

On the non-planar beta-deformed $N=4$ super-Yang-Mills theory

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Based on 1201.5012 with Q. Jin

$\mathcal{N} = 4$ super-Yang-Mills theory has a number of remarkable properties:

- super-conformal invariance
- dual super-conformal invariance at planar level
- integrable planar dilatation operator
- remarkable **planar amplitudes** / **Wilson loops** relation
Alday, Maldacena
- remarkable **planar amplitudes** / **correlation function** relation
Eden, Korchemsky, Sokatchev

➡ **Planar amplitudes have a very constrained structure**

Unregularized d=4 integrand determined by symmetries and a small number of unitarity properties
Arkani-Hamed, Bourjaily, Cachazo, Caron-Huot, Trnka

➡ **Far fewer constraints for non-planar amplitudes**

Yet...

... non-planar amplitudes are simpler than what they could have been

U(1) decoupling: • 1-loop sub-leading color i.t.o. leading color

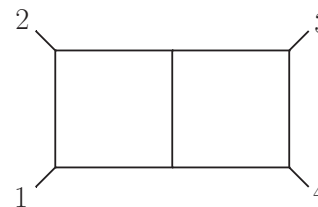
→ combination of box integrals

Bern, Kosower

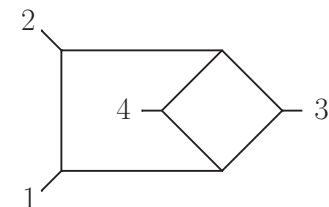
• parts of 2-loop 2-trace related to leading color

Bern, Rozowsky, Yan; Bern, de Freitas, Dixon

→ combination of



and



• Inversion of higher loop U(1) decouplings

Naculich

Higher loops:

• 3 & 4 loops: 2-trace better in UV than rest

$$D_c = 4 + \frac{6}{L} \quad \text{vs} \quad D_c = 4 + \frac{8}{L}$$

Bern, Carrasco, Dixon, Johansson, RR

→ Susy-based explanation

Berkovits, Green, Russo, Vanhove

Exhibits **color-kinematic duality**; $c_i + c_j + c_k = 0 \iff n_i + n_j + n_k = 0$

Bern, Carrasco, Johansson

→ (potential) all-order relation between l. and sub-l. color

→ simple and structured expressions

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→ simple and structured expressions

Question: How much of this structure relies on supersymmetry?

Can there be more structure that is hidden?

Analyze a QFT which shares most of the properties of $\mathcal{N} = 4$ sYM
except for supersymmetry

The supersymmetric β -deformed $\mathcal{N} = 4$ super-Yang-Mills theory

- $W = \text{Tr}[\Phi_1[\Phi_2, \Phi_3]] \longrightarrow f(\beta, N)\text{Tr}[\Phi_1(e^{i\beta}\Phi_2\Phi_3 - e^{-i\beta}\Phi_3\Phi_2)]$
Leigh, Strassler

- a pattern for the deformation:

noncommutative deformation: $\varphi_I\varphi_J \mapsto e^{i\hat{\beta}_{ij}q_I^i q_J^j} \varphi_I\varphi_J$ Lunin, Maldacena

$$\hat{\beta}_{ij} = -\hat{\beta}_{ji} ; \text{ if } \hat{\beta}_{12} = \hat{\beta}_{23} = \hat{\beta}_{31} = \beta \mapsto \text{susy}$$

$\mathcal{N} = 4$
R-charge vectors

	ϕ^{14}	ϕ^{24}	ϕ^{34}	A_μ	ψ^1	ψ^2	ψ^3	ψ^4	Q^1	Q^2	Q^3	Q^4
J_{12}	1	0	0	0	1/2	-1/2	-1/2	1/2	-1/2	1/2	1/2	-1/2
J_{34}	0	1	0	0	-1/2	1/2	-1/2	1/2	1/2	-1/2	1/2	-1/2
J_{56}	0	0	1	0	-1/2	-1/2	1/2	1/2	1/2	1/2	-1/2	-1/2

Some consequences:

- most non-commutative results survive; planar amplitudes are inherited
Filk (space-time noncommutativity); Khoze; ...
- vector U(1) factors decouple; chiral superfield U(1) factors are coupled
 $U(N)$ vs $SU(N)$?
- both f_{abc} and d_{abc} couplings

What is the coefficient of the superpotential and
what is the UV behavior of the theory?

- Leigh/Strassler: not constructive but guarantees that there exists a coefficient $h = f(\beta, N_c, g_{YM})$ that leads to a finite theory

- 1- and 2-loop expressions are known
- when/how are they corrected?

Freedman, Gursoy
Penati, Santambrogio, Zanon

- 4-point amplitudes: Manifest susy \longrightarrow formally diverge in

$$D_c = 4 + \frac{2\mathcal{N}}{L} \longrightarrow D_c = 4 + \frac{2}{L}$$

Is this really true?

- ▶ Try to answer by evaluating 4-point amplitudes

- Single-trace amplitudes:

$$A^{(0)} = \sum_{\rho \in S_n / Z_n} \text{Tr}[T^{a_{\rho(1)}} \dots T^{a_{\rho(n)}}] A^{(0)}(k_{\rho(1)} \dots k_{\rho(n)})$$



$$A^{(0)}(k_1 \dots k_n) \mapsto e^{i\Theta(1, \dots, n)} A^{(0)}(k_1 \dots k_n); \quad \Theta(1, \dots, n) = \sum_{1 \leq i < j \leq n} q_i \cdot \hat{\beta} \cdot q_j$$

- A simple example -- 3-point amplitudes: $\phi^{ij} \psi^{ikl} \psi^{jkl}$, $i \neq j \neq k = 1, 2, 3, 4$

$$\mathcal{A}^{(0)}(1\phi^{ij}, 2\psi^{ikl}, 3\psi^{jkl}) = \frac{h}{g} \langle 23 \rangle e^{i\beta} \text{Tr}_{123} + \frac{h}{g} \langle 32 \rangle e^{-i\beta} \text{Tr}_{321}$$

- Single-trace amplitudes:

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- Account for the $\mathcal{O}(1/N^2)$ deformation of the coefficient of the superpotential

Here: focus on double-trace terms; ignore $\mathcal{O}(1/N^2)$ corrections

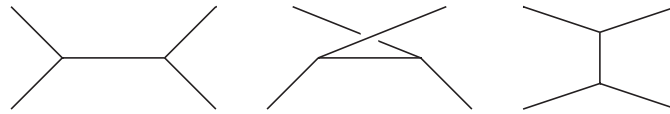
- non-vanishing tree-level double-trace amplitudes

$$\mathcal{L}_{2\text{tr}} = \frac{1}{2N} |f(\beta, N)|^2 |\epsilon_{ijk} \epsilon^{ilm} \text{Tr}[[\phi^j, \phi^k]_\beta] \text{Tr}[[\bar{\phi}_l, \bar{\phi}_m]_\beta]$$

→ crucial for finiteness; also $|f(\beta, N)|^2 = \frac{g_{YM}^2}{1 - \frac{4}{N^2} \sin^2 \beta}$

With same planar properties, differences appear at subleading color in dim. reg.

