$\mathcal{N} = 2$ Superconformal Index and Ruijsenaars-Schneider models

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Objectives

- Pre-existing condition : Existance of a huge class (class $\mathcal{S}$) of $\mathcal{N} = 2$ SCFTs which one can obtain by compactifying the $(2, 0)$ theory on a Riemann surface. Most of these theories are non-Lagrangian and intrinsically strongly interacting thus rendering direct computations unlikely.

- Problem: To find an explicit form for the superconformal index for ALL these theories.

- Treatment : "bottom-up", "experimental math" approach; fully exploit the intuition about the hidden $6d$ origin of the $4d$ theories to generalize directly computable results for Lagrangian theories to non-Lagrangian ones.

- Side effects : An AGT-like relation between the superconformal index of the $4d$ theories to $2d$ gauge theories and to integrable systems.

$\mathcal{N} = 2$ quiver gauge theories

- $\mathcal{N} = 2$ SCFTs obtained by compactifying the $(2,0)$ theory on a punctured Riemann surface. (Gaiotto 2009, GMN 2009)
- The moduli of the Riemann surface map to gauge couplings of the corresponding $4d$ theory.
- The punctures are associated with flavor symmetries.
- Basic building blocks: theories corresponding to spheres with three punctures (no moduli=no tunable couplings)
  - Free hypermultiplets of $SU(k)$ theories correspond to spheres with two "maximal" punctures and one $U(1)$ puncture.
  - *All* the three-punctured spheres which are not free hypers do not have Lagrangian description.
  - An example of interacting theory corresponding to three-punctured spheres is the $SU(3)$ theory with three maximal punctures is an SCFT with E_6 flavor symmetry.
- "Gluing" three-punctured spheres at the punctures corresponds to gauging an $SU(k)$ flavor symmetry factor.
- Different "pair-of-pants" decompositions correspond to different S-duality frames.
- S-duality is extremely constraining!! All the information about the $4d$ theories we need to solve our problem is listed on this slide!!

The superconformal index

- The superconformal index (Kinney-Maldacena-Minwalla-Raju 2006) encodes the information about the protected spectrum of a SCFT that can be obtained from representation theory alone.
- It is evaluated by a trace formula of the schematic form

$$\mathcal{I}(\mu_i) = \mathrm{Tr}(-1)^F \, e^{-\sum_i \mu_i T_i} \, e^{-\beta\,\delta}\,, \qquad \delta = 2\left\{\mathcal{Q}, \mathcal{Q}^\dagger\right\} \; (\geq 0)\,,$$

  where $\mathcal{Q}$ is the supercharge "with respect to which" the index is calculated and $\{T_i\}$ a complete set of generators that commute with $\mathcal{Q}$ and with each other.
- The trace is over the states of the theory on S^3 (in the radial quantization). States with $\delta \neq 0$ cancel pairwise, so the index counts states with $\delta = 0$ and it is independent of β.
- For a theory with a Lagrangian description one can compute the index in the free limit of the theory using simple matrix integral techniques.
- For $\mathcal{N} = 2$ superconformal theories the most generic index takes the following from

$$\boxed{\mathcal{I}(p, q, t, a_\ell) = \mathrm{Tr}(-1)^F \left(\frac{t}{p\,q}\right)^r p^{j_2+j_1}\, q^{j_2-j_1}\, t^R \prod_\ell a_i^{f_\ell}}\,,$$

  and we take $\delta = E - 2j_2 - 2R + r = 0$.
- Unitarity implies that charges coupled to p and q are non negative.

TQFT structure

- The superconformal index does not depend on the tunable parameters/coupling of the theory.
- For Gaiotto theories this means that the index does not depend on the moduli of the underlying Riemann surface.
- Thus, it is expected that the index will be given by a $2d$ TQFT computation.
- The structure constants of this TQFT are the indices of the three-punctured spheres,

$$\mathcal{I}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$$

where $\mathbf{x}_i$ are fugacities of the Cartan subgroup of the flavor symmetry.

- A basic property of a TQFT is that the different pair-of-pants decompositions of the riemann surface give the same result - the algebra defined by the structure constants is associative:

$$\oint \prod_{i=1}^{k-1} \frac{dx^i}{2\pi i x_i} \Delta(\mathbf{x}) \mathcal{I}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}) \mathcal{I}_V(\mathbf{x}) \mathcal{I}(\mathbf{x}^{-1}, \mathbf{x}_3, \mathbf{x}_4) .$$

The associativity implies that this index is invariant under permutations of $\mathbf{x}_i$.

