Dynamical critical phenomena in strongly coupled gauge theory plasma

Alex Buchel

(Perimeter Institute & University of Western Ontario)

<u>Based on:</u> arXiv:1005.0819, also arXiv:0912.3212 (with Chris Pagnutti), and to appear (with Chris Pagnutti)

Motivation:

$\Rightarrow$ There was been many studies were the gauge/string theory correspondence framework was been used to extract transport coefficients of strongly coupled gauge theory plasma.

however...

$\Rightarrow$ real QCD is not in any one of the models studied

(it is possible to reach QCD as a particular limit in some of the models, but the price to pay is too big: the truncation of the full string theory to a supergravity sector is inconsistent)

thus...

$\Rightarrow$ one attempts to discover common/universal features of hydrodynamics of strongly coupled gauge theories (by looking at the explicit string theory models as well as phenomenological models) and

hope...

$\Rightarrow$ that QCD is in the universality class of the models studied

Examples:

- the shear viscosity ratio

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

- the bulk viscosity ratio

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right) , \qquad c_s^2 = \frac{\partial \mathcal{P}}{\partial \mathcal{E}}$$

$\Rightarrow$ It is not clear why and how this universality arises, or how to properly “define” the corresponding universality classes: while the shear viscosity ratio is universal in 2-derivative supergravity (or a phenomenological model of thereof), it can be violated in full string theory ; while the bulk viscosity bound is satisfied in all models of supergravity derived from string theory, it can be violated in some phenomenological models of gauge/gravity correspondence.

$\Rightarrow$ A more common notion of ’universality’ arises in the theory of continuous critical phenomena.

$\Rightarrow$ We are going to use gauge theory/string theory correspondence of Maldacena to study static and dynamic critical phenomena of strongly coupled (non-conformal) gauge theories in various dimensions

“Holographic-”, ”Maldacena-”, ”gauge theory/string theory-”

correspondence

Consider $\mathcal{N} = 4\ SU(N)$ SYM:

- $g^2_{YM}N \ll 1$ (weak effective coupling) $\Longrightarrow$ perturbative gauge theory description
- $g^2_{YM}N \gg 1$ (strong effective coupling) $\Longrightarrow$ IIB string theory on $AdS_5 \times S^5$

$\Rightarrow$ The duality can be extended to non-conformal gauge theories; it is a very effective tool to compute correlation functions of gauge-invariant operators in QFT at strong coupling, in the presence of finite temperature and/or chemical potentials for the conserved $U(1)$ charges.

Outline of the talk:

- Static critical phenomena
- Dynamical critical phenomena
- Holographic second order phase transitions in $\mathcal{N} = 4$ SYM $(T \neq 0\,, \mu \neq 0)$
- Bulk viscosity at criticality
  - experiment — or why it is interesting?
  - Karsch-Kharzeev-Tuchin model ([arXiv:0711.0914])
  - Quasiparticle models in relaxation time approximation (Sasaki-Redlich [arXiv:0806.4745])
  - bulk viscosity in dynamical critical phenomena (Onuki, PRE **55** 403 (1997))
  - relevance to QCD

- Bulk viscosity in mass deformed $\mathcal{N}=4$ SYM plasma at criticality
- Holographic second order phase transitions at $(T\neq 0\,,\mu=0)$
  - $\mathcal{N}=2^*$ plasma
  - cascading gauge theory plasma
- Holographic bulk viscosity at criticality
  - $\mathcal{N}=2^*$ plasma
  - cascading gauge theory plasma
- Conclusions and future directions

Static critical phenomena

$\Rightarrow$ consider ferromagnetic phase transition

$$\text{magnetization} \quad \mathcal{M} \quad \Leftrightarrow \quad \text{order parameter}$$

$$\text{external magnetic field} \quad \mathcal{H} \quad \Leftrightarrow \quad \text{a control parameter}$$

$$\mathcal{M} = -\left(\frac{\partial \mathcal{W}}{\partial \mathcal{H}}\right)_T$$

where $\mathcal{W} = \mathcal{W}(T, \mathcal{H})$ is the Gibbs free energy

$$\mathcal{M}\Big|_{\mathcal{H}=0} = \begin{cases} 0\,, & \text{disordered [unbroken] phase} \\ \neq 0\,, & \text{ordered [broken] phase} \end{cases}$$

$\Rightarrow$ Basic thermodynamic relations

$$\mathcal{W} = \epsilon - s\,T - \mathcal{M}\mathcal{H}\,, \qquad d\mathcal{W} = -s\,dT - \mathcal{M}\,d\mathcal{H}$$