- supersymmetry → more double-trace amplitudes
- structure constant color factors combine 1- and 2-trace terms



$$\mathcal{A}_4^{\beta,(0)}(1g^+, 2\phi^{23}, 3f^{134}, 4f^{124}) = \frac{n_{12}}{s_{12}} f^{12a} f_{\beta}^{34}{}_a + \frac{n_{23}}{s_{23}} f_{\beta}^{23a} f^{14}{}_a + \frac{n_{13}}{s_{13}} f^{31a} f_{\beta}^{24}{}_a$$

$$\mathcal{A}_4^{\beta,(0)}(1\phi^{23}, 2\phi^{14}, 3\phi^{13}, 4\phi^{24}) = \frac{n_{12}}{s_{12}} f^{12a} f^{34}{}_a + \frac{n_{23}}{s_{23}} f^{23a} f^{14}{}_a + \frac{n_{13}}{s_{13}} f_{\beta}^{31a} f_{\beta}^{24}{}_a$$

$$f_{\beta}^{abc} = \text{Tr}[T^a [T^b, T^c]_{\beta}] = e^{i\Phi(a,b,c)} \text{Tr}[T^a T^b T^c] - e^{i\Phi(a,c,b)} \text{Tr}[T^a T^c T^b]$$

Numerator factors -- same as in $\mathcal{N}=4$ sYM:

$$n_{12} + n_{23} + n_{13} = 0 \quad \text{-- BCJ-like numerator relations}$$

Color factors – different; generically no Jacobi identity
involving only d-structure constants:

$$f^{[12}{}_a f^{3]4a} = 0 \quad f^{[12}{}_a d^{3]4a} = 0$$

Some all-order results

- planar cusp anomaly is independent of the deformation
 - consequence of integrability; testable at weak and strong coupling
 - same leading IR divergences as in the un-deformed theory
- 2-trace β -dependence: iff nontrivial R-charge flow between traces



Generically:

$$q_1 + q_2 + q_3 + q_4 = 0$$

If $q_1 + q_2 = 0 = q_3 + q_4$ all β -dependence drops out

- 4-gluon double-trace terms: same properties as in $\mathcal{N} = 4$ sYM
 - diverge in $D_c = 4 + 8/L$
 - same for $\text{Tr}[\phi_i \bar{\phi}^i] \text{Tr}[\phi_j \bar{\phi}^j]$ to first nontrivial order in $1/N$
- Generalization to higher-point multi-trace terms

Some explicit examples 4-point loop amplitudes: 1 loop

Jin, RR

- Construct using generalized unitarity

- use color-dressed cuts

- supersums: use pictorial rules

Bern, Carrasco, Ita, Johansson, RR

- dressed with the extra phase factors

- focus on 3 terms: $\text{Tr}[T^{a_1} T^{a_i}] \text{Tr}[T^{a_j} T^{a_k}] \quad i \neq j \neq k = 2, 3, 4$

- Classify following the number of vector multiplets

- 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM

- 3 vector multiplets + 1 chiral multiplet: vanish identically

- 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

- 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$

- 4 chiral multiplets: $A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi_1}, 4_{\psi_1}), A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi_1}, 4_{\psi_2})$

Some explicit examples 4-point loop amplitudes: 1 loop

- 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

$$\mathcal{A}(1234)_4^{(1)\beta} = \mathcal{A}(1234)_4^{(1)\mathcal{N}=4} - 8 \sin^2 \beta (\text{Tr}_{13} \text{Tr}_{24} + \text{Tr}_{14} \text{Tr}_{23}) \mathcal{A}(1234)_{4;3}^{(1)\text{extra}}$$

$$\begin{aligned} \mathcal{A}(1234)_{4;3}^{(1)\text{extra}} &= \frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} \left[-s_{12} s_{23} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} + s_{12} \left(\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \right) + s_{23} \left(\begin{array}{c} 2 \quad 3 \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} \right) \right] \\ &= -\frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} \frac{G[l, 1, 2, 3]}{s_{12} s_{23}} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} = -\frac{1}{(4\pi)^2} \frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} \left[\left(\ln \frac{s_{12}}{s_{23}} \right)^2 + \pi^2 \right] \end{aligned}$$

- IR finite

- UV divergent in 6 dimensions; standard expectation for a conformal $\mathcal{N} = 1$ theory

Some explicit examples 4-point loop amplitudes: 1 loop

- 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

$$\mathcal{A}(1234)_4^{(1)\beta} = \mathcal{A}(1234)_4^{(1)\mathcal{N}=4} - 8 \sin^2 \beta (\text{Tr}_{13} \text{Tr}_{24} + \text{Tr}_{14} \text{Tr}_{23}) A(1234)_{4;3}^{(1)\text{extra}}$$

$$A(1234)_{4;3}^{(1)\text{extra}} = \frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} \left[-s_{12} s_{23} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} + s_{12} \left(\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \right) + s_{23} \left(\begin{array}{c} 2 \quad 3 \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} \right) \right]$$

$$= -\frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} \frac{G[l, 1, 2, 3]}{s_{12} s_{23}} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} = -\frac{1}{(4\pi)^2} \frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} \left[\left(\ln \frac{s_{12}}{s_{23}} \right)^2 + \pi^2 \right]$$

- complete deformation-dependent amplitude

$$\mathcal{A}_4^{\text{extra},(1)}(1234) = -8\tilde{h}^2 \sin^2 \beta \frac{\langle 13 \rangle^2}{\langle 23 \rangle^2} \frac{G[l, 1, 2, 3]}{s_{12} s_{23}} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ 1 \quad 4 \end{array} \times \left[\left(1 - \frac{2}{N^2} \right) (\text{Tr}_{13} \text{Tr}_{24} + \text{Tr}_{14} \text{Tr}_{23}) - \frac{1}{N} (\text{Tr}_{1324} + \text{Tr}_{1432}) - \frac{2}{N^2} \text{Tr}_{12} \text{Tr}_{34} \right]$$

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Some 1-loop comments:

- results consistent with expected structure of IR divergences
 - most corrections are in fact IR-finite; consistent with structure of IR div's
 - only small changes in the soft anomalous dimension matrix
- no real improvement over a **finite** “garden variety” $\mathcal{N} = 1$ theory
 - except perhaps absence of incomplete cancellations (of bubbles)
- some details are as if there were more than $\mathcal{N} = 1$ supersymmetry
 - some supersums are perfect squares
 - yet, no noticeable effect

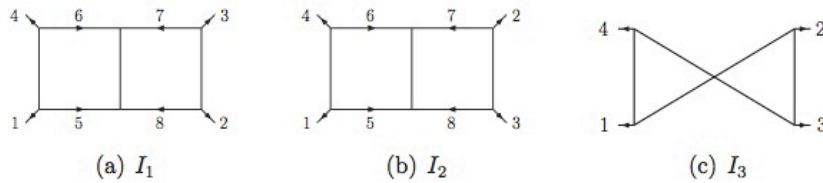
More explicit examples 4-point loop amplitudes: 2 loops

Jin, RR

Same classification:

- 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

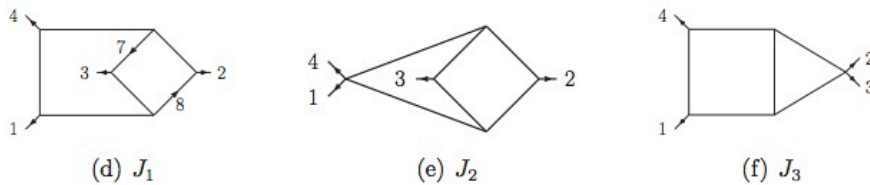
$$A(1234)_{4; 2\text{tr}}^{(2)\beta} = A(1234)_{4; 2\text{tr}}^{(2)\mathcal{N}=4} - 8 \sin^2 \beta \text{Tr}_{13} \text{Tr}_{24} A_{13;24}^{(2)\text{extra}} - 8 \sin^2 \beta \text{Tr}_{14} \text{Tr}_{23} A_{14;23}^{(2)\text{extra}}$$



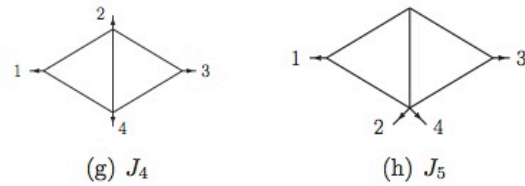
$C U$ Symmetries of $\frac{\langle 13 \rangle^2}{\langle 23 \rangle^2} A_{14;23}^{(2)\text{extra}}$

$$C : (1 \leftrightarrow 2, 3 \leftrightarrow 4)$$

$$U : (1 \leftrightarrow 4, 2 \leftrightarrow 3)$$



U



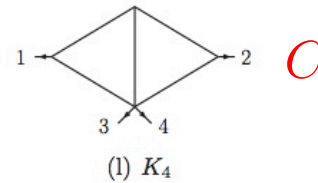
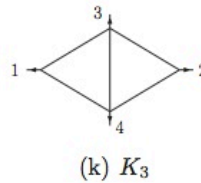
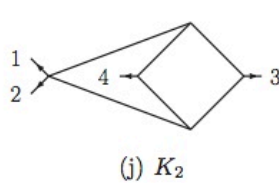
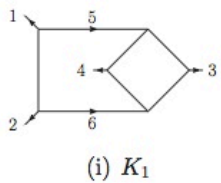
U'

$$M_{14;23}^{(2)\text{extra}} = \sum_i \alpha_i I_i$$

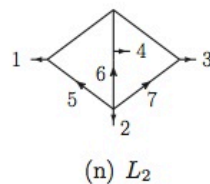
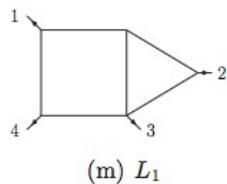
$$+ (1 + C) \sum_i \beta_i J_i$$

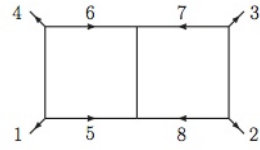
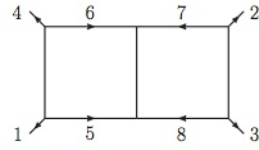
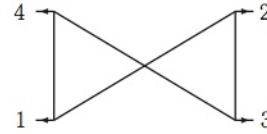
$$+ (1 + U) \sum_i \gamma_i K_i$$

$$+ (1 + U)(1 + C) \sum_i \delta_i L_i$$



C



(a) I_1 (b) I_2 (c) I_3

$$\alpha_1 = \tau_{1,4} (\tau_{1,8}^2 + \tau_{2,5}^2 + \tau_{4,7}^2 + \tau_{3,6}^2 + \tau_{1,2}(\tau_{1,8} + \tau_{2,5} + \tau_{4,7} + \tau_{3,6} - 2\tau_{1,4}) - \tau_{1,8}\tau_{2,5} - \tau_{4,7}\tau_{3,6})$$

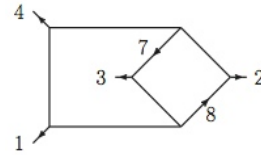
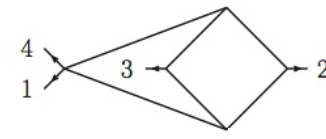
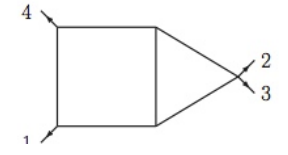
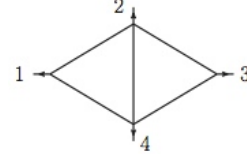
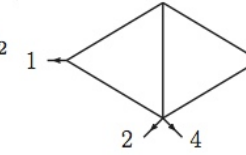
$$\alpha_2 = \tau_{1,4} (\tau_{1,8}^2 + \tau_{2,6}^2 + \tau_{4,7}^2 + \tau_{3,5}^2 + \tau_{1,2}(\tau_{1,8} + \tau_{2,6} + \tau_{4,7} + \tau_{3,5}) + \tau_{1,8}\tau_{2,6} + \tau_{4,7}\tau_{3,5})$$

$$\alpha_3 = -4\tau_{1,3}\tau_{1,4}$$

$$\beta_1 = 2\tau_{1,4} (\tau_{1,8}^2 + \tau_{4,7}^2 + \tau_{1,3}(\tau_{1,8} + \tau_{4,7}))$$

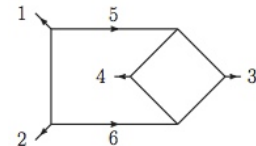
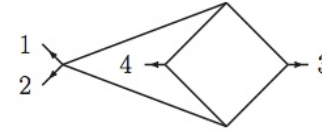
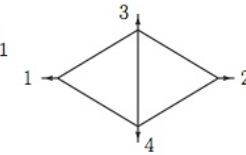
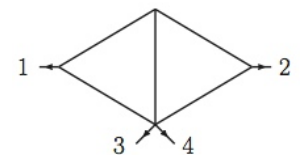
$$\beta_2 = -2\tau_{1,4}^2 \quad \beta_3 = 2\tau_{1,4}^2$$

$$\beta_4 = 2\tau_{1,2} \quad \beta_5 = (\tau_{1,4} - 2\tau_{1,2})$$

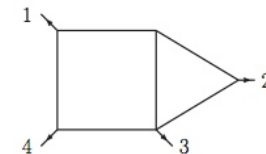
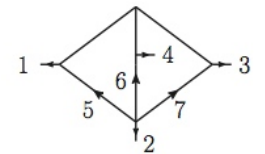
(d) J_1 (e) J_2 (f) J_3 (g) J_4 (h) J_5

$$\gamma_1 = \tau_{1,2} (\tau_{1,5}^2 + \tau_{2,6}^2 + \tau_{1,2}(\tau_{1,5} + \tau_{2,6}))$$

$$\gamma_2 = 2\tau_{1,2}^2 \quad \gamma_3 = 2\tau_{1,2} \quad \gamma_4 = (\tau_{1,3} - \tau_{1,2})$$