Our strategy I - Change perspective!!

- We want to obtain the superconformal index for ALL the $\mathcal{N} = 2$ generalized quivers.
- Our strategy in solving the problem is to rewrite the index of the Lagrangian theories in such a way that the Riemann surface underlying the theory will be clearly visible in the expressions. Thus, allowing for generalizations to arbitrary rank and Riemann surface.
- Choose an orthonormal basis for symmetric functions $f^{\lambda}(a_1, \dots, a_n)$.
- Define structure constants

$$\mathcal{I}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = \sum_{\mu,\nu,\lambda} C_{\mu\nu\lambda}\, f^{\mu}(\mathbf{a}_1)\, f^{\nu}(\mathbf{a}_2)\, f^{\lambda}(\mathbf{a}_3).$$

- Gluing two spheres is then just multiplying the structure constants

$$\oint \prod_{i=1}^{k-1} \frac{da^i}{2\pi i a_i}\, \Delta(\mathbf{a})\, \mathcal{I}_V(\mathbf{a}) \mathcal{I}(\mathbf{a}, \mathbf{a}_1, \mathbf{a}_2) \mathcal{I}(\mathbf{a}^{-1}, \mathbf{a}_3, \mathbf{a}_4) =$$
$$\sum_{\mu,\nu,\lambda,\rho} C_{\mu\nu\alpha}\, \delta^{\alpha\beta}\, C_{\beta\lambda\rho}\, f_{\mu}(\mathbf{a}_1)\, f_{\nu}(\mathbf{a}_2)\, f_{\lambda}(\mathbf{a}_3)\, f_{\rho}(\mathbf{a}_4)\,.$$

- S-duality implies that the structure constants are associative: $C_{\alpha\beta}{}^{\gamma} C_{\gamma\delta\rho} = C_{\alpha\delta}{}^{\gamma} C_{\gamma\beta\rho}$.
- "Diagonalize" the basis such that the only non-zero structure constants will be $C_{\alpha\alpha\alpha}$.
- It turns out that in this diagonal basis representation of the index of Lagrangian three-punctured spheres is naturally generalizable to arbitrary rank and punctures.

Hall-Littlewood index

- We take the limit $p,\ q \to 0$ of the full index

$$\mathcal{I}(p,q,t,a_\ell) = \mathrm{Tr}(-1)^F \left(\frac{t}{p\,q}\right)^r p^{j_2+j_1}\, q^{j_2-j_1}\, t^R \prod_\ell a_i^{f_\ell}$$

- The quiver theories with $SU(2)$ gauge and flavor groups are the simplest: all the relevant theories have Lagrangian description.
- The basic building block corresponding to a sphere with three punctures is a free hypermultiplet,

$$\mathcal{I}(a_1,a_2,a_3) = \frac{1}{\prod_{\pm 1}(1 - t^{\frac{1}{2}}\ a_1^{\pm 1}\ a_2^{\pm 1}\ a_3^{\pm 1})}\,.$$

- This index can be written as

$$\mathcal{I}(a_1,a_2,a_3) \quad \sim \quad \prod_{i=1}^{3}\mathcal{K}(a_i)\sum_{\lambda=0}^{\infty}\frac{1}{P_\lambda^{HL}(t^{\frac{1}{2}},t^{-\frac{1}{2}}|\ t)}\prod_{i=1}^{3}P_\lambda^{HL}(a_i,a_i^{-1}|\ t)$$

where

$$P_\lambda^{HL}(a,a^{-1}|t) = \mathcal{N}_\lambda(\mathrm{t})\,(\chi_\lambda(a) - t\,\chi_{\lambda-2}(a))$$

are $SU(2)$ Hall-Littlewood polynomials.