At a second order phase transition the first derivatives of $\mathcal{W}$ are continuous while the higher derivatives are not. Under the static scaling hypothesis we have:

$$\mathcal{W}(t,\mathcal{H}) = \lambda^{-p}\,\mathcal{W}\left(\lambda^{y_T}t, \lambda^{y_\mathcal{H}}\mathcal{H}\right)\,, \qquad t \equiv \frac{T - Tc}{T_c}$$

for the free energy, and

$$\tilde{G}(\vec{q}, t, \mathcal{H}) = \lambda^{2y_\mathcal{H} - p}\,\tilde{G}(\lambda\vec{q}, \lambda^{y_T}t, \lambda^{y_\mathcal{H}}\mathcal{H})$$

for the Fourier transform of the equilibrium two-point correlation function of the magnetization

$$G(\vec{r}) = \langle\mathcal{M}(\vec{r})\mathcal{M}(\vec{0})\rangle \propto \frac{\partial^2\mathcal{W}}{\partial\mathcal{H}(\vec{r})\partial\mathcal{H}(\vec{0})}$$

p is the number of spatial dimensions.

The static critical exponents

$$\{\alpha, \beta, \gamma, \delta, \nu, \eta\}$$

are defined as

$$\text{specific heat}: \qquad c_{\mathcal{H}} = -T \left(\frac{\partial^2 \mathcal{W}}{\partial T^2} \right)_{\mathcal{H}} = \frac{s}{c_s^2} \propto |t|^{-\alpha}$$

$$\text{spontaneous magnetization}: \qquad \mathcal{M} \propto |t|^{\beta}$$

$$\text{magnetic susceptibility}: \qquad \chi_T = \left(\frac{\partial \mathcal{M}}{\partial \mathcal{H}} \right)_T \propto |t|^{-\gamma}$$

$$\text{critical isotherm}: \qquad \mathcal{M}(t=0) \propto |\mathcal{H}|^{1/\delta}$$

$$\text{correlation function}: \qquad G(\vec{r}) \propto \begin{cases} e^{-|\vec{r}|/\xi}, & t \neq 0 \\ |\vec{r}|^{-p+2-\eta}, & t = 0 \end{cases}$$

$$\text{correlation length}: \qquad \xi \propto |t|^{-\nu}$$

Note: η is the anomalous critical exponent

$\Rightarrow$ Given the scaling hypothesis we can compute

$$\alpha = 2 - \frac{p}{y_T}\,, \qquad \beta = \frac{p - y_{\mathcal{H}}}{y_T}\,, \qquad \gamma = \frac{2y_{\mathcal{H}} - p}{y_T}$$

$$\delta = \frac{y_{\mathcal{H}}}{p - y_{\mathcal{H}}}\,, \qquad \nu = \frac{1}{y_T}\,, \qquad \eta = p - 2y_{\mathcal{H}} + 2$$

which implies 4 scaling relations:

$$\alpha + 2\beta + \gamma = 2\,, \qquad \gamma = \beta(\delta - 1) = \nu(2 - \eta)\,, \qquad 2 - \alpha = \nu p$$

Some mean-field results (LG model for uniaxial ferromagnet in $p = 3$)

Free energy:

$$\mathcal{W} = \int d\vec{x} \left[\frac{c}{2} \left(\nabla \mathcal{M} \right)^2 + \frac{a}{2} \mathcal{M}^2 - \mathcal{M}\mathcal{H} \right]$$

with

$$c > 0 \,, \qquad a = a_0 \left(T - T_c \right)$$

$\Rightarrow$ minimum is achieved for constant $\mathcal{M}$; solving for $\mathcal{M}$,

$$\frac{\partial \mathcal{W}}{\partial \mathcal{M}} = 0 \qquad \Rightarrow \qquad \mathcal{W} = \mathcal{W}(t, \mathcal{H})$$

$$\{\alpha, \beta, \gamma, \delta, \nu, \eta\} = \left\{ 0, \frac{1}{2}, 1, 3, \frac{1}{2}, 0 \right\}$$

Dynamical universality classes and z-exponent

$\Rightarrow$ depends on additional properties of the system:

$$\text{same static universality class} \quad \Rightarrow \quad \text{different dynamical universality class}$$

$\Rightarrow$ crucial question is whether or not the order parameter is conserved

$\Rightarrow$ relaxation to equilibrium is described by time-dependent Landau-Ginsburg (TDLG) equation

In case of Brownian motion $\Rightarrow$ Langevin equation:

$$\frac{dv(t)}{dt} = -\Gamma v(t) + \xi(t) = -\Gamma \frac{\delta H}{\delta v} + \xi(t)$$

where $\Gamma > 0$ is a friction coefficient, $\xi(t)$ is the random force with $\langle \xi(t) \rangle = 0$ and

$$H = \frac{v^2}{2}$$

is the Hamiltonian of the system.