(i) K_1 (j) K_2 (k) K_3 (l) K_4

$$\delta_1 = \tau_{1,4}^2 \quad \delta_2 = -\tau_{1,4}\tau_{2,5} + \frac{1}{2}\tau_{12}(3\tau_{1,3} + \tau_{2,5} - 2\tau_{3,5} + 2\tau_{2,7})$$

(m) L_1 (n) L_2

The UV behavior of this color structure

Slightly problematic to phrase it in terms of a critical dimension: higher-dimensional theory has a different field content from the four dimensional one

Nevertheless, analytically continuing the result to $d > 4$ is a measure of the degree of divergence of the amplitude; reduces to standard critical dimension as $\beta \rightarrow 0$

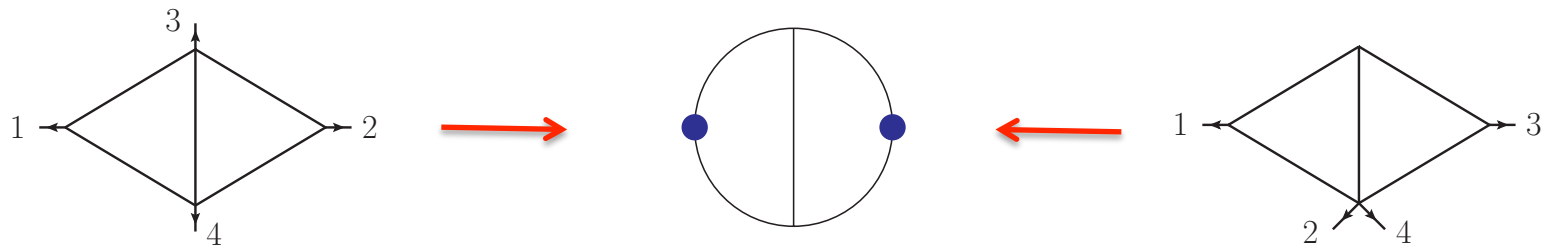
Strategy: same as for $\mathcal{N} = 4$ sYM and for $\mathcal{N} = 8$ supergravity:

- expand at small external momenta
- use Lorentz-invariance to reorganize tensor integrals

7-prop. integrals: at most 2 loop mom. num. factors

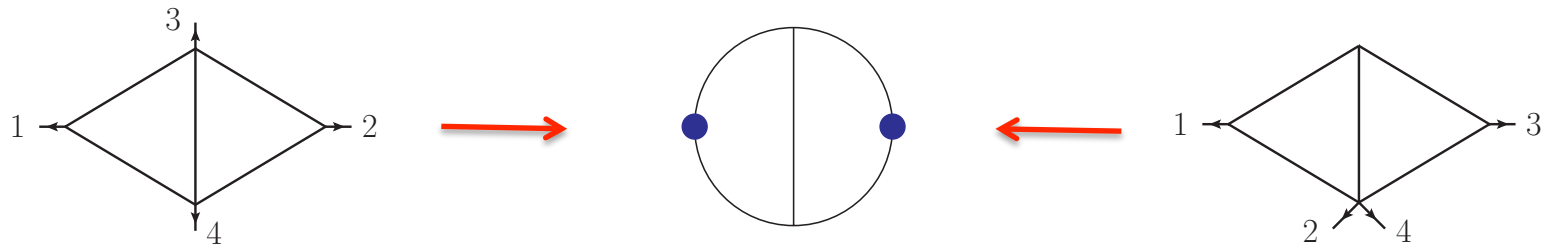
Types of integrals: 6-prop. integrals: at most 1 loop mom. num. factors

5-prop. integrals: no loop mom. num. factors \rightarrow leading UV



The UV behavior of this color structures

7-prop. integrals: at most 2 loop mom. num. factors
 Types of integrals: 6-prop. integrals: at most 1 loop mom. num. factors
 5-prop. integrals: no loop mom. num. factors \rightarrow leading UV



$$M_{14;23}^{\beta,(2)} : 2\beta_4 + 2\beta_5 + 2\gamma_3 + 2\gamma_4 = 0 \quad M_{13;24}^{\beta,(2)} : 2\beta'_2 + 2\beta'_3 + 4\rho'_1 + 4\eta'_1 = 0$$

- Double-trace part of β -deformed 2-loop amplitude is finite in $d=5$;
 diverges only in $d=6$ – better than manifest $\mathcal{N} = 1$ supersymmetry suggests:

$$D_c = 4 + \frac{2\mathcal{N}}{L} \quad \rightarrow \quad D_c = 4 + \frac{4}{L}$$

- Superficially similar behavior to 2-trace terms at $L \geq 3$ in $\mathcal{N} = 4$ sYM

$$D_c = 4 + \frac{6}{L} \quad \text{vs.} \quad D_c^{2 \text{ trace}} = 4 + \frac{8}{L}$$

Summary and some questions

- Despite extensive planar similarity with $\mathcal{N} = 4$ sYM, the β -deformed theory is not “simple” at the non-planar level
- certain all-loop $\mathcal{O}(1/N)$ structures inherited from the $\mathcal{N} = 4$ theory
- at 1-loop: expected properties of a finite $\mathcal{N} = 1$ theory
- at 2-loops: $\mathcal{N} = 1$ susy seems more powerful than it should be
 - Better UV convergence properties
 - Is this an accident or a sign of further structure?
 - Is “this” present in the $\mathcal{N} = 4$ theory but obscured by maximal supersymmetry?
- Do the BCJ-like numerator relations play any role? Is there a generalization of color/kinematic duality to symmetric couplings? Is it possible to break susy with the $\mathcal{N} = 4$ sYM field content while preserving conformal and dual conformal inv. and w/o d-str constants?

Extra slides

The supersymmetric β -deformed $\mathcal{N} = 4$ super-Yang-Mills theory

- the same field content as $\mathcal{N} = 4$ sYM
- real β : almost the same properties except for supersymmetry
- a pattern for the deformation:

Lunin, Maldacena

noncommutative deformation: $\varphi_I \varphi_J \mapsto e^{i\hat{\beta}_{ij} q_I^i q_J^j} \varphi_I \varphi_J$

$\hat{\beta}_{ij} = -\hat{\beta}_{ji}$; if $\hat{\beta}_{12} = \hat{\beta}_{23} = \hat{\beta}_{31} = \beta \mapsto$ susy

$\mathcal{N} = 4$
R-charge vectors

	ϕ^{14}	ϕ^{24}	ϕ^{34}	A_μ	ψ^1	ψ^2	ψ^3	ψ^4	Q^1	Q^2	Q^3	Q^4
J_{12}	1	0	0	0	1/2	-1/2	-1/2	1/2	-1/2	1/2	1/2	-1/2
J_{34}	0	1	0	0	-1/2	1/2	-1/2	1/2	1/2	-1/2	1/2	-1/2
J_{56}	0	0	1	0	-1/2	-1/2	1/2	1/2	1/2	1/2	-1/2	-1/2

Some consequences:

- most non-commutative results survive; planar amplitudes are inherited
Filk (space-time noncommutativity); Khoze; ...
- vector U(1) factors decouple; chiral superfield U(1) factors are coupled
 $U(N)$ vs $SU(N)$?
- both f_{abc} and d_{abc} couplings

