Time Crystals

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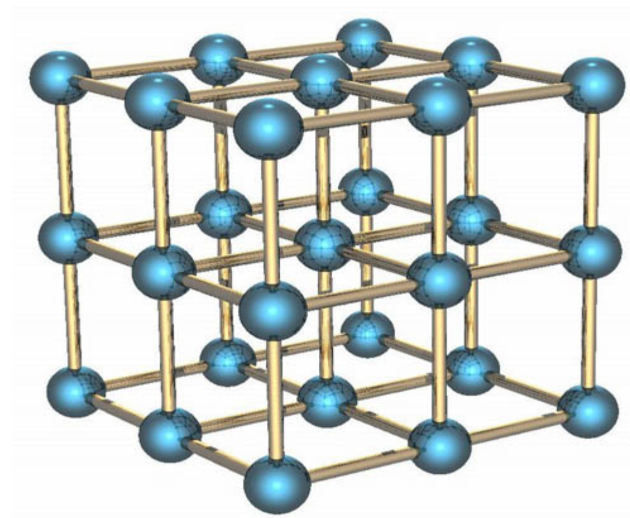
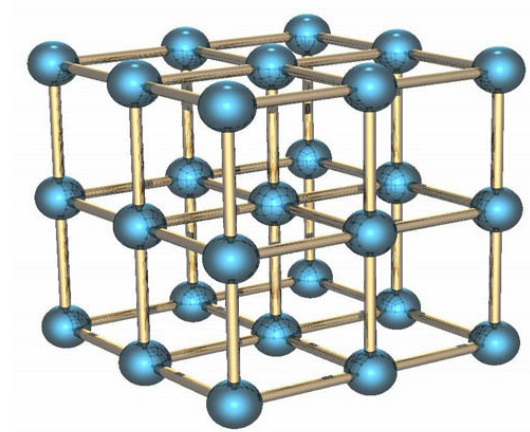
Time crystals?

- Crystals: ground state a periodic array in space
  - in ground state
  - spontaneously break space-translation symmetry
- Time crystals: ground state exhibits periodic behavior in time
  - spontaneously break time-translation symmetry in ground state
- Is this even possible?
  - It's not obvious…



Space Crystals



- Field $\phi(t, x)$ angle-valued
- Potential

$$V(\phi) = -\frac{\kappa}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{\lambda}{4}\left(\frac{d\phi}{dx}\right)^4$$

- Minimized by $\left.\frac{d\phi}{dx}\right|_{\min} = \sqrt{\frac{\kappa}{\lambda}}$
- Min-energy solution $\phi = \sqrt{\frac{\kappa}{\lambda}}x + \phi_0$
- spontaneously breaks x-translation down to

$$x \to x + 2\pi\sqrt{\frac{\lambda}{\kappa}}\,n$$

… a crystal!

Try to break t-symmetry

- by doing a similar thing with kinetic term

$$H(p, \phi) = -\frac{\kappa}{2} p^2 + \frac{\lambda}{4} p^4$$

- Minimized by $p_{\min} = \sqrt{\frac{\kappa}{\lambda}}$

- But velocity $\dot{\phi} = \frac{\partial H}{\partial p} = -\kappa p + \lambda p^3 \Big|_{p=p_{\min}} = 0$

- So minimum-energy solution is static, t-translation symmetry is **unbroken**.

- This is a **theorem**…

No spontaneous t-breaking

- Minimize Hamiltonian $H(p,q)$

$$\frac{\partial H}{\partial q} = 0 \qquad \frac{\partial H}{\partial p} = 0$$

- Hamilton's equations

$$\dot{p} = -\frac{\partial H}{\partial q} = 0 \qquad \dot{q} = \frac{\partial H}{\partial p} = 0$$

⇨ Minimum-energy solution is **static**.