TDLG equation is multi-body generalization of the Langevin equation:

$$\frac{\partial \mathcal{M}(t,\vec{x})}{\partial t} = -\int d\vec{y}\ \Gamma(|\vec{x}-\vec{y}|)\ \frac{\delta \mathcal{W}(\mathcal{M})}{\delta \mathcal{M}(t,\vec{y})}\ +\ \xi(t,\vec{x})$$

$\Gamma|\vec{x}-\vec{y}|$ is a dynamical transport coefficient (friction in Langevin equation)

$\Rightarrow$ Go back to LG model:

$$\mathcal{W} = \int d\vec{x} \left[\frac{c}{2}\left(\nabla \mathcal{M}\right)^2 + \frac{a}{2}\mathcal{M}^2 - \mathcal{M}\mathcal{H}\right]$$

Fourier transform of TDLG, plus averaging

$$-i\omega\langle \mathcal{M}_{\omega,\vec{q}}\rangle = -(cq^2+a)\cdot\Gamma_q\cdot\langle \mathcal{M}_{\omega,\vec{q}}\rangle + \Gamma_q\cdot\mathcal{H}_{\omega,\vec{q}}$$

Consider dynamical susceptibility:

$$\chi_{\omega,\vec{q}} = \frac{\partial \langle \mathcal{M}_{\omega,\vec{q}} \rangle}{\partial \mathcal{H}_{\omega,\vec{q}}} = \frac{\Gamma_q}{i\omega + (cq^2 + a)\Gamma_q}$$

$\Rightarrow$ the response function has a pole

$$\omega = -i\,\tau_q^{-1}\,, \qquad \tau_q^{-1} = (cq^2 + a)\Gamma_q$$

where τ_q is the dynamical relaxation time. In hydro limit ($q \longrightarrow 0$) and for $\Gamma_0 \neq 0$

$$\tau_{q=0}^{-1} \propto t \qquad \Rightarrow$$

the relaxation time diverges (critical slow-down)

We can now introduce a new dynamical exponent z as

$$\tau_{q=0} \propto \xi^z \propto |t|^{-z\nu}$$

Let's look @ LG model:

- $\mathcal{M}$ is not a conserved quantity

$$\tau_{q=0} \propto t^{-1} \qquad \Rightarrow \qquad \xi^{z} \propto t^{-\frac{1}{2}z} \qquad \Rightarrow \qquad z = 2$$

- $\mathcal{M}$ is a conserved quantity

$$\frac{\partial \mathcal{M}_{q=0}}{\partial t} = 0 \qquad \Rightarrow$$

from TDLG:

$$\Gamma(q = 0) = 0 \qquad \Rightarrow \qquad \Gamma_q \propto q^2$$

Thus, in the hydrodynamic limit $q \propto \xi^{-1}$

$$\tau_q \propto \frac{1}{aq^2} \propto \frac{\xi^4}{(\xi q)^2} \qquad \Rightarrow \qquad z = 4$$

More formally, the critical exponent z is introduced by looking at the scaling of the near-equilibrium correlation function of the magnetization

$$\tilde{G}(\omega, \vec{q}, t, \mathcal{H}) = \lambda^{2y_{\mathcal{H}}-p+z}\, \tilde{G}(\lambda^z \omega, \lambda \vec{q}, \lambda^{y_T} t, \lambda^{y_{\mathcal{H}}} \mathcal{H})$$

Note: the dynamics associated with off-equilibrium relaxation, in principle, has nothing to do with Minkowski time evolution, thus z should not be identified with 1, even if we are dealing with relativistic critical phenomena

Likewise: the critical phenomena with $z \neq 1$ does not necessarily have to have a Lifshitz-like holographic scaling

Critical phenomena in $\mathcal{N}=4$ SYM

$\Rightarrow$ Consider strongly coupled $\mathcal{N}=4$ SYM with a (single, non-diagonal) $U(1)\subset SU(4)$ R-symmetry chemical potential