Analyze QFT-s which share most of the properties of $\mathcal{N} = 4$ sYM

→ Deform it in a controlled way

1. orbifolds

Inheritance principle: Bershadsky, Johansen

Bershadsky, Kakushadze, Vafa

$$\varphi_i^I = R^I{}_J g^{-1} \varphi_i^J g \quad R \in SU(4) \quad g \in SU(4) \subset SU(N)$$

2. the h deformation

Leigh, Strassler

$$W = \text{Tr}[\Phi_1[\Phi_2, \Phi_3]] \longrightarrow f(h, N)(\text{Tr}[\Phi_1[\Phi_2, \Phi_3]] + h(\text{Tr}[\Phi_1^3] + \text{Tr}[\Phi_2^3] + \text{Tr}[\Phi_3^3]))$$

3. the β deformation

Leigh, Strassler

$$W = \text{Tr}[\Phi_1[\Phi_2, \Phi_3]] \longrightarrow f(\beta, N)\text{Tr}[\Phi_1(e^{i\beta}\Phi_2\Phi_3 - e^{-i\beta}\Phi_3\Phi_2)]$$

	super-conf.	dual super-conf.	planar integrable	Amp/W.L.
1.	yes; N=2, 1, 0	yes; inherited	yes	quite likely
2.	yes; N=1	not known	sometimes	not clear
3.	yes; N=1, 0	yes	yes	yes

General deformation: $\varphi_I \varphi_J \mapsto e^{i\hat{\beta}_{ij} q_I^i q_J^j} \varphi_I \varphi_J$ $\hat{\beta}_{12} \neq \hat{\beta}_{23} \neq \hat{\beta}_{31}$

Frolov

- Supersymmetry is completely broken;

deformation of the $\mathcal{N} = 4$ component Lagrangian

- Renormalizability requires some 2-trace terms

Dymarsky, Klebanov, RR
(unpublished)

$$\delta S_{eff} = -\frac{\lambda^2}{16\pi^2} \ln \frac{\Lambda^2}{M^2} \left[\begin{aligned} &8 (\cos 4\beta_{ij} - \cos 4\beta_{ik})^2 \text{Tr}[\phi^i \phi^i] \text{Tr}[\phi^{\bar{i}} \phi^{\bar{i}}] \\ &+ 8 (\cos 4\beta_{ij} - \cos 2(\beta_{ik} + \beta_{jk}))^2 \text{Tr}[\phi^i \phi^j] \text{Tr}[\phi^{\bar{i}} \phi^{\bar{j}}] \\ &+ 8 (\cos 4\beta_{ij} - \cos 2(\beta_{ik} - \beta_{jk}))^2 \text{Tr}[\phi^i \phi^{\bar{j}}] \text{Tr}[\phi^{\bar{i}} \phi^j] \end{aligned} \right] \Big|_{i \neq j \neq k}$$



$$\delta S_{tree} = f_{1ij} \text{Tr}[\phi^i \phi^i] \text{Tr}[\phi^{\bar{i}} \phi^{\bar{i}}] + f_{2ij} \text{Tr}[\phi^i \phi^j] \text{Tr}[\phi^{\bar{i}} \phi^{\bar{j}}] + f_{3ij} \text{Tr}[\phi^i \phi^{\bar{j}}] \text{Tr}[\phi^{\bar{i}} \phi^j] \Big|_{i \neq j \neq k}$$

- Unstable RG evolution of 2-trace couplings (Landau pole)

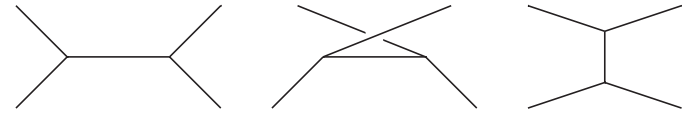
- $\mathcal{N} = 1$ limit: unique 2-trace coupling with RG fixed point

$$f_{2ij} = \frac{2}{N} |h|^2 \sin^2 2\beta$$

Value required to project out the U(1)
auxiliary fields in chiral multiplets

Expectedly, supersymmetry cures instability; does it do anything else?

- BCJ-like numerator relations



$$\mathcal{A}_4^{\beta,(0)}(1g^+, 2\phi^{23}, 3f^{134}, 4f^{124}) = \frac{n_{12}}{s_{12}} f^{12a} f_{\beta}^{34}{}_a + \frac{n_{23}}{s_{23}} f_{\beta}^{23a} f^{14}{}_a + \frac{n_{13}}{s_{13}} f^{31a} f_{\beta}^{24}{}_a$$

$$\mathcal{A}_4^{\beta,(0)}(1\phi^{23}, 2\phi^{14}, 3\phi^{13}, 4\phi^{24}) = \frac{n_{12}}{s_{12}} f^{12a} f^{34}{}_a + \frac{n_{23}}{s_{23}} f^{23a} f^{14}{}_a + \frac{n_{13}}{s_{13}} f_{\beta}^{31a} f_{\beta}^{24}{}_a$$

$$f_{\beta}^{abc} = \text{Tr}[T^a [T^b, T^c]_{\beta}] = e^{i\Phi(a,b,c)} \text{Tr}[T^a T^b T^c] - e^{i\Phi(a,c,b)} \text{Tr}[T^a T^c T^b]$$

Numerator factors -- same as in $\mathcal{N}=4$ sYM:

$$n_{12} + n_{23} + n_{13} = 0$$

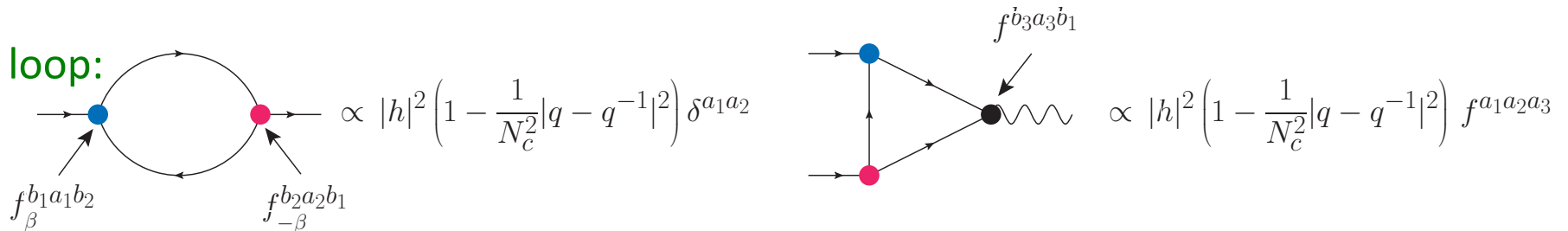
Color factors – different; generically no Jacobi identity involving only d-structure constants:

$$f^{[12}{}_a f^{3]4a} = 0 \quad f^{[12}{}_a d^{3]4a} = 0$$

On corrections to the coefficient of the superpotential

Fairly accurate picture from color structure of supergraphs with only scalar vertices

- 1 loop:



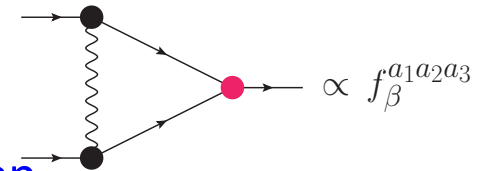
Finiteness as $q \rightarrow 1$ requires they are cancelled by vector multiplet interactions

$$|f(\beta, N_c)|^2 \equiv |h|^2 = \frac{g_{YM}^2}{1 - \frac{1}{N_c^2} |q - q^{-1}|^2}$$

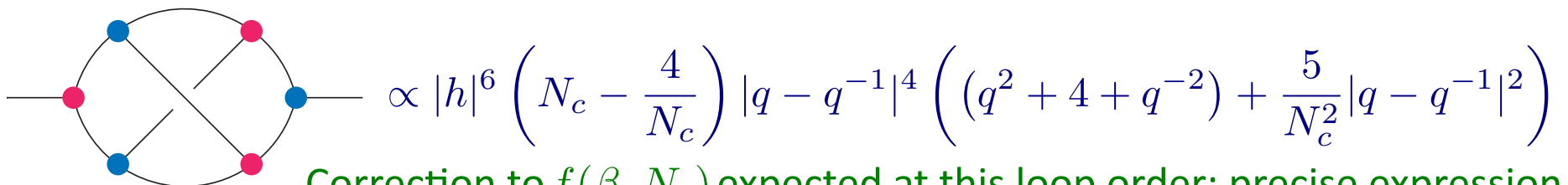
- 2 loops: All 2-point and 3-point graphs contain a triangle

→ color structure reduces to 1-loop analysis

→ finiteness at $q \rightarrow 1$: same condition as at 1 loop



- 3 loops: Argument no longer works; there exist graphs with only box subintegrals



Correction to $f(\beta, N_c)$ expected at this loop order; precise expression of the 2-loop divergence is important.

Some all-order results

- cusp anomaly is independent of the deformation
 - consequence of integrability; testable at weak and strong coupling
 - same leading IR divergences as in the un-deformed theory
- 2-trace β -dependence: iff nontrivial R-charge flow between traces



Generically:

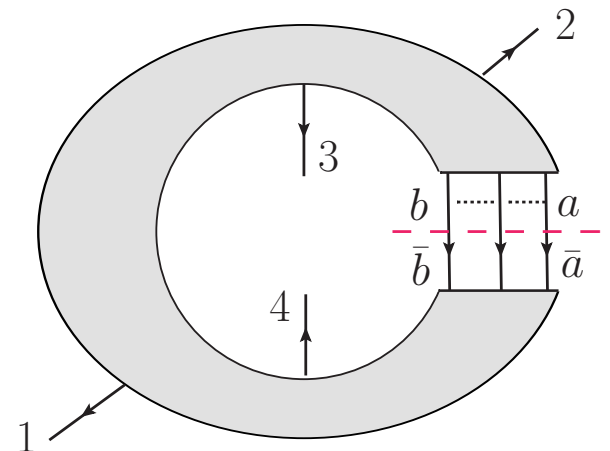
$$q_1 + q_2 + q_3 + q_4 = 0$$

If $q_1 + q_2 = 0 = q_3 + q_4$ all β -dependence drops out

To see this: follow charge flow in generalized unitarity cuts

Charge conservation:

$$\Theta(1, 2, a, \dots, b, 3, 4, \bar{b}, \dots, \bar{a}) = \Theta(1, 2) + \Theta(3, 4) + 2\Theta(1 + 2, a, \dots, b)$$



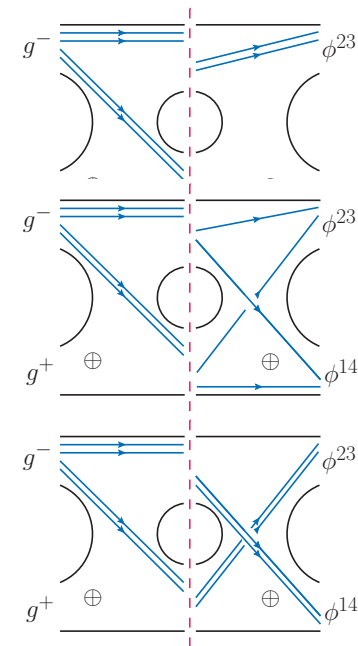
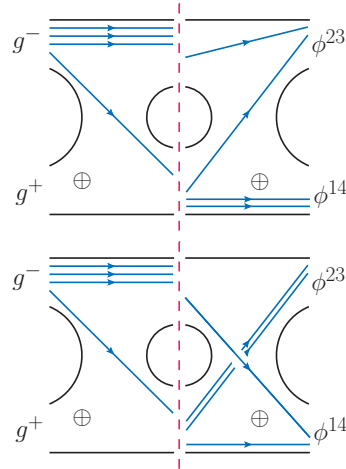
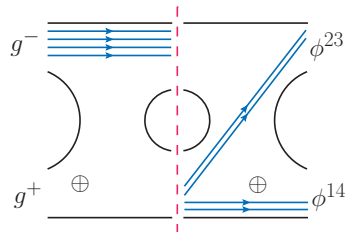
Some explicit examples 4-point loop amplitudes: 1 loop

Jin, RR

- Construct using generalized unitarity
 - use color-dressed cuts
 - supersums: use pictorial rules dressed with the extra phase factors

Bern, Carrasco, Ita, Johansson, RR

$$\mathcal{A}_4^{\beta,(1)}(1g^+, 2g^-, 3\phi^{23}, 4\phi^{14})$$



$$A = \frac{\langle l_a 2 \rangle}{\langle l_b 2 \rangle} \quad B = \frac{\langle l_b 3 \rangle}{\langle l_a 3 \rangle} \quad C = \frac{\langle l_b 4 \rangle}{\langle l_a 4 \rangle}$$

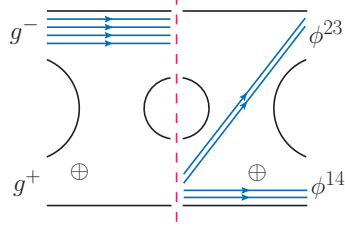
$$\rho_{\mathcal{N}=4}^2 = A^2 BC - (2AB + 2AC) + \left(4 + \frac{B}{C} + \frac{C}{B}\right) - \left(\frac{2}{AB} + \frac{2}{AC}\right) + \frac{1}{A^2 BC}$$

Some explicit examples 4-point loop amplitudes: 1 loop

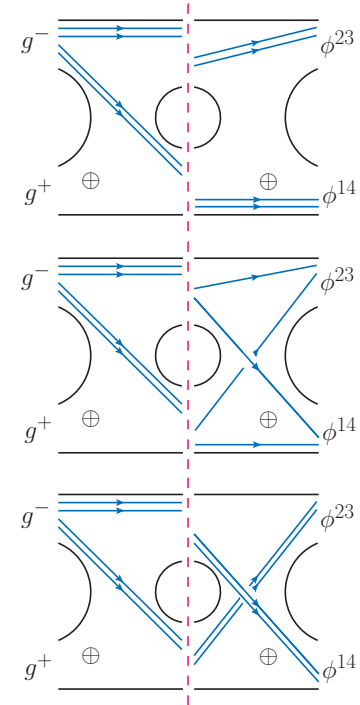
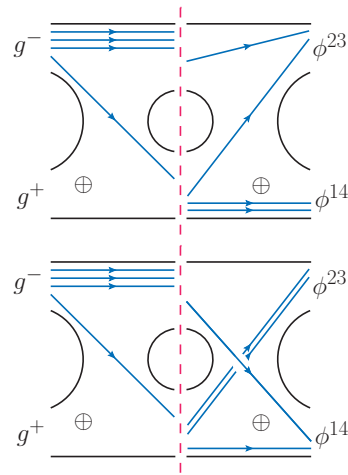
Jin, RR

