Semiclassical approach to string spectrum in AdS/CFT

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R. Roiban, AT, arXiv:0906.4294, arXiv:1102.1209

M. Beccaria, S. Giombi, R. Roiban, AT, to appear

General aims:

- understand quantum gauge theories at any coupling
- understand string theories in non-trivial backgrounds

Maximally symmetric case of gauge-string duality:

$\mathcal{N} = 4$ SYM — $AdS_5 \times S^5$ superstring

Integrability:

allows "in principle" to solve the problem of spectrum of anomalous dimensions / string energies

Spectrum of states

I. Spectrum of "long" operators = "semiclassical" string states

determined by Asymptotic Bethe Ansatz (2002-2007)

• its final (BES) form found after intricate superposition of information from perturbative gauge theory (spin chain, BA,...) and perturbative string theory (classical and 1-loop phase,...), symmetries (S-matrix), assumption of exact integrability

• consequences checked against available gauge and string data

Key example: cusp anomalous dimension $\mathrm{Tr}(\Phi D^S \Phi)$

$$\Delta = S + 2 + f(\lambda) \ln S + ... \, , \qquad S \gg 1$$

$$f(\lambda \ll 1) = \frac{\lambda}{2\pi^2} \Big[1 - \frac{\lambda}{48} + \frac{11\lambda^2}{2^8 \cdot 45} - \big(\frac{73}{630} + \frac{4\zeta^2(3)}{\pi^6} \big) \frac{\lambda^3}{2^7} + ... \Big]$$

$$f(\lambda \gg 1) = \frac{\sqrt{\lambda}}{\pi} [1 - \frac{3 \ln 2}{\sqrt{\lambda}} - \frac{K}{(\sqrt{\lambda})^2} - ...]$$

exact equation [Basso, Korchemsky, Kotanski]

II. Spectrum of "short" operators = quantum string states

Thermodynamic Bethe Ansatz (2005-...)

[Ambjorn, Janik, Kristjansen; Arutyunov, Frolov;
Gromov, Kazakov, Vieira; Banjok, Janik; ...]

• reconstructed from ABA using solely methods/intuition of 2-d integrable QFT, i.e. string-theory side

• highly non-trivial construction – lack of 2-d Lorentz invariance in the standard "BMN-vacuum-adapted" l.c. gauge

• in few cases ABA "improved" by Luscher corrections is enough: 4 and 5-loop Konishi dimension, 4-loop minimal twist op. dim

• need more data to check predictions at $\lambda \ll 1$ and $\lambda \gg 1$: against perturbative gauge-theory and string-theory data

Key example:

dimension $\Delta = \Delta_0 + \gamma(\lambda)$ of Konishi operator $\mathrm{Tr}(\bar{\Phi}_i \Phi_i)$

$$\Delta(\lambda \ll 1) = 2 + \frac{12\lambda}{(4\pi)^2}\Big[1 - \frac{4\lambda}{(4\pi)^2} + \frac{28\lambda^2}{(4\pi)^4} - [208 - 48\zeta(3) + 120\zeta(5)]\frac{\lambda^3}{(4\pi)^6} + 8[158 + 72\zeta(3) - 54\zeta^2(3) - 90\zeta(5) + 315\zeta(7)]\frac{\lambda^4}{(4\pi)^8} + ...\Big]$$

5-loop result from integrability confirmed [Eden et al 2012]

Suppose can sum up $\lambda \ll 1$ expansion and re-expand at $\lambda \gg 1$

String theory suggests structure of strong-coupling expansion:

[Roiban, AT 09]

$$\begin{aligned}\Delta(\lambda \gg 1) &= 2\sqrt[4]{\lambda} + \mathrm{b}_0 + \frac{b_1}{\sqrt[4]{\lambda}} + \frac{b_2}{(\sqrt[4]{\lambda})^3} + ... \\ &= 2\sqrt[4]{\lambda}\Big[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + ...\Big] + \mathrm{b}_0\end{aligned}$$

Recent progress:

values of $b_1, \ b_2, ...$ matched between TBA and string theory

Extracting direct TBA predictions at strong coupling is hard

start at weak coupling for sl(2) descendant $\mathrm{Tr}(\Phi D^2 \Phi) \ \ (\Delta_0 = 4)$;

plot numerically $\Delta(\lambda)$;

match to expected strong-coupling expansion – extract b_i

[Gromov, Kazakov, Vieira 09, Frolov 10]

$$\mathrm{b}_0 = 0 \ , \qquad b_1 \approx 1.988 \ , \qquad b_2 \approx -3.1$$

string theory computation

[Roiban, AT 09, 11; Gromov,Serban, Shenderovich, Volin 11]

$$\mathrm{b}_0 = 0 \ , \qquad b_1 = 2$$

more recent [Gromov, Valatka 11]

$$b_2 = \frac{1}{2} - 3\zeta(3)$$

using “2-loop” result of [Basso 11] (see below)

Many open questions:

Analytic form of strong-coupling expansion from TBA/Y-system?

Other quantum states?

How to match to string spectrum in near-flat-space expansion?

general structure of the spectrum?

Dimensions of short operators

= energies of quantum string states:

Aims:

• compute leading $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ correction to energy of "lightest" massive string states on first massive level dual to operators in Konishi multiplet in SYM theory

– check against predictions of TBA approach

• understand structure of energy of higher spin states on leading Regge trajectory

Konishi operator multiplet:

long multiplet related to singlet by susy

$[J_2 - J_3, J_1 - J_2, J_2 + J_3]_{(s_L, s_R)} = [0, 0, 0]_{(0,0)}$

$s_{L,R} = \frac{1}{2}(S_1 \pm S_2)$

$SO(6)$ (J_1, J_2, J_3) and $SO(4)$ (S_1, S_2) labels

of $SO(2,4) \times SO(6)$ global symmetry

[Andreanopoli,Ferrara 98; Bianchi,Morales,Samtleben 03]

see table

$\Delta = \Delta_0 + \gamma(\lambda), \qquad \Delta_0 = 2, \frac{5}{2}, 3, ..., 10$

– same anomalous dimension γ

singlet eigen-state of anom. dim. matrix with lowest eigenvalue examples of gauge theory operators in Konishi multiplet:

$[0,0,0]_{(0,0)}$:

$\mathrm{Tr}(\bar{\Phi}_i \Phi_i), \quad i = 1,2,3, \qquad \Delta_0 = 2$

$[2,0,2]_{(0,0)}$:

$\mathrm{Tr}([\Phi_1, \Phi_2]^2)$ in $su(2)$ sector, $\Delta_0 = 4$

$[0,2,0]_{(1,1)}$:

$\mathrm{Tr}(\Phi_1 D^2 \Phi_1)$ in $sl(2)$ sector, $\Delta_0 = 4$

AdS/CFT duality:

Konishi operator dual to

"lightest" among massive $AdS_5 \times S^5$ string states

large $\sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$:

– "small" string at "center" of AdS_5 – in nearly flat space

Comparison between gauge and string theory states:

GT ($\lambda \ll 1$): operators built out of free fields,
canonical dim. Δ_0 determines operators that can mix

ST ($\lambda \gg 1$): near-flat-space string states built out of
free oscillators, level n determines states that can mix

(i) relate states with same global charges
(ii) assume "non-intersection principle"

no level crossing for states with same quantum numbers
as λ changes from strong to weak coupling

Flat space case:

$m^2 = \frac{4n}{\alpha'}, \quad n = \frac{1}{2}(N + \bar{N}) = 0, 1, 2, ..., \quad N = \bar{N}$

$n = 0$: massless IIB supergravity (BPS) level

l.c. vacuum $|0>$: $(8 + 8)^2 = 256$ states

$n = 1$: first massive level (many states, highly degenerate)

$[(a^i_{-1} + S^a_{-1})|0>]^2 = [(8 + 8) \times (8 + 8)]^2$

in $SO(9)$ reps:

$([2, 0, 0, 0] + [0, 0, 1, 0] + [1, 0, 0, 1])^2 = (44 + 84 + 128)^2$

e.g. $44 \times 44 = 1 + 36 + 44 + 450 + 495 + 910$

$84 \times 84 = 1 + 36 + 44 + 84 + 126 + 495 + 594 + 924 + 1980 + 2772$

switching on $AdS_5 \times S^5$ background fields lifts degeneracy

states with "lightest mass" at first excited string level

should correspond to Konishi multiplet

string spectrum in $AdS_5 \times S^5$:

long multiplets $\mathcal{A}^{\Delta}_{[k,p,q](s,s')}$ of $PSU(2,2|4)$

highest weight states: $[k,p,q]_{(s,s')}$

Remarkably, flat-space string spectrum can be re-organized
in multiplets of $SO(2,4) \times SO(6) \subset PSU(2,2|4)$
[Bianchi, Morales, Samtleben 03; Beisert et al 03]
$SO(4) \times SO(5) \subset SO(9)$ rep.
lifted to $SO(4) \times SO(6)$ rep. of $SO(2,4) \times SO(6)$

Konishi long multiplet
$\widehat{T}_1 = (1 + Q + Q \wedge Q + ...)[0,0,0]_{(0,0)}$
determines the KK "floor" of 1-st excited string level
$H_1 = \sum_{J=0}^{\infty} [0,J,0]_{(0,0)} \times \widehat{T}_1$

What one should expect for energy of scalar massive state in AdS_5:

$$(-\nabla^2 + m^2)\Phi + ... = 0$$
$$\Delta(\Delta - 4) = (m\mathrm{R})^2 + O(\alpha') = 4n\frac{\mathrm{R}^2}{\alpha'} + O(\alpha')$$
$$\Delta = 2 + \sqrt{(m\mathrm{R})^2 + 4 + O(\alpha')}$$

$$\Delta(\lambda \gg 1) = \sqrt{4n\sqrt{\lambda}} + ... \,, \qquad \sqrt{\lambda} = \frac{\mathrm{R}^2}{\alpha'}$$

[Gubser, Klebanov, Polyakov 98]

e.g., for first massive level:

$n = 1: \quad \Delta = 2\sqrt[4]{\lambda} + ...$

Subleading corrections?

Approaches to computation of corrections to string energies:

(i) vertex operator approach:
use $AdS_5 \times S^5$ string sigma model perturbation theory to find leading terms in anomalous dimension of corresponding vertex operators [Polyakov 01; AT 03]

(ii) space-time effective action approach:
use near-flat-space expansion and NSR vertex operators to reconstruct $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ corrections to corresponding massive string state equation of motion
[Burrington, Liu 05]

(iii) “light-cone” quantization approach:
start with light-cone gauge $AdS_5 \times S^5$ string action
and compute corrections to energy of
corresponding flat-space oscillator string state
[Metsaev, Thorn, AT 00]

(iv) semiclassical approach:
identify short string state as small-spin limit of
semiclassical string state
– reproduce the structure of strong-coupling corrections
to short operators
[Tirziu, AT 08; Roiban, AT 09, 11]

Spectrum of quantum string states

from target space anomalous dimension operator

Flat space: $k^2 = m^2 = \frac{4(n-1)}{\alpha'}$ (bosonic string)

e.g. leading Regge trajectory $(\partial x \bar{\partial} x)^{S/2} e^{ikx}, \quad n = S/2$

spectrum in (weakly) curved background:

solve marginality (1,1) conditions on vertex operators

e.g. scalar anomalous dimension operator $\widehat{\gamma}(G)$

on $T(x) = \sum c_{n...m} x^n ... x^m$ or on coefficients $c_{n...m}$

differential operator in target space

found from β-function for the corresponding perturbation

$$I = \frac{1}{4\pi\alpha'} \int d^2z [G_{mn}(x) \partial x^m \bar{\partial} x^n + T(x)]$$

$$\beta_T = -2T - \tfrac{\alpha'}{2} \, \widehat{\gamma} \, T + O(T^2)$$

$$\widehat{\gamma} = \Omega^{mn} D_m D_n + ... + \Omega^{m...k} D_m ... D_k + ...$$

$$\Omega^{mn} = G^{mn} + O(\alpha'^3), \qquad \Omega^{...} \sim \alpha'^n R^p_{....}$$

Solve $-\widehat{\gamma}\, T + m^2 T = 0$: diagonalize $\widehat{\gamma}$

similarly for massless (graviton, ...) and massive states
e.g. $\beta^G_{mn} = \alpha' R_{mn} + O(\alpha'^3)$
gives Lichnerowitz operator as anomalous dimension operator

$$(\widehat{\gamma} h)_{mn} = -D^2 h_{mn} + 2R_{mknl}h^{kl} - 2R_{k(m}h^k_{n)} + O(\alpha'^3)$$

Massive string states in curved background:

$$\int d^D x \sqrt{g}\, \left[\Phi_{...}(-D^2 + m^2 + X)\Phi_{...} + ...\right]$$
$$m^2 = \tfrac{4}{\alpha'} n\,, \qquad X = R_{....} + O(\alpha')$$

case of $AdS_5 \times S^5$ background

$$R_{mn} - \tfrac{1}{96}(F_5 F_5)_{mn} = 0, \qquad R = 0\,, \qquad F_5^2 = 0$$

Find leading-order term in X ?

leading α' correction to scalar string state mass =0 (?!)

$$[-D^2 + m^2 + O(\tfrac{1}{\sqrt{\lambda}})]\Phi \ = 0$$
$$\Delta = 2 + \sqrt{4n + 4 + O(\tfrac{1}{\sqrt{\lambda}})}$$
$$\Delta_{(n=1)} = 2 + 2\sqrt[4]{\lambda}\left[1 + \tfrac{1}{2\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\right]$$

prediction for leading term in strong-coupling expansion

of singlet Konishi state dimension?

