = \mathbf{P}^1$. Fix a gauge instanton background with $c_1 = d$
- $\overline{\mathcal{D}}\Phi^i = 0$ implies $\bar{\partial}\phi^i = 0$
- ϕ^i is a holomorphic section of $\mathcal{O}_{\mathbf{P}^1}(d)$, a degree d polynomial
- These are identified with degree d holomorphic maps (worldsheet instantons)

$$\phi : \mathbf{P}^1 \to \mathbf{P}^n, \phi(x_0, x_1) = (\phi^0(x_0, x_1), \ldots, \phi^n(x_0, x_1))$$

- This gives the gauge theory/string theory dictionary: gauge instantons $\leftrightarrow$ worldsheet instantons

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$\mathbf{P}^1 \times \mathbf{P}^1$

- Now consider $G = U(1)^2$, charged chirals $\Phi^1, \ldots, \Phi^4$
- Charge matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

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$$|\Phi^1|^2 + |\Phi^2|^2 = r_1, \qquad |\Phi^3|^2 + |\Phi^4|^2 = r_2$$

- After gauge equivalence, this is $\mathbf{P}^1 \times \mathbf{P}^1$
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The (0, 2) Deformation

- For the rest of this talk, we specialize to the case where the Υ^i are in one to one correspondence with the Φ^i, same charges, i.e. E is a deformation of the tangent bundle
- Convenient to let W be 2-dimensional vector space containing the lattice of charges (in $\mathbf{P}^1 \times \mathbf{P}^1$ case)
- W also generates the ring of operators (arising in the gauge sector)
- Since Υ^i and $E^i(\Phi)$ have the same charges, we must have for $a_{ij}, \tilde{a}_{ij} \in W$
  $E^1(\Phi) = a_{11}\Phi^1 + a_{12}\Phi^2,\ E^2(\Phi) = a_{21}\Phi^1 + a_{22}\Phi^2,$
  $E^3(\Phi) = \tilde{a}_{11}\Phi^3 + \tilde{a}_{12}\Phi^4,\ E^4(\Phi) = \tilde{a}_{21}\Phi^1 + \tilde{a}_{22}\Phi^4$

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The $(0, 2)$ Deformation

- For the rest of this talk, we specialize to the case where the Υ^i are in one to one correspondence with the Φ^i, same charges, i.e. E is a deformation of the tangent bundle
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Bundle E.

- The bundle E on $\mathbf{P}^1 \times \mathbf{P}^1$ determined by the Υ^i is the quotient of $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0)^2 \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(0,1)^2$ by the subbundle spanned by the two-dimensional space of sections $(E^1(\phi),\ldots,E^4(\phi))$ of $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1,0)^2 \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(0,1)^2$
- In the language of algebraic geometry, this corresponds to the exact sequence of vector bundles on $\mathbf{P}^1 \times \mathbf{P}^1$

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Relation to Heterotic String Theory.

- Can rewrite by dualizing as

$$0 \to E^* \to \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2 \to W \otimes \mathcal{O} \to 0$$

- Since $H^1(\mathcal{O}(-1,0)) = H^1(\mathcal{O}(0,-1)) = 0$, we conclude $W \simeq H^1(E^*)$
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Outline

$\mathbf{P}^1 \times \mathbf{P}^1$.

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- Let W be generated by $\psi, \tilde{\psi}$
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- $\langle P \rangle = \sum_{d,\tilde{d}} \langle P \rangle_{d,\tilde{d}} q^d \tilde{q}^{\tilde{d}}$ where $q, \tilde{q}$ depend on FI parameters
- Analogous to instanton sum in string theory
- Let $Q = \det(A), \tilde{Q} = \det(\tilde{A}) \in \mathrm{Sym}^2 W$
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- Can compute $\langle P \rangle_{d,\tilde{d}}$ by algebraic geometry
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$$\otimes^k H^1(E^*) \to H^k(\Lambda^k E^*) \simeq H^k(\Omega^k_{M_{d,\tilde{d}}}) \simeq \mathbf{C}$$
$$k = 2d + 2\tilde{d} + 2, M_{d,\tilde{d}} = \mathbf{P}^{2d+1} \times \mathbf{P}^{2\tilde{d}+1}$$
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Relations.

- *Generalized Koszul complex* of

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is

$$0 \to \Lambda^2 E^* \to \Lambda^2 Z \to W \otimes Z \to \mathrm{Sym}^2 W \otimes \mathcal{O} \to 0$$

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Gauge Instanton Moduli Space.

- In the $(d, \tilde{d})$ instanton background, the zero modes of Φ^1 and Φ^2 are polynomials $f^{1,2}$ of degree d in the homogeneous coordinates (x_0, x_1) on the worldsheet
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- The FI constraints prevent either (f^1, f^2) or (f^3, f^4) from being identically zero.
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- The general case relies on toric geometry rather than the geometry of projective space
- One new ingredient in general: if a field lives in a bundle with negative chern class in an instanton background, there are no zero modes. To compensate, new terms enter into the calculation from the four-fermi terms in the action
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A little toric geometry

- In general, M_{vac} is a toric variety.
- To each field Φ^i, we associate the divisor $D_i \subset M_{\text{vac}}$ defined by $\phi^i = 0$
- Have line bundles $\mathcal{O}(D_i)$ associated with these divisors by general algebraic geometry
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- Labelling each collection by an index c, these expressions for the E^i can be arranged into a W-valued square matrix A_c
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Four-Fermi Terms

- Suppose Φ^i lives in a bundle with negative chern class d_i in an instanton background β
- Then any Φ^j in the same class c lives in the same bundle
- The four-fermi terms are $Q_c^{-d_c-1}$
- In GLSM, four-fermi terms are generated by Yukawa couplings

$$\sigma^a \psi^i \bar{\psi}_i, \qquad \sigma^a \lambda^i \bar{\lambda}_i$$

- $\dim H^1(O_{\mathbf{P}^1}(d_c)) = -d_c - 1$
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