Supersum in terms of charge-flow diagrams:

$$A = \frac{\langle l_a 2 \rangle}{\langle l_b 2 \rangle} \quad B = \frac{\langle l_b 3 \rangle}{\langle l_a 3 \rangle} \quad C = \frac{\langle l_b 4 \rangle}{\langle l_a 4 \rangle}$$

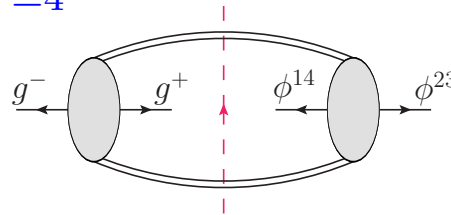


$$\mathcal{A}_4^{\beta, (1)}(1g^+, 2g^-, 3\phi^{23}, 4\phi^{14})$$



- Planar cut topology: same $\rho_{\mathcal{N}=4}^2$

- Non-planar cut topology:



$$\rho_\beta^2 = A^2 BC - (2AB + 2\cos(2\beta)AC) + (4\cos(2\beta) + \frac{B}{C} + \frac{C}{B}) - \frac{2}{AB} - \frac{2\cos(2\beta)}{AC} + \frac{1}{A^2 BC}$$

$$C = \rho^2 \frac{\langle l_a 2 \rangle^2 \langle l_b 2 \rangle^2}{\langle 12 \rangle \langle 2l_a \rangle \langle l_a l_b \rangle \langle l_b 1 \rangle} \frac{\langle l_b 4 \rangle \langle l_b 3 \rangle \langle l_a 4 \rangle \langle l_a 3 \rangle}{\langle 3l_a \rangle \langle l_a 4 \rangle \langle 4l_b \rangle \langle l_b 3 \rangle}$$

Some explicit examples 4-point loop amplitudes: 1 loop

- Classify following the number of vector multiplets
 - 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
 - 3 vector multiplets + 1 chiral multiplet: vanish identically
 - 2 vector multiplets + 2 chiral multiplets: $\mathcal{A}(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

$$\mathcal{A}(1234)_4^{(1)\beta} = \mathcal{A}(1234)_4^{(1)\mathcal{N}=4} + \mathcal{A}(1234)_4^{\text{extra},(1)}$$

$$\mathcal{A}_4^{\text{extra},(1)}(1234) = -8\tilde{h}^2 \sin^2 \beta \frac{\langle 13 \rangle^2}{\langle 23 \rangle^2} \frac{G[l, 1, 2, 3]}{s_{12}s_{23}} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \times \left[\left(1 - \frac{2}{N^2} \right) (\text{Tr}_{13}\text{Tr}_{24} + \text{Tr}_{14}\text{Tr}_{23}) - \frac{1}{N} (\text{Tr}_{1324} + \text{Tr}_{1432}) - \frac{2}{N^2} \text{Tr}_{12}\text{Tr}_{34} \right]$$

- IR finite

- UV divergent in 6 dimensions; standard expectation for a conformal $\mathcal{N} = 1$ theory

Some explicit examples 4-point loop amplitudes: 1 loop

- Classify following the number of vector multiplets
 - 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
 - 3 vector multiplets + 1 chiral multiplet: vanish identically
 - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$
 - 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$

$$\mathcal{A}^{(1)2\text{tr}}(1234) = \cos \beta A_{\mathcal{N}=4}^{(1)2\text{tr}}(1234) (\text{Tr}_{12}\text{Tr}_{34} + \text{Tr}_{13}\text{Tr}_{24} + \text{Tr}_{14}\text{Tr}_{23})$$

$$A_{\mathcal{N}=4}^{(1)2\text{tr}} = -2 s_{12} s_{23} \frac{[23][34]}{[12][13]} \left(\begin{array}{ccc} \begin{array}{c} 2 \swarrow \quad \searrow 3 \\ \square \\ 1 \swarrow \quad \searrow 4 \end{array} & + & \begin{array}{c} 4 \swarrow \quad \searrow 2 \\ \square \\ 1 \swarrow \quad \searrow 3 \end{array} & + & \begin{array}{c} 3 \swarrow \quad \searrow 4 \\ \square \\ 1 \swarrow \quad \searrow 2 \end{array} \end{array} \right)$$

Some explicit examples 4-point loop amplitudes: 1 loop

- Classify following the number of vector multiplets

- 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
- 3 vector multiplets + 1 chiral multiplet: vanish identically
- 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$
- 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$
- 4 chiral multiplets: $A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi_1}, 4_{\psi_1}), A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi_1}, 4_{\psi_2})$

$$\mathcal{A}(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^{234}}, 4_{\psi^{234}})_4^{(1)\beta} = \mathcal{A}(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^{234}}, 4_{\psi^{234}})_4^{(1)\mathcal{N}=4} - 8 \sin^2 \beta \text{Tr}_{12} \text{Tr}_{34} A(1234)^{(1)\text{extra}}$$

$$\frac{A(1234)^{(1)\text{extra}}}{\cos^2 \beta} = \frac{\langle 34 \rangle}{\langle 12 \rangle} \left[s_{13} s_{14} \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \square \\ \swarrow \quad \searrow \\ 1 \quad 3 \end{array} - s_{13} \left(\begin{array}{c} 3 \\ \swarrow \quad \searrow \\ \triangle \\ \swarrow \quad \searrow \\ 1 \quad 4 \end{array} + \begin{array}{c} 3 \\ \swarrow \quad \searrow \\ \triangle \\ \swarrow \quad \searrow \\ 1 \quad 4 \end{array} \right) - s_{14} \left(\begin{array}{c} 2 \quad 3 \\ \swarrow \quad \searrow \\ \triangle \\ \swarrow \quad \searrow \\ 1 \quad 4 \end{array} + \begin{array}{c} 2 \quad 3 \\ \swarrow \quad \searrow \\ \triangle \\ \swarrow \quad \searrow \\ 1 \quad 4 \end{array} \right) \right]$$

$$= -\frac{\langle 34 \rangle}{\langle 12 \rangle} \frac{G[l, 1, 4, 2]}{s_{13} s_{14}} \begin{array}{c} 4 \\ \swarrow \quad \searrow \\ \square \\ \swarrow \quad \searrow \\ 1 \quad 3 \end{array}$$

l

Some explicit examples 4-point loop amplitudes: 1 loop

- Classify following the number of vector multiplets

- 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
- 3 vector multiplets + 1 chiral multiplet: vanish identically
- 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$
- 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$
- 4 chiral multiplets: $A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi_1}, 4_{\psi_1}), A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi_1}, 4_{\psi_2})$