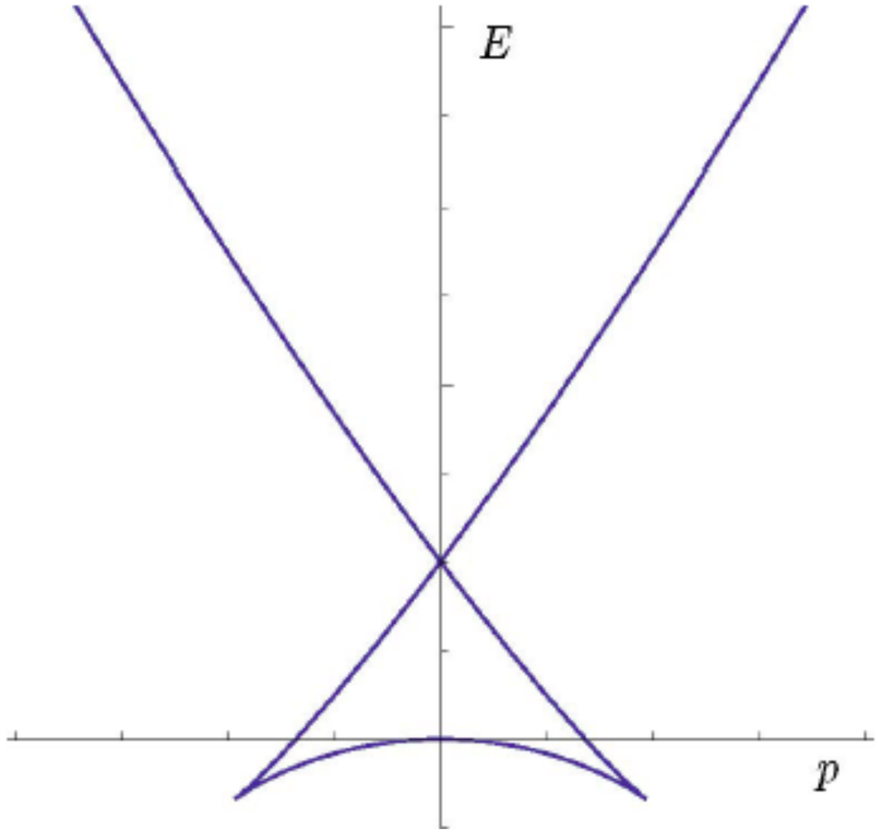
∴ t-translation symmetry breaking by minimum energy state is **impossible**

… or is it?

Lagrangian approach

- Try Lagrangian $L = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{\lambda}{4}\dot{\phi}^4$
- Energy function $E = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{3\lambda}{4}\dot{\phi}^4$
- Minimized by $\dot{\phi} = \sqrt{\frac{\kappa}{3\lambda}}$
- This is **not** static. How did we evade theorem?
- Momentum is not well-defined $p = -\kappa\dot{\phi} + \lambda\dot{\phi}^3$
- Hamiltonian a multivalued function of p

E vs. p



swallowtail catastrophe

- Hamiltonian $H(p,\phi)$ not differentiable at minima!

Add potential energy

- So we have a Lagrangian with minimum energy solutions that are periodic in time (if ϕ is angular)
- What happens if we add $V(\phi)$?

$$L = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{\lambda}{4}\dot{\phi}^4 - V(\phi)$$

- Then ϕ wants to minimize KE and PE at the same time – incompatible: can't satisfy

$$\phi = \phi_{\min} \quad \text{and} \quad \dot{\phi} = \sqrt{\frac{\kappa}{3\lambda}}$$

at the same time.

- Does a minimum energy solution even exist?

Equation of motion

- Another problem:
- Equation of motion

$$(3\lambda\dot{\phi}^2 - \kappa)\ddot{\phi} = -V'(\phi)$$

is problematic when $\dot{\phi} = \pm\sqrt{\frac{\kappa}{3\lambda}}$

- Would require *infinite acceleration*.
- Try to solve the equations anyway…

Quadratures

- Solve energy equation

$$E = -\frac{\kappa}{2}\dot{\phi}^2 + \frac{3\lambda}{4}\dot{\phi}^4 + V(\phi)$$

(absorb λ)