Too naive:

various subtleties (10d scalar vs singlet state, mixing, etc.)

What about non-singlet (susy descendant) Konishi states?

should have the same dimension

$\mathrm{Tr}[\Phi_1, \Phi_2]^2$ corresponds to $SO(6)$ state $J_1 = J_2 = 2$

tensor wave function $\Phi_{mn;kl}$

or vertex operator $\sim Y_+^{-\Delta}\partial X_x \bar{\partial} X_x \partial X_y \bar{\partial} X_y$

Vertex operator approach

calculate 2d anomalous dimensions from "first principles"–
superstring theory in $AdS_5 \times S^5$:

$$I = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \Big[\partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$Y_+ Y_- - Y_u Y_u^* - Y_v Y_v^* = 1 \, , \qquad X_x X_x^* + X_y X_y^* + X_z X_z^* = 1$$

$$Y_\pm = Y_0 \pm i Y_5, \quad Y_u = Y_1 + i Y_2, \ ..., \quad X_x = X_1 + i X_2, ...$$

construct marginal (1,1) operators in terms of Y_p and X_k
e.g. vertex operator for dilaton mode (NSR framework)

$$\mathrm{V}_J = \ (Y_+)^{-\Delta} \, (X_x)^J \, \Big[- \partial Y_p \bar{\partial} Y^p + \partial X_k \bar{\partial} X_k + \text{fermions} \Big]$$

$$Y_+ \equiv Y_0 + i Y_5 = \tfrac{1}{z} (z^2 + x_m x_m) \sim e^{it}$$
$$X_x \equiv X_1 + i X_2 \sim e^{i\varphi}$$

$$2 = 2 + \frac{1}{2\sqrt{\lambda}} [\Delta(\Delta - 4) - J(J+4)] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

i.e. $\Delta = 4 + J$ (BPS)

Vertex operator for bosonic string state
on leading Regge trajectory in flat space: $\alpha' E^2 = 2(S-2)$

$$\mathrm{V}_S = e^{-iEt}\big(\partial x \bar\partial x\big)^{S/2} , \quad x = x_1 + i x_2$$

candidate operators for states on leading Regge trajectory:

$$\mathrm{V}_J = (Y_+)^{-\Delta}\big(\partial X_x \bar\partial X_x\big)^{J/2} , \qquad X_x \equiv X_1 + i X_2$$

$$\mathrm{V}_S(\xi) = (Y_+)^{-\Delta}\big(\partial Y_u \bar\partial Y_u\big)^{S/2} , \qquad Y_u \equiv Y_1 + i Y_2$$

+ fermionic terms

+ $\alpha' \sim \frac{1}{\sqrt{\lambda}}$ terms from diagonalization of anom. dim. op.

mix with operators with same charges and dimension

in general $\big(\partial X_x \bar\partial X_x\big)^{J/2}$ mixes with singlets

$$(X_x)^{2p+2q}(\partial X_x)^{J/2-2p}(\bar\partial X_x)^{J/2-2q}(\partial X_m \partial X_m)^p(\bar\partial X_k \partial X_k)^q$$

true vertex operators

= eigenstates of 2d anomalous dimension matrix

– particular linear combinations

operators for states on leading Regge trajectory

$$O_{\ell,s} = f_{k_1...k_\ell m_1...m_{2s}} X_{k_1}...X_{k_\ell} \partial X_{m_1} \bar{\partial} X_{m_2} ... \partial X_{m_{2s-1}} \bar{\partial} X_{m_{2s}}$$

their renormalization studied before [Wegner 90]

simplest case: $f_{k_1...k_\ell} X_{k_1}...X_{k_\ell}$ with traceless $f_{k_1...k_\ell}$

same anom. dim. $\widehat{\gamma}$ as its highest-weight rep $V_J = (X_x)^J$

$$\widehat{\gamma} = 2 - \frac{1}{2\sqrt{\lambda}} J(J+4) + ...$$

scalar spherical harmonic that solves Laplace eq. on S^5

Example of higher-level scalar operator:

$$Y_+^{-\Delta}[(\partial X_k \bar{\partial} X_k)^r + ...] \, , \qquad r = 1, 2, ...$$

[Kravtsov, Lerner, Yudson 89; Castilla, Chakravarty 96]

$$0 = -2(r-1) + \frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta-4) + 2r(r-1)\Big] + \frac{1}{(\sqrt{\lambda})^2}\Big[\tfrac{2}{3}r(r-1)(r-\tfrac{7}{2}) + 4r\Big] + ...$$

$r = 1$: ground level

fermionic contributions should make $r = 1$ exact zero of $\widehat{\gamma}$

$r = 2$: excited level – candidate for singlet Konishi state $\Delta_0 = 2$

$$\Delta(\Delta-4) = 4\sqrt{\lambda} - 4 + O(\tfrac{1}{\sqrt{\lambda}}) \, ,$$

$$\Delta - \Delta_0 = 2\sqrt[4]{\lambda}\Big[1 + 0 \times \tfrac{1}{\sqrt{\lambda}} + O(\tfrac{1}{(\sqrt{\lambda})^2})\Big]$$

fermionic contribution may change this

Bosonic operators with two spins $J_1 = J,\ J_2 \equiv K$ in S^5:

$$V_{K,J} = Y_+^{-\Delta} \sum_{u,v=0}^{K/2} c_{uv} M_{uv}$$
$$M_{uv} \equiv X_y^{J-u-v} X_x^{u+v} (\partial X_y)^u (\partial X_x)^{K/2-u} (\bar\partial X_y)^v (\bar\partial X_x)^{K/2-v}$$

highest and lowest eigen-values of 1-loop anom. dim. matrix

$$\widehat{\gamma}_{\min} = 2 - K + \frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta-4) - J(J+4) - \frac{1}{2}K(K+10) - 2JK\Big] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

$$\widehat{\gamma}_{\max} = 2 - K + \frac{1}{2\sqrt{\lambda}}\Big[\Delta(\Delta-4) - J(J+4) - \frac{1}{2}K(K+6)\Big] + O(\tfrac{1}{(\sqrt{\lambda})^2})$$

fermions may alter terms linear in K

How to take fermionic contributions into account?