$$\mathcal{A}(1_{\psi^1}, 2_{\psi^2}, 3_{\psi_1}, 4_{\psi_2})_4^{(1)\beta} = \mathcal{A}_4^{(1)\mathcal{N}=4} - 8 \sin^2 \beta \text{Tr}_{12} \text{Tr}_{34} A(1234)_{12;34}^{(1)\text{extra}} - 8 \sin^2 \beta \text{Tr}_{14} \text{Tr}_{23} A(1234)_{14;23}^{(1)\text{extra}}$$

$$A(1234)_{12;34}^{(1)\text{extra}} = \cos^2 \beta \frac{\langle 34 \rangle}{\langle 12 \rangle} \left(\frac{1}{2} s_{12} s_{13} \begin{array}{c} 2 \quad 4 \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} - s_{12} \left(\begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \triangle \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} - \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \triangle \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \right) \right)$$

$$A(1234)_{14;23}^{(1)\text{extra}} = \frac{\langle 34 \rangle}{\langle 12 \rangle} \left[\cos^2 \beta \frac{G[l, 1, 2, 4]}{2s_{12}s_{13}} \begin{array}{c} 2 \quad 4 \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} + s_{12}s_{14} \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} - s_{12}s_{13} \begin{array}{c} 2 \quad 4 \\ \diagdown \quad \diagup \\ \square \\ \diagup \quad \diagdown \\ 1 \quad 3 \end{array} \right]$$

- The other trace structure $A_{13;24}^{(2)\text{extra}}$: similar structure with $A_{14;23}^{(2)\text{extra}}$ with a few twists

- planar double-boxes are absent

- additional symmetries: $C : (1 \leftrightarrow 2, 3 \leftrightarrow 4)$

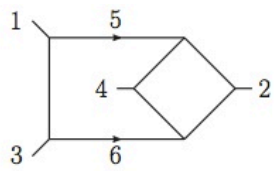
$U : (1 \leftrightarrow 3, 2 \leftrightarrow 4)$

$E : (1 \leftrightarrow 3)$

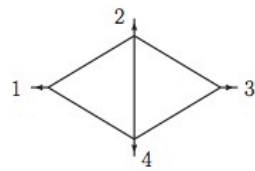
$$A_{**}^{(2)\text{extra}} = \frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} M_{**}^{(2)\text{extra}}$$

$$M_{13;24}^{(2)\text{extra}} = (1 + C) \sum_i \beta'_i J'_i + (1 + U) \sum_i \gamma'_i K'_i + (1 + U)(1 + C) \delta'_1 L'_1$$

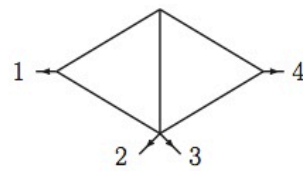
$$+ (1 + U')(1 + C)(1 + E) \epsilon'_1 M'_1 + (1 + C)(1 + E) \rho'_1 X'_1 + (1 + U')(1 + E) \eta'_1 Y'_1$$



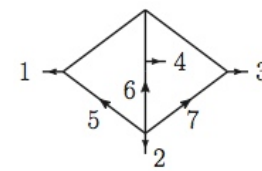
J'_1



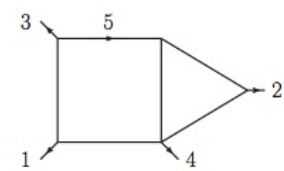
J'_2



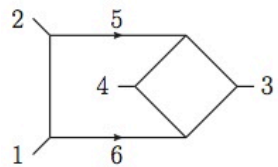
J'_3



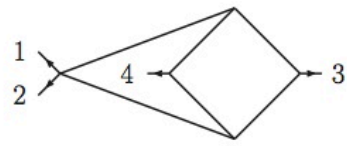
L'_1



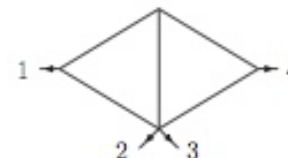
M'_1



K'_1



K'_2



X'_1



Y'_1

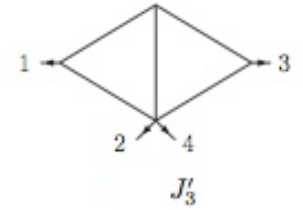
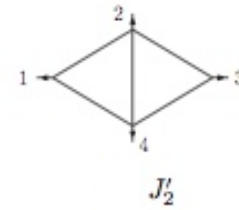
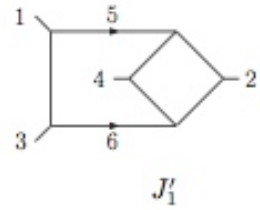
$$M_{13;24}^{(2)\text{extra}} = (1 + C) \sum_i \beta'_i J'_i + (1 + U) \sum_i \gamma'_i K'_i + (1 + U)(1 + C) \delta'_1 L'_1$$

$$+ (1 + U')(1 + C)(1 + E) \epsilon'_1 M'_1 + (1 + C)(1 + E) \rho'_1 X'_1 + (1 + U')(1 + E) \eta'_1 Y'_1$$

$$\beta'_1 = -2\tau_{1,3} (\tau_{2,5}\tau_{2,6} + \tau_{4,5}\tau_{4,6})$$

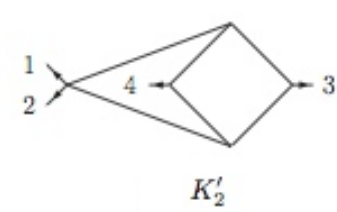
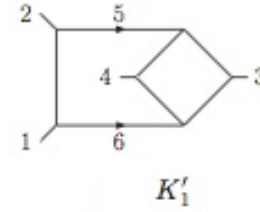
$$\beta'_2 = -2\tau_{1,3}$$

$$\beta'_3 = 2\tau_{1,3}$$



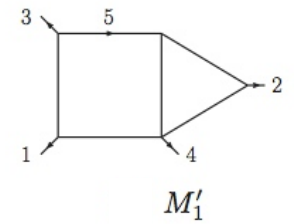
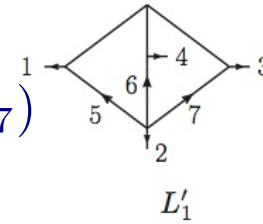
$$\gamma'_1 = \tau_{1,2} (\tau_{4,5}^2 + \tau_{3,6}^2 + \tau_{1,3}(\tau_{4,5} + \tau_{3,6}))$$

$$\gamma'_2 = \tau_{1,2}^2$$



$$\delta'_1 = -\frac{1}{2}\tau_{1,3}\tau_{2,6}$$

$$+ \frac{1}{2}(\tau_{1,4} - \tau_{1,2})(\tau_{2,5} - \tau_{2,7} + 2\tau_{4,5} - 2\tau_{4,7})$$



$$\epsilon'_1 = 2\tau_{1,3}\tau_{2,5}$$

$$\rho'_1 = 4\tau_{1,3}$$

$$\eta'_1 = -4\tau_{1,3}$$