HL index - higher rank generalization

- Using the orthogonality of the HL polynomials we can immediately write the index of any $SU(2)$ quiver

$$\mathcal{I}_{\mathfrak{g},s}(a_i) = (1-t)^{\mathfrak{g}-1}\,(1+t)^{2\mathfrak{g}-2+s}\prod_{i=1}^{s}\mathcal{K}(a_i)\sum_{\lambda=0}^{\infty}\frac{\prod_{i=1}^{s}P_{\lambda}^{HL}(a_i,a_i^{-1}|\,t)}{\left[P_{\lambda}^{HL}(t^{\frac{1}{2}},t^{-\frac{1}{2}}|\,t)\right]^{2\mathfrak{g}-2+s}}\,.$$

- The expression for the $SU(2)$ index is tightly tied to the underlying Riemann surface and to the rank of the group. We thus can conjecture a simple generalizion to higher ranks.
- The HL polynomials can be defined for $U(k)$ groups

$$P_{\lambda}^{HL}(x_1,\dots,x_k|\,t) = \mathcal{N}_{\lambda}(t)\,\sum_{\sigma\in S_k}\sigma\left(x_1^{\lambda_1}\dots x_k^{\lambda_k}\prod_{i<j}\frac{x_i-t\,x_j}{x_i-x_j}\right)\,.$$

and thus for higher rank building blocks, the T_k theories, the HL index is given by

$$\boxed{\mathcal{I}(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3) = \mathcal{N}_k(t)\,\prod_{l=1}^{3}\mathcal{K}(\mathbf{a}_l)\,\sum_{\lambda}\frac{1}{P_{\lambda}^{HL}(t^{\frac{k-1}{2}},\dots,t^{\frac{1-k}{2}})}\prod_{l=1}^{3}P_{\lambda}^{HL}(\mathbf{a}_l)}\,.$$

- Simple generalization of this formula exists for the index of theories with arbitrary types of punctures.

Comments

- We can "guess" the HL index for any $\mathcal{N} = 2$ generalized quiver for arbitrary rank and types of punctures.
- This guess can be subjected to numerous checks:
  S-duality, Argyres-Seiberg dualities, Symmetry enhancements ($E_{6,7,8}$) SCFTs, ...
- An immediate generalization is to index with more superconformal fugacities turned on. It so happens that with $p = 0$ and q, t generic all one has to do is to to exchange HL polynomials with Macdonald polynomials.
- In another simple specialization of parameters, $t = q$, the relevant functions are Schur polynomials and the index is directly related to 2d qYM!!
- Where do these special polynomials come from?
- Macdonald polynomials are simultaneous eigenfunctions of a set commuting "Hamiltonians" defining an integrable quantum mechanics: the trigonometric Ruijsenaars-Schneider (RS) model .
- In particular it admits an elliptic version with three parameters direcly analogous to our p, q, and t.
- In what follows we will comment on how these "Hamiltonians" emerge from the index.

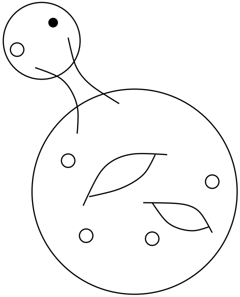
Strategy II - "Magic" (analytic properties of the index)

- The index has many poles in flavor fugacities.
- The index of the free hyper is

$$\mathcal{I}_{hyp.}(b,c;a)=\prod_{i,j=1}^{N}\prod_{m,n\geq 0}\frac{1-p^{n+1}q^{m+1}t^{-\frac{1}{2}}(ab_ic_j)^{-1}}{1-p^nq^mt^{\frac{1}{2}}ab_ic_j}\frac{1-p^{n+1}q^{m+1}t^{-\frac{1}{2}}ab_ic_j}{1-p^nq^mt^{\frac{1}{2}}(ab_ic_j)^{-1}}\,.$$

- A natural question is what are the residues?

- Consider a general quiver associated to Riemann surface $\mathcal{C}$ with index $\mathcal{I}^{\mathcal{C}}$ and couple a free hyper-multiplet to it.
- It is possible to compute the residue of the full theory $\mathcal{I}$ at a pole of the $U(1)$ fugacity without explicitly knowing the index of the theory associated to the Riemann surface $\mathcal{C}$, $\mathcal{I}^{\mathcal{C}}$.
- The residue can be presented as a difference operator acting on the $SU(N)$ flavor fugacity living "on the tube" $\mathcal{I}^{\mathcal{C}}$.
- This operator is one of the RS "Hamiltonians" !!