General structure of dimension Δ = energy E

vertex operators on $R^2 \ \leftrightarrow\ $ string states on $R \times S^1$
aim: understand structure of dependence of string energy on
string tension and quantum numbers (spins)
guided by form of string vertex op. marginality condition
structure of dependence of energy E of quantum string state on
quantum charges Q_i in the large string tension expansion $\sqrt{\lambda} \gg 1$
from α' expansion of 2d anomalous dimensions
of $AdS_5 \times S^5$ vertex ops $\to$ solution of marginality condition
should give $E = E(Q, \sqrt{\lambda})$ in the form [Roiban, AT 09, 11]

$$E^2 = 2\sqrt{\lambda} \sum_i a_i Q_i + \sum_{i,j} b_{ij} Q_i Q_j + \sum_i c_i Q_i$$
$$+ \frac{1}{\sqrt{\lambda}} \Big(\sum_{i,j,k} d_{ij} Q_i Q_j Q_k + \sum_{i,j} e_{ij} Q_i Q_j Q_k + \sum_i f_i Q_i \Big) + ...$$

Q_i – fixed in the limit $\sqrt{\lambda} \gg 1$

string state with S^5 orbital momentum J and quantum number N

N= effective string level, e.g., spin component $S = N$

E^2 from the 2d marginality condition

(ignore shifts of N and E by integers: depend on choice of vac.)

$$0 = N + \frac{1}{2\sqrt{\lambda}}\Big(- E^2 + J^2 + n_{02}N^2 + n_{11}N\Big)$$
$$+ \frac{1}{2(\sqrt{\lambda})^2}\Big(n_{01}NJ^2 + n_{03}N^3 + n_{12}N^2 + n_{21}N\Big) + O(\tfrac{1}{(\sqrt{\lambda})^3})$$

then E^2 takes form:

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N$$
$$+ \frac{1}{\sqrt{\lambda}}\Big(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N\Big)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}\Big(\widetilde{n}_{11}J^2N + n_{04}N^4 + ...\Big) + O(\tfrac{1}{(\sqrt{\lambda})^3})$$

expanding in large $\sqrt{\lambda}$ for fixed N, J

$$E = \sqrt{2\sqrt{\lambda}N}\Big[1 + \frac{A_1}{\sqrt{\lambda}} + \frac{A_2}{(\sqrt{\lambda})^2} + O(\tfrac{1}{(\sqrt{\lambda})^3})\Big] \ ,$$

$$A_1 = \frac{1}{4N}J^2 + \frac{1}{4}(n_{02}N + n_{11}) \ ,$$

$$A_2 = -\frac{1}{2}A_1^2 + \frac{1}{4}(n_{01}J^2 + n_{03}N^2 + n_{12}N + n_{21})$$

Gives for particular quantum string state values of N and J
strong-coupling expansion of energy/dimension
of the corresponding gauge-theory operator

Plan: determine the coefficients n_{km}
using semiclassical “short string” expansion approach

Approach based on interpolation of semiclassical expansion

start with a solitonic string carrying same charges

as vertex operator representing particular quantum string state

(i) first perform semiclassical expansion $\sqrt{\lambda} \gg 1$

for fixed classical parameters

$\mathcal{Q}_i = \frac{1}{\sqrt{\lambda}} Q_i$, i.e. $(\mathcal{N}, \mathcal{J}) = \frac{1}{\sqrt{\lambda}}(N, J)$

(ii) then expand E in small values of $\mathcal{Q}_i$

(iii) re-interpret the resulting expression in terms of N, J

limit $\mathcal{Q}_i = \frac{Q_i}{\sqrt{\lambda}} \to 0$ should correspond to $\frac{1}{\sqrt{\lambda}} \to 0$ for fixed

values of quantum charges Q_i

same coefficients n_{km} should be found

in direct vertex operator approach

E in terms of $\mathcal{N}, \mathcal{J}$:

$$(\frac{E}{\sqrt{\lambda}})^2 = (2\mathcal{N} + \mathcal{J}^2 + n_{01}\mathcal{J}^2\mathcal{N} + n_{02}\mathcal{N}^2 + n_{03}\mathcal{N}^3 + n_{04}\mathcal{N}^4 + ...)$$
$$+ \frac{1}{\sqrt{\lambda}}(n_{11}\mathcal{N} + \widetilde{n}_{11}\mathcal{J}^2\mathcal{N} + n_{12}\mathcal{N}^2 + ...)$$
$$+ \frac{1}{(\sqrt{\lambda})^2}(n_{21}\mathcal{N} + ...) + O(\tfrac{1}{(\sqrt{\lambda})^3}) \ ,$$

interpret n_{km} as semiclassical k-loop contribution to $\mathcal{N}^m$ term

• quantum string loop (i.e. $\alpha' \sim \frac{1}{\sqrt{\lambda}} \ll 1$) expansion
in 2d anom. dim. is different from semiclassical loop expansion:
n_{km} in general appear at different orders in two expansions
(but n_{11} and n_{21} are 1-loop and 2-loop in both expansions)

• each loop term in exact expansion polynomial in charges
but in semiclassical expansion each term may contain
infinite series in small $\mathcal{J}, \mathcal{N}$ expansion

• to relate two expansions need to reorganize them

Semiclassical expansion of E^2 organized as expansion in small $\mathcal{N}$ formally looks like an expansion in powers of N:

$$E^2 = J^2 + h_1(\lambda, J)N + h_2(\lambda, J)N^2 + h_3(\lambda, J)N^3 + ...$$

where for fixed J and large λ

$$h_1 = 2\sqrt{\lambda} + n_{11} + \frac{n_{21}}{\sqrt{\lambda}} + \frac{n_{31}}{(\sqrt{\lambda})^2} + ... + J^2(\frac{n_{01}}{\sqrt{\lambda}} + \frac{\widetilde{n}_{11}}{(\sqrt{\lambda})^2} + ...) + ...$$

$$h_2 = n_{02} + \frac{n_{12}}{\sqrt{\lambda}} + ...$$

$$h_3 = \frac{n_{03}}{\sqrt{\lambda}} + ... , \qquad h_4 = \frac{n_{03}}{(\sqrt{\lambda})^2} + ...$$

[exact computation of h_1 for folded string state: Basso 11]

Will consider examples of “small” semiclassical string states corresponding to quantum string states with angular momentum J and few oscillator modes excited

For $N=2,\ J=2$ they represent particular states
in the Konishi multiplet on gauge theory side
– should have same 4d anomalous dimension
= same E (modulo constant shifts)

$$E = 2\sqrt[4]{\lambda}\Big[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + O(\tfrac{1}{(\sqrt{\lambda})^3})\Big] \ ,$$

$$b_1 = 2(A_1)_{N=J=2} = 1 + n_{02} + \frac{1}{2}n_{11}$$

$$b_2 = 2(A_2)_{N=J=2} = -\frac{1}{4}b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \frac{1}{2}n_{21}$$

find the coefficients n_{km} using semiclassical approach
check this universality (implied by susy)
identify general patterns in the structure of n_{km}