- Get

$$t(\phi) = \int^{\phi} \frac{d\phi}{\pm\sqrt{\frac{\kappa}{3} \pm \sqrt{\left(\frac{\kappa}{3}\right)^2 + \frac{4}{3}(E - V(\phi))}}}$$

- Can express in terms of elliptic functions.
- Qualitatively: two types of turning points.
  - focus on inner square root…

Turning points

$$t(\phi) = \int^{\phi} \frac{d\phi}{\pm\sqrt{\frac{\kappa}{3} \pm \sqrt{\left(\frac{\kappa}{3}\right)^2 + \frac{4}{3}(E - V(\phi))}}}$$

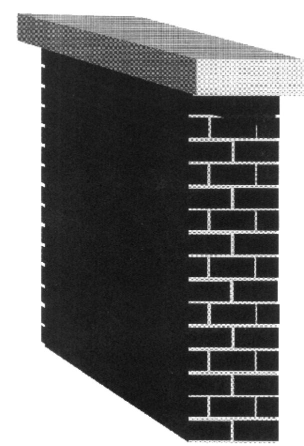
- Argument of inner square root is non-negative iff

$$V(\phi) \;\le\; \frac{\kappa^2}{12} + E = \Delta$$

- At turning point $\dot{\phi} = \pm\sqrt{\frac{\kappa}{3\lambda}}$

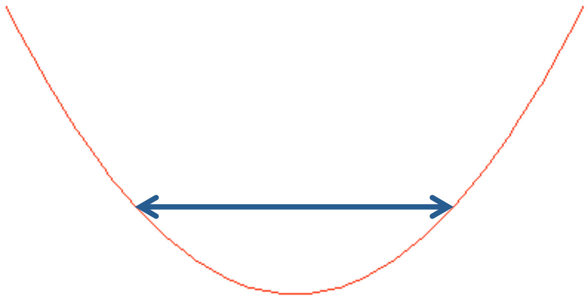
ϕ runs right into potential wall, flips over

from $\dot{\phi} = +\sqrt{\frac{\kappa}{3}}$ to $\dot{\phi} = -\sqrt{\frac{\kappa}{3}}$



- Infinite acceleration! Like a brick wall.

Nearly minimum-energy solutions



- Close to the bottom of potential velocity is $$\dot{\phi} \sim \pm\sqrt{\frac{\kappa}{3}}$$ with equality at turning points.
- ϕ oscillates back and forth with nearly constant speed.
- At bottom, ϕ oscillates with infinite frequency.
- Reconciles apparently contradictory conditions $$\phi = \phi_{\min} \quad \text{and} \quad \dot{\phi} = \pm\sqrt{\frac{\kappa}{3\lambda}}$$
- Quantum effects will lift minimum energy state away from minimum.

Semiclassical quantization

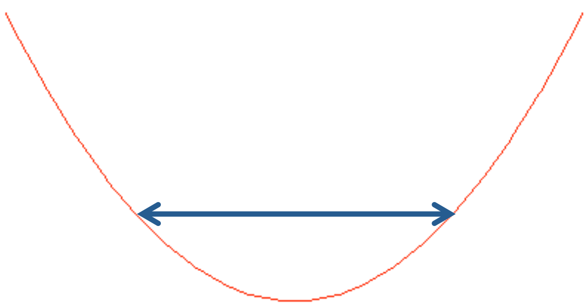
- BS formula

$$S = \oint p\, d\phi = \int (\dot{\phi}^3 - \kappa\dot{\phi})d\phi = 2\pi\hbar(n+\delta)$$

  - use $\delta = 1$ for hard-wall potential
  - approximate $\dot{\phi} = \pm\sqrt{\frac{\kappa}{3\lambda}}$
  - turning points for $V(\phi) \approx \frac{1}{2}\mu(\phi - \phi_0)^2$

- Ground state energy

$$\boxed{E_{\min} = \frac{27\pi^2}{128}\frac{\hbar^2\mu}{\kappa^3}}$$



Generalizations

- Allow κ to depend on ϕ

$$L = -\frac{\kappa(\phi)}{2}\dot{\phi}^2 + \frac{1}{4}\dot{\phi}^4 - V(\phi) = \frac{1}{4}(\dot{\phi}^2 - \kappa(\phi))^2 - \frac{1}{4}\kappa(\phi)^2 - V(\phi)$$