The appearance of the RS "hamiltonians"

- One can show that the index of the theory corresponding ti surface $\mathcal{C}$ with a free hypermultiplet coupled to it has poles at

$$a = t^{\frac{1}{2}}\, q^{\frac{1}{N}r}\, p^{\frac{1}{N}r'}\,, \qquad r, r' \in \mathbb{N}\,.$$

- The contour integrals involved in gluing the sphere to the Riemann surface $\mathcal{C}$ are "pinched" at these values of a and that is why the poles appear.
- The residue at $a = t^{\frac{1}{2}}$ is given simply by $\mathcal{I}_{\mathcal{C}}$, i.e. by the index on the Riemann surface. That is computing this resideue simply amounts to removing the $U(1)$ puncture.
- The residue at $a = t^{\frac{1}{2}}\, q^{\frac{1}{N}}$ is given by

$$\mathfrak{S}_{(1,0)}(a)\, \mathcal{I}_{\mathcal{C}}(a, \cdots)\,,$$

with

$$\mathfrak{S}_{(1,0)}\, \mathcal{I}_{\mathcal{C}} = \frac{\theta(t;p)}{\theta(q^{-1};p)} \sum_{i=1}^{N} \prod_{j\neq i} \frac{\theta(\frac{t}{q} b_i/b_j;p)}{\theta(b_j/b_i;p)}\, \mathcal{I}_{\mathcal{C}}(b_i \to q^{\frac{1-N}{N}} b_i,\; b_{j\neq i} \to q^{\frac{1}{N}} b_j)\,.$$

- This operator, up to trivial manipulations, IS the basic elliptic RS difference operator.
- Higher RS operators are obtained from other residues. In particular the $N-1$ independent Hamiltonians are encoded inside $\mathfrak{S}_{(r,0)}$ $(r = 1, \cdots, N-1)$.

A construction of the general index

- S-duality is very constraining!! We can exploite it to write the index of the generic quivers.
- Although the operators act on a given flavor fugacity, any choice of the flavor fugacity will give the same result due to S-duality,

$$\mathfrak{S}_{(1,0)}(a)\,\mathcal{I}_{\mathcal{C}}(a,b,\cdots) = \mathfrak{S}_{(1,0)}(b)\,\mathcal{I}_{\mathcal{C}}(a,b,\cdots)\,.$$

- Defining the eigenfunctions of the RS difference operators by ψ^λ and also defining the eigenvalues as

$$\mathfrak{S}_{(1,0)}\,\cdot\,\psi^\lambda = E_\lambda\,\psi^\lambda\,,$$

  we obtain

$$\begin{aligned}\mathfrak{S}_{(1,0)}\,\mathcal{I}_{0,3} &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\,E_\alpha\,\psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c) = \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\,E_\beta\,\psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c)\\ &= \sum_{\alpha,\beta,\gamma} C_{\alpha\beta\gamma}\,E_\gamma\,\psi^\alpha(a)\psi^\beta(b)\psi^\gamma(c)\,.\end{aligned}$$

- This implies that the index is diagonal in the basis of ψ^α

$$\boxed{\mathcal{I}_{0,3} = \sum_\alpha C_\alpha\,\psi^\alpha(a)\psi^\alpha(b)\psi^\alpha(c)\,,}$$

  Thus, rederiving again the result of strategy I!!
  (With some more work the structure constants C_α can be also fixed from S-duality)

Summary and Comments

- We have obtained explicit expression for the (two parameter) superconformal index of all Gaiotto's theories.
- The expressions for the index are manifestly S-duality invariant and have a uniform form for all types of punctures.
- The basic trick of the argument I was to write the index in a convenient discrete basis.
- This basis is related to a very generic family of symmetric functions: Macdonald polynomials and their elliptic generalizations.
- These functions are eigenfunctions of RS difference operators. We have commented on how these operators are encoded in the index through residue computations.

Summary and Comments cont.

- Although looking on residues seems ad hoc they actually have physical meaning.
- One can argue that the residues of the index of the type we discussed today give the index of a theory in presence of surface defects.
- The expressions we get for the index are suggestive of a 2d YM interpretation analogous to the AGT conjecture.
- 2d gauge theories are related to Calogero-Moser-Sutherland type of models and it will be interesting to understand these relations further in this context.
- Another interesting question for further research is whether there is a direct physical derivation of our results. E.g. whether starting from the $(2, 0)$ 6d theory and compactifying on $S^3 \times S^1$ one can obtain the 2d gauge theory and/or the integrable quantum mechanical systems
- We have an answer for the index for a large class of interacting SCFTs: it is an opportunity to look for interesting physical information encoded in the index ...
- . . .

Thank You!!

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