Semiclassical expansion:

$\sqrt{\lambda} \gg 1, \ \mathcal{J} = \frac{J}{\sqrt{\lambda}}$=fixed (e.g. for $J = 0$):

$$E(\frac{N}{\sqrt{\lambda}}, \sqrt{\lambda}) = \sqrt{\lambda}\mathcal{E}_0(\mathcal{N}) + \mathcal{E}_1(\mathcal{N}) + \frac{1}{\sqrt{\lambda}}\mathcal{E}_2(\mathcal{N}) + ...$$

$$\mathcal{E}_n = \sqrt{\mathcal{N}} \left(a_{n0} + a_{n1}\mathcal{N} + a_{n2}\mathcal{N}^2 + ...\right), \qquad \mathcal{N} \ll 1$$

if know all terms in this expansion – express $\mathcal{N}$ in terms of N

fix it to finite value and re-expand in $\sqrt{\lambda}$

$$E = \sqrt{2\sqrt{\lambda}N}\Big[1 + \frac{a_{01}N + a_{10}}{\sqrt{\lambda}} + \frac{a_{02}N^2 + a_{11}N + a_{20}}{(\sqrt{\lambda})^2} + ...\Big]$$

a_{km} – k-loop string corrections – related to n_{km}

$a_{01} = \frac{1}{4}n_{02}, \ \ a_{10} = \frac{1}{4}n_{11}, ...$ etc

to trust the coeff of $\frac{1}{(\sqrt{\lambda})^n}$ need coeff of up to n-loop terms

e.g. classical a_{01} and 1-loop a_{10} sufficient to fix $\frac{1}{\sqrt{\lambda}}$ term

[cf. "fast string" expansion $\mathcal{N} \gg 1$ for fixed N

– positive powers of $\sqrt{\lambda}$ – need to resum]

"Short" string: probing flat-space limit of $AdS_5 \times S^5$

(i) start with classical string solutions in flat space

representing states at 1-st excited string level

(ii) embed into $AdS_5 \times S^5$ and find 1-loop correction to E

(iii) interpolate result to finite values N, i.e. $\mathcal{N} = \frac{N}{\sqrt{\lambda}} \to 0$

Two basic classes of examples (N= spin, J= orbital momentum):

- circular string with 2 spins in two orthogonal planes
- folded spinning string

Rigid circular string rotating in two planes of R^4

$$t = \kappa\tau \ , \ \ \mathrm{x}_x \equiv x_1 + i x_2 = \ a\, e^{i(\tau+\sigma)}, \ \ \ \mathrm{x}_y \equiv x_3 + i x_4 = \ a\, e^{i(\tau-\sigma)}$$

$$E_{\rm flat} = \frac{\kappa}{\alpha'} = \sqrt{\frac{4}{\alpha'} J} \ , \qquad\qquad J_1 = J_2 = \frac{a^2}{\alpha'} \ .$$

Identifying oscillator modes that are excited
associate it with the quantum string state created by

$$e^{-iEt}\left[(\partial \mathrm{x}_x \bar{\partial} \mathrm{x}_x)^{\frac{J_1}{2}} (\partial \mathrm{x}_y \bar{\partial} \mathrm{x}_y)^{\frac{J_2}{2}} + ...\right]$$

$$\alpha' E^2 = 2N = 2(J_1 + J_2 - 2)$$

$J_1 = J_2$ in bosonic string:

$$E_{\rm flat} = \sqrt{\frac{4}{\alpha'}(J-1)}\ .$$

Folded string rotating in a plane

$$t = \kappa\tau\ , \qquad \mathrm{x}_1 \equiv x_1 + i x_2 = \ a \sin\sigma\, e^{i\tau}$$

$$E_{\rm flat} = \sqrt{\frac{2}{\alpha'} S}\ , \qquad\qquad S = \frac{a^2}{2\alpha'}\ ,$$

semiclassical counterpart of quantum string state on
leading Regge trajectory

$$e^{-iEt}\left[(\partial \mathrm{x}_x \bar{\partial} \mathrm{x}_x)^{\frac{S}{2}} + ...\right], \qquad\qquad \alpha' E^2 = 2N = 2(S-2)$$

3 obvious choices how to embed these solutions into $AdS_5 \times S^5$:
(i) the two 2-planes may belong to S^5: $J_1 = J_2$ “small string”
(ii) the two 2-planes may belong to AdS_5: $S_1 = S_2$ “small string”
(iii) one plane in AdS_5 and the other in S^5: $S = J$ “small string”

similar choices for folded string
1. study each case in $AdS_5 \times S^5$; interpolate to fixed values of N
2. match to states in Konishi table
3. verify universality of strong-coupling expansion of
4d anom. dim of dual gauge theory operators
in same supermultiplet

Results:

for several solutions for states on leading Regge trajectory

(maximal spin for given energy in flat limit)

$$E^2 = 2\sqrt{\lambda}N + J^2 + n_{02}N^2 + n_{11}N + \frac{1}{\sqrt{\lambda}}(n_{01}J^2N + n_{03}N^3 + n_{12}N^2 + n_{21}N) + \frac{1}{(\sqrt{\lambda})^2}(\widetilde{n}_{11}J^2N + n_{04}N^4 + ...) + ...$$

- $n_{01} = 1$

follows from near-BMN expansion of classical energy ($J \ll \sqrt{\lambda}$)

$E^2 = J^2 + 2N\sqrt{\lambda + J^2} + ... = J^2 + N(2\sqrt{\lambda} + \frac{J^2}{\sqrt{\lambda}} + ...)$

- tree-level $n_{02}, n_{03}, ...$ are rational
- leading 1-loop n_{11} rational [Roiban, AT 09; Gromov et al 11]
- $\widetilde{n}_{11} = -n_{11}$

$h_1 = 2\sqrt{\lambda}\sqrt{1 + \mathcal{J}^2} + \frac{n_{11}}{1+\mathcal{J}^2} + ...$ [Basso; BGRT]

- $n_{12} = n'_{12} - 3\zeta(3)$, $\quad n'_{12}$ is rational

[Tirziu, AT 08; Roiban, AT 09; Gromov-Valatka 11]

$\zeta(3)$ term is universal for states on leading Regge trajectory

n_{1k} contain $\zeta(5)$, ... etc; likely to be universal too

– universality of "short-distance" ($n \gg 1$) behaviour

- leading 2-loop coefficient n_{21} is rational and universal:

$n_{21} = -\frac{1}{4}$

found for folded string state [Basso 11]

evidence from universality [BGRT]

of the Konishi state energy ($J = N = 2$)

$$E_{_{N=J=2}} = 2\sqrt[4]{\lambda}\Big[1 + \frac{b_1}{2\sqrt{\lambda}} + \frac{b_2}{2(\sqrt{\lambda})^2} + O(\tfrac{1}{(\sqrt{\lambda})^3})\Big]$$

$$b_1 = 1 + n_{02} + \frac{1}{2}n_{11} = 2$$

$$b_2 = -\frac{1}{4}b_1^2 + 2n_{01} + 2n_{03} + n_{12} + \frac{1}{2}n_{21} = \frac{1}{2} - 3\zeta(3)$$

matching TBA predictions interpolated to $\lambda \gg 1$

$$2n_{02} + n_{11} = 2 \, , \qquad 4n_{03} + 2n'_{12} + n_{21} = -1$$

Need to confirm universality of n_{21} by direct computation
generalize exact result for h_1 [Basso]
for sl(2) sector state to other string states

Summary of results for n_{km}

I. Folded strings with one spin N and orbital momentum J

• folded string in AdS_5 with (S, J), $N = S$

[Tirziu,AT08; Gromov,Serban, Shenderovich,Volin 11;

Basso 11; Gromov,Valatka 11]

$$n_{01} = 1\ , \quad n_{02} = \frac{3}{2}\ , \quad n_{03} = -\frac{3}{8}\ ,$$

$$n_{11} = -1\ , \quad \widetilde{n}_{11} = 1\ , \quad n'_{12} = \frac{3}{8}\ , \quad n_{21} = -\frac{1}{4}$$

• folded string in S^5 with $(J_1, J_3 = J)$, $N = J_1$

[Beccaria, Marconi 11; BGRT 12]

$$n_{01} = 1\ , \quad n_{02} = \frac{1}{2}\ , \quad n_{03} = \frac{1}{8}\ ,$$

$$n_{11} = 1\ , \quad \widetilde{n}_{11} = -1\ , \quad n'_{12} = -\frac{5}{8}\ , \quad n_{21} = -\frac{1}{4}(?)$$

II. Circular strings with two spins and orbital momentum J

[Roiban, AT 09, 11; BGRT 12]

• "small" circular string with 2 spins in S^5:

$(J_1 = J_2, J_3 = J)$, $N = J_1 + J_2 = 2J_1$

$$n_{01} = 1 , \quad n_{02} = 0 , \quad n_{03} = 0 ,$$
$$n_{11} = 2 , \quad \widetilde{n}_{11} = -2 , \quad n'_{12} = -\frac{3}{8} , \quad n_{21} = -\frac{1}{4} (?)$$

• "small" circular string with 2 spins in AdS_5:

$(S_1 = S_2, J)$, $N = S_1 + S_2 = 2S_1$

$$n_{01} = 1 , \quad n_{02} = 2 , \quad n_{03} = -1 ,$$
$$n_{11} = -2 , \quad \widetilde{n}_{11} = 2 , \quad n'_{12} = \frac{13}{8} , \quad n_{21} = -\frac{1}{4} (?)$$

• "small" circular string with S in AdS_5 and J_1 in S^5:

$(S = J_1, J_3 = J)$, $N = S + J_1 = 2S$

$$n_{01} = 1 , \quad n_{02} = 1 , \quad n_{03} = -\frac{1}{2} ,$$
$$n_{11} = 0 , \quad \widetilde{n}_{11} = 0 , \quad n'_{12} = \frac{5}{8} \, (?) , \quad n_{21} = -\frac{1}{4} \, (?)$$

III. Circular pulsating strings:

[Beccaria,Dunne,Macorini,Tirziu,AT 10]

- pulsating string in AdS_3: N= oscillation number

$$n_{01} = 1\ , \quad n_{02} = \frac{5}{2}\ , \quad n_{03} = -\frac{13}{8}\ ,$$
$$n_{11} = -3\ , \quad \widetilde{n}_{11} = 3(?)\ , \quad n'_{12} = \frac{23}{8}(?)\ , \quad n_{21} = -\frac{1}{4}(?)$$

- pulsating string in $R \times S^2$

$$n_{01} = 1\ , \quad n_{02} = -\frac{1}{2}\ , \quad n_{03} = -\frac{1}{8}\ ,$$
$$n_{11} = 3\ , \quad \widetilde{n}_{11} = -3(?)\ , \quad n'_{12} = -\frac{1}{8}(?)\ , \quad n_{21} = -\frac{1}{4}(?)$$

for $N = J = 2$ pulsating strings should also represent states on the first excited string level, i.e. from Konishi multiplet predict the same b_1, b_2 with above n_{nk}

Examples of states on subleading Regge trajectories

- m-folded spinning string
- spinning string with n spikes [Kruczenski 04]

1-loop corrections:

[Gromov, Valatka 11; Beccaria, Ratti, AT 11]

$$E^2_{\rm folded} = 2m\sqrt{\lambda}S\Big[1 + \frac{1}{\sqrt{\lambda}}b_{\rm folded} + ...\Big] + J^2 + \frac{3}{2}S^2 + ...$$

$$E^2_{\rm spiky} = 4(1-\frac{1}{n})\sqrt{\lambda}S\Big[1 + \frac{1}{\sqrt{\lambda}}b_{\rm spiky} + ...\Big] + J^2 + 4(1-\frac{5}{2n}+\frac{5}{2n^2})S^2 + ...$$

$$b_{\rm folded} = 2F(m)\ , \qquad b_{\rm spiky} = -\frac{1}{4} + F(n-1)$$

$$F(r) \equiv -\frac{3}{4r} + 2H_r - H_{2r}\ , \qquad H_r \equiv \sum_{k=1}^{r}\frac{1}{k}$$

coincide in 1-fold string case: $E_{\rm folded}(m=1) = E_{\rm spiky}(n=1)$

$$a_{\rm folded}(m=1) = a_{\rm spiky}(n=2) = F(1) = -\frac{1}{4}$$

Some details

Circular rotating string in S^5 with $J_1 = J_2 \equiv J'$:

cf. Konishi descendant with $J_1 = J_2 = 2$: $\quad \mathrm{Tr}([\Phi_1, \Phi_2]^2)$

represent it by "short" classical string with same charges

flat space $R_t \times R^4$: circular string solution

$$x_1 + ix_2 = \ a\, e^{i(\tau+\sigma)}\ , \qquad x_3 + ix_4 = \ a\, e^{i(\tau-\sigma)}$$

$$E = \sqrt{\frac{4}{\alpha'} J'}, \qquad J' = \frac{a^2}{\alpha'}$$

can be directly embedded into

$R_t \times S^5$ in $AdS_5 \times S^5$ [Frolov, AT 03] :

string on small sphere inside S^5: $X_1^2 + ... + X_6^2 = 1$

$$X_1 + iX_2 = a\, e^{i(\tau+\sigma)}, \quad X_3 + iX_4 = a\, e^{i(\tau-\sigma)}\ ,$$
$$X_5 + iX_6 = \sqrt{1 - 2a^2}, \qquad t = \kappa\tau$$
$$\mathcal{J}' = \mathcal{J}_1 = \mathcal{J}_2 = a^2, \qquad \mathcal{E}^2 = \kappa^2 = 4\mathcal{J}'$$