- Special case $\frac{1}{4}\kappa(\phi)^2 + V(\phi) = \text{constant}$

  gives min-energy solution

$$\dot{\phi} = \pm\sqrt{\kappa(\phi)/3}$$

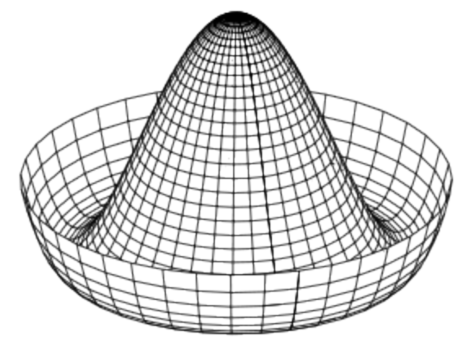
Can choose $\kappa(\phi)$ to achieve any desired time-dependence for ϕ.

Generalizations

- Two degrees of freedom

$$L = \tfrac{1}{4}(\dot{\psi}_1^2 + \dot{\psi}_2^2 - \kappa)^2 - V(\psi_1, \psi_2)$$

- e.g. “Double Mexican Hat”



$$V = -\frac{\mu}{2}(\psi_1^2 + \psi_2^2) + \frac{\lambda}{4}(\psi_1^2 + \psi_2^2)^2$$

$$L = \tfrac{1}{4}(\dot{\rho}^2 + \rho^2\dot{\phi}^2 - \kappa)^2 + \frac{\mu}{2}\rho^2 - \frac{\lambda}{4}\rho^4$$

- Has minimum-energy solutions without turning points:
  - just go around bottom of hat at constant velocity

$$\dot{\phi} = \pm\sqrt{\kappa(\phi)/3\rho_0}$$

Generalizations

- Fields $\phi(t, x)$
- Can consider higher-order gradient terms
  - with ordinary time derivatives, these lead to Lifschitz theories
    - get fixed point where quadratic gradient terms vanish, or where number of real roots jumps…
- Couple these to higher-order time derivatives
- *e.g.*

$$E_{\text{kinetic}}(\phi) = \frac{\kappa_3}{2}\Big(\big(\frac{d\phi}{dx}\big)^2 - \frac{1}{v^2}\dot{\phi}^2\Big)^2$$

  - Get propagating waves with minimum energy.
  - Can engineer charge-density waves etc.

Relativistic fields

- Quartic in derivatives

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)^2 + \lambda(\partial_\mu \phi)^4 + \cdots$$

- Energy density

$$\mathcal{E} = \frac{1}{2}\left((\partial_0 \phi)^2 + (\nabla \phi)^2\right) + 3\lambda\left((\partial_0 \phi)^2 + (\nabla \phi)^2\right)(\partial_\mu \phi)^2$$

- Not bounded below: wrong sign of $(\nabla \phi)^4$
- However this problem is cured at sixth order! nth-order term gives

$$((2n-1)(\partial_0 \phi)^2 + (\nabla \phi)^2)((\partial_0 \phi)^2 - (\nabla \phi)^2)^{n-1}$$

- So highest n should be $4k+2$.

Cosmology

- Such L's have been proposed as a source of inflationary vacuum energy:
  - *k*-inflation (Armendariz-Picon, Damour, Mukhanov 99)
  - ghost condensation (Arkani-Hamed, Cheng, Luty, Mukokyama 04)
  - interesting but different: non-equilibrium, external *t*-dependent background

- Would be interesting to explore whether some of the characteristic features of higher-derivative Lagrangians (such as hard-wall turning points) could lead to observable signatures.

Strings?

- In string theory higher-derivative spacetime effective lagrangians are unavoidable.
- What are the positivity constraints on coefficients of higher-derivative terms?
- If negative coefficients are allowed, some of the phenomenology we are finding could be relevant….