E_0 is just as in flat space

$$E_0 = \sqrt{\lambda}\mathcal{E} = \sqrt{4\sqrt{\lambda}J'}\,, \qquad J' = \sqrt{\lambda}\mathcal{J}'$$

1-loop quantum string correction to the energy:

sum of bosonic and fermionic fluctuation frequencies ($n = 0, 1, 2, ...$)

Bosons (2 massless + massive):

$$AdS_5: \quad 4\times \qquad \omega_n^2 = n^2 + 4\mathcal{J}'$$
$$S^5: \qquad 2\times \qquad \omega_{n\pm}^2 = n^2 + 4(1-\mathcal{J}') \pm 2\sqrt{4(1-\mathcal{J}')n^2 + 4\mathcal{J}'^2}$$

Fermions:

$$4\times \qquad \omega_{n\pm}^{2\,f} = n^2 + 1 + \mathcal{J}' \pm \sqrt{4(1-\mathcal{J}')n^2 + 4\mathcal{J}'}$$

$$E_1 = \frac{1}{2\kappa}\sum_{n=-\infty}^{\infty}\Big[4\omega_n + 2(\omega_{n+} + \omega_{n-}) - 4(\omega_{n+}^f + \omega_{n-}^f)\Big]$$

expand in small $\mathcal{J}'$ and do sums (UV divergences cancel)

$$E_1 = \frac{1}{\sqrt{\mathcal{J}}}\Big(\mathcal{J} - [3+\zeta(3)]\mathcal{J}'^2 - \tfrac{1}{4}\big[5+6\zeta(3)+30\zeta(5)\big]\mathcal{J}'^3 + \ldots\Big)$$

$$E = E_0 + E_1 = 2\sqrt{\sqrt{\lambda}J'}\Big(1 + \frac{1}{2\sqrt{\lambda}} - \frac{3}{4}[1+2\zeta(3)]\frac{J'}{(\sqrt{\lambda})^2} + ...\Big)$$

include orbital momentum J dependence:

value of b_1 is shifted by "classical" contribution $\sim J^2$ as

$$b_1(J) = b_1(0) + \frac{1}{4}J^2$$

universal value $b_1(0) = 1$ [Roiban, AT 2009] implies $b_1(2) = 2$

i.e. same as value for the Konishi multiplet state in the $sl(2)$ sector

(having $S = J = 2$) found from TBA [Gromov et al 2009]

J^2 term has simple classical origin

$$E_0^2 = 2\sqrt{\lambda}N + aN^2 + J^2 + ... \,, \qquad N, J \ll \sqrt{\lambda}$$

$$E_0 = \sqrt{2\sqrt{\lambda}N}\,\Big[1 + \frac{1}{4\sqrt{\lambda}}\Big(aN + \frac{J^2}{N}\Big) + ...\Big]$$

For each solution (with values of spins representing a state on the first excited string level) there will be a state in the corresponding representation in the Konishi multiplet table

– universality of the predicted value $b_1 = 2$

- Small circular string with $J_1 = J_2$ and $J_3 \neq 0$

$$X_1 = \ a\, e^{i(w\tau+\sigma)} \ , \qquad X_2 = \ a\, e^{i(w\tau-\sigma)} \ , \qquad X_3 = \sqrt{1-2a^2}\, e^{i\nu\tau} \ ,$$
$$\mathcal{E}_0^2 = \kappa^2 = 4a^2 + \nu^2 = \nu^2 + \frac{4\mathcal{J}'}{\sqrt{1+\nu^2}} \ , \qquad w^2 = 1+\nu^2 \ ,$$
$$\mathcal{J}' \equiv \mathcal{J}_1 = \mathcal{J}_2 = a^2 w \ , \qquad \mathcal{J} \equiv \mathcal{J}_3 = \sqrt{1-2a^2}\,\nu$$

for $\mathcal{J}' = \frac{J'}{\sqrt{\lambda}} \ll 1, \ \mathcal{J} = \frac{J}{\sqrt{\lambda}} \ll 1$

$$E_0 = 2\sqrt{\sqrt{\lambda}J'} \left[1 + \frac{1}{\sqrt{\lambda}}\frac{J^2}{8J'} - \frac{1}{(\sqrt{\lambda})^2}\frac{J^4}{128J'^2} + \dots\right]$$

leading term in 1-loop correction expanded in $\mathcal{J} = \frac{J}{\sqrt{\lambda}} \ll 1$ does not depend on J – has same value as for $J = 0$

To get a state on the first excited string level

we should choose $J' = 1$, i.e. $J_1 = J_2 = 1$

for minimal non-trivial value of $J = J_3 = 2$

there is unique corresponding state in Konishi multiplet table:

$[0, 1, 2]_{(0,0)}$ at level $\Delta_0 = 6$ and thus

$$b_1 = 2\Big(\frac{J^2}{8J'} + \frac{1}{2}\Big)_{J=2, J'=1} = 2$$

- Small circular spinning string with $S_1 = S_2$ and $J \neq 0$

rigid circular string with two equal spins in AdS^5

and orbital momentum $J = J_1$ in S^5

$$Y_0 + iY_5 = \sqrt{1+2r^2}\, e^{i\kappa t}\ , \ \ Y_1 + iY_2 = r\ e^{i(w\tau+\sigma)}\ , \ \ Y_3 + iY_4 = r\ e^{i(w\tau-\sigma)}\ ,$$
$$X_1 + iX_2 = e^{i\nu\tau}\ , \qquad w^2 = \kappa^2 + 1\ , \qquad \kappa^2(1+2r^2) = 2r^2(1+w^2) + \nu^2\ ,$$
$$\mathcal{E}_0 = (1+2r^2)\kappa = \kappa + \frac{2\kappa\mathcal{S}}{\sqrt{1+\kappa^2}}\ , \qquad \mathcal{S} = \mathcal{S}_1 = \mathcal{S}_2 = r^2 w\ , \qquad \mathcal{J} = \nu\ .$$

“short” string expansion of the classical energy ($E_0 = \sqrt{\lambda}\mathcal{E}_0$):

$$\mathcal{E}_0 = 2\sqrt{\mathcal{S}}\left(1 + \mathcal{S} + \frac{\mathcal{J}^2}{8\mathcal{S}} + ...\right)$$

including 1-loop correction:

$$E_0 + E_1 = 2\sqrt{\sqrt{\lambda}S}\left[1 + \frac{1}{\sqrt{\lambda}}\Big(S + \frac{J^2}{8S} - \frac{1}{2}\Big) + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2})\right]$$

the state on the first excited level associated to string
with two equal spins in AdS_5
has two excited oscillators, i.e. should have $S = S_1 = S_2 = 1$
for $J = 2$ the dual state should be in representation $[0, 2, 0]_{(1,0)}$
there is just one state in Konishi table with $\Delta_0 = 6$

$$b_1 = 2\Big(S + \frac{J^2}{8S} - \frac{1}{2}\Big)_{S=1,J=2} = 2$$

• Small circular spinning string with $S = J_1$ and $J_2 \neq 0$

rigid circular solution with one spin in AdS_5 and one spin in S^5
and orbital momentum in S^5

$$Y_0 + iY_5 = \sqrt{1+r^2}\, e^{i\kappa t} , \qquad Y_1 + iY_2 = r\, e^{i(w\tau+\sigma)} , \qquad w^2 = \kappa^2 + 1 ,$$
$$X_1 + iX_2 = a\, e^{i(w'\tau-\sigma)} , \qquad X_3 + iX_4 = \sqrt{1-a^2} e^{i\nu\tau} , \qquad w'^2 = \nu^2 + 1 ,$$
$$\mathcal{E}_0 = 2\sqrt{\mathcal{S}} \left(1 + \frac{1}{2}\mathcal{S} + \frac{\mathcal{J}_2^2}{8\mathcal{S}} + ...\right)$$

The leading 1-loop correction to the energy vanishes
(cancellation of AdS and sphere contributions)

$$E_0 + E_1 = 2\sqrt{\sqrt{\lambda} S}\, \Big[1 + \frac{1}{\sqrt{\lambda}} \Big(\frac{1}{2} S + \frac{J_2^2}{8S}\Big) + \mathcal{O}(\frac{1}{(\sqrt{\lambda})^2})\Big]$$

state on the first excited level: $S = J_1 = 1$
for $J_2 = 2$ get state $[1,1,1]_{(\frac{1}{2},\frac{1}{2})}$ at $\Delta_0 = 6$ level

$$b_1 = 2\Big(\frac{1}{2} S + \frac{J_2^2}{8S}\Big)_{S=1, J_2=2} = 2$$

Conclusions

• beginning of understanding quantum string spectrum
in $AdS_5 \times S^5$ = spectrum of “short” SYM operators

• agreement with numerical prediction of TBA:
non-trivial check of existing TBA equations at strong coupling

• observation of universality of some coefficients
in strong coupling expansion of dimension
for states on leading Regge trajectory

• need of systematic study of quantum string theory in $AdS_5 \times S^5$
in particular, in near flat space expansion

Δ_0	
2	$[0,0,0]_{(0,0)}$
$\frac{5}{2}$	$[0,0,1]_{(0,\frac{1}{2})} + [1,0,0]_{(\frac{1}{2},0)}$
3	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{7}{2}$	$[0,0,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [0,1,1]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [1,0,0]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [1,0,2]_{(\frac{1}{2},0)}$ $+[1,1,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [2,0,1]_{(0,\frac{1}{2})}$
4	$[0,0,0]_{(0,0)+(0,2)+(1,1)+(2,0)} + [0,0,2]_{(\frac{1}{2},\frac{1}{2})+(\frac{3}{2},\frac{1}{2})} + [0,1,0]_{2(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})} + [2,0,2]_{(0,0)}$ $+[0,1,2]_{(1,0)} + [0,2,0]_{2(0,0)+(1,1)} + [1,0,1]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [1,1,1]_{2(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(\frac{1}{2}}$
6	$[0,0,0]_{3(0,0)+3(1,1)+(2,2)} + [0,0,2]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})} + [0,1,0]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})+}$ $+[0,1,2]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [0,2,0]_{3(0,0)+(0,1)+(0,2)+(1,0)+3(1,1)+(2,0)} + [0,2,2]_{(\frac{1}{2},\frac{1}{2})}$ $+[0,3,0]_{2(\frac{1}{2},\frac{1}{2})} + [0,4,0]_{(0,0)} + [1,0,1]_{(0,0)+3(0,1)+3(1,0)+4(1,1)+(1,2)+(2,1)} + [1,0,3]_{(\frac{1}{2},\frac{1}{2})} + [0,$ $+[1,1,1]_{4(\frac{1}{2},\frac{1}{2})+2(\frac{1}{2},\frac{3}{2})+2(\frac{3}{2},\frac{1}{2})} + [1,2,1]_{(0,0)+(0,1)+(1,0)} + [2,0,0]_{3(\frac{1}{2},\frac{1}{2})+(\frac{1}{2},\frac{3}{2})+(\frac{3}{2},\frac{1}{2})+(\frac{3}{2},\frac{3}{2})}$ $+[2,0,2]_{(0,0)+(1,1)} + [2,1,0]_{(0,0)+2(0,1)+2(1,0)+(1,1)} + [2,2,0]_{(\frac{1}{2},\frac{1}{2})} + [3,0,1]_{(\frac{1}{2},\frac{1}{2})} + [4,0,0]_{(0,0}$
$\frac{17}{2}$	$[0,0,1]_{(0,\frac{1}{2})+(0,\frac{3}{2})+(1,\frac{1}{2})} + [0,1,1]_{(\frac{1}{2},0)+(\frac{1}{2},1)} + [1,0,0]_{(\frac{1}{2},0)+(\frac{1}{2},1)+(\frac{3}{2},0)} + [1,0,2]_{(0,\frac{1}{2})}$ $+[1,1,0]_{(0,\frac{1}{2})+(1,\frac{1}{2})} + [2,0,1]_{(\frac{1}{2},0)}$
9	$[0,0,0]_{(\frac{1}{2},\frac{1}{2})} + [0,0,2]_{(0,0)} + [0,1,0]_{(0,1)+(1,0)} + [1,0,1]_{(\frac{1}{2},\frac{1}{2})} + [2,0,0]_{(0,0)}$
$\frac{19}{2}$	$[0,0,1]_{(\frac{1}{2},0)} + [1,0,0]_{(0,\frac{1}{2})}$
10	$[0,0,0]_{(0,0)}$