$\Rightarrow$ The dual holographic model is

$$S_5=\frac{1}{16\pi G_5}\int_{\mathcal{M}_5}d^5\xi\sqrt{-g}\left(R-\frac{1}{4}\phi^{4/3}F^2-\frac{1}{3}\phi^{-2}\left(\partial\phi\right)^2+4\phi^{2/3}+8\phi^{-1/3}\right)$$

$\Rightarrow$ It is straightforward to construct RN black hole solution, describing the equilibrium state of finite temperature and density $\mathcal{N}=4$ SYM plasma

$\Rightarrow$ we find:

$$s = \frac{4\pi^2(1+\kappa)^2 T^3 N^2}{(\kappa+2)^3}\,, \qquad \epsilon = 3P = \frac{6N^2T^4(1+\kappa)^3\pi^2}{(\kappa+2)^4}$$

$$\rho = \frac{2\pi(1+\kappa)^2\kappa^{1/2}T^3N^2}{(\kappa+2)^3}\,, \qquad \frac{2\pi T}{\mu} = \sqrt{\kappa} + \frac{2}{\sqrt{\kappa}}$$

$\Rightarrow$ it is easy to verify that

$$\Omega = \epsilon - Ts - \mu\rho = -P\,, \qquad d\epsilon = Tds + \mu d\rho\,, \qquad dP = sdT + \rho d\mu$$

$\Rightarrow$ we see that $\frac{T}{\mu}$ achieves a minimum at $\kappa = \kappa_c = 2$, corresponding to the critical temperature $T_c = \sqrt{2}\mu/\pi$ and the critical chemical potential $\mu_c = \pi T/\sqrt{2}$. Introducing

$$t = \frac{T}{T_c} - 1\,, \qquad \bar{\mu} = 1 - \frac{\mu}{\mu_c} \qquad \Longrightarrow \qquad \bar{\mu} = \frac{t}{t+1}$$

we find

$$\Omega_{\pm}(\mu, t) = -\frac{27N^2\mu^4}{32\pi^2}\left(1 + \frac{8}{3}\,t \mp \frac{16\sqrt{2}}{27}\,t^{3/2} + \frac{68}{27}\,t^2 + \mathcal{O}\left(t^{5/2}\right)\right)$$

$$\Omega_{\pm}(T, \bar{\mu}) = -\frac{27N^2T^4\pi^2}{128}\left(1 - \frac{4}{3}\,\bar{\mu} \mp \frac{16\sqrt{2}}{27}\,\bar{\mu}^{3/2} + \frac{14}{27}\,\bar{\mu}^2 + \mathcal{O}\left(\bar{\mu}^{5/2}\right)\right)$$

$$\kappa = \kappa_{\pm}(t) = 2 \pm 4\sqrt{2}\,t^{1/2} + 8\,t \pm 5\sqrt{2}\,t^{3/2} + 4\,t^2 + \mathcal{O}\left(t^{5/2}\right)$$

$\Rightarrow$ **Remember:**

$$\kappa - \kappa_c \propto t^{1/2}$$

Thus for a given temperature t there are two thermodynamic phases of the system, with Ω_- being the stable one.

For Ω_- phase:

$$C = T\left(\frac{\partial s}{\partial T}\right)\bigg|_{\mu} \propto -\frac{\partial^2\Omega_-(\mu,t)}{\partial t^2} \propto +t^{-1/2} \qquad \Rightarrow \qquad \alpha = \frac{1}{2}$$

$$\chi_T = \left(\frac{\partial \rho}{\partial \mu}\right)\bigg|_{T} \propto -\frac{\partial^2\Omega_-(T,\bar{\mu})}{\partial \bar{\mu}^2} \propto +\bar{\mu}^{-1/2} \propto +t^{-1/2} \qquad \Rightarrow \qquad \gamma = \frac{1}{2}$$

$\Rightarrow$ thus, assuming the scaling relations we find the static universality class of strongly coupled RN plasma to be

$$(\alpha,\beta,\gamma,\delta,\nu,\eta) = \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},2,\frac{1}{2},1\right)$$

$\Rightarrow$ All I said about $\mathcal{N}=4$ plasma is old result

I want to claim that these naive identification of the [static] universality class is in fact incorrect

$\Rightarrow$ A hint that something is wrong can be seen from the fact that the anomalous scaling exponent $\eta = 0$, as one would expect in the large-N (equivalently mean-field) limit

$\Rightarrow$ At a technical level, the hyperscaling relation between the critical exponents

$$2 - \alpha = \nu p = 3\nu$$

is quite often is violated

$\Rightarrow$ To proceed we need to compute the dynamical susceptibility $\chi(\mathfrak{w}, \mathfrak{q})$:

- $$\chi(\mathfrak{w}=0, \mathfrak{q}=0) = \chi_T \qquad \Rightarrow \qquad \text{a test on computations}$$

- $$\chi(\mathfrak{w}=0, \mathfrak{q})\Big|_{t\neq 0} \propto \frac{1}{\mathfrak{q}^2 + (2\pi T\xi)^2} \qquad \Rightarrow \qquad \xi \propto t^{-\nu}$$

- $$\chi(\mathfrak{w}=0, \mathfrak{q})\Big|_{t=0} \propto \mathfrak{q}^{-2+\eta}$$

- $$\frac{1}{\chi(\mathfrak{w}, \mathfrak{q})} = 0 \qquad \Rightarrow \qquad i\mathfrak{w} = (2\pi T\tau)^{-1} \qquad \Rightarrow \qquad \tau \propto \xi^z$$

$\Rightarrow$ I will now present the results of the analysis.

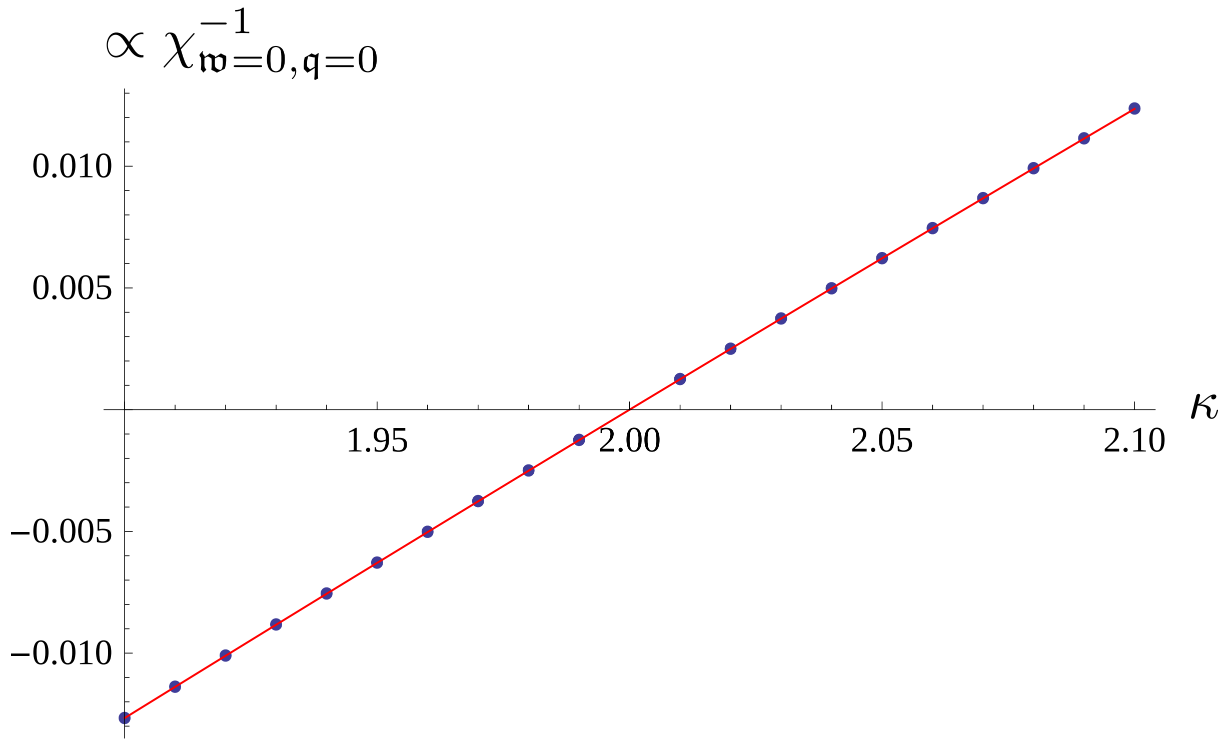

Figure 1: The scaling (blue dots) of the inverse of the static susceptibility $\chi_{\mathfrak{w}=0,\mathfrak{q}=0}$ in the vicinity of the critical point. The solid red line is a quadratic fit to the data. The red line intersects the κ axis at $\kappa_c = 1.999999(6)$ in excellent agreement with the expected value $\kappa_c = 2$.

$$\chi_{\mathfrak{w}=0,\mathfrak{q}=0} = \chi_T \propto \frac{1}{\kappa - \kappa_c} \propto +t^{-1/2}, \qquad |\kappa - \kappa_c| \ll \kappa_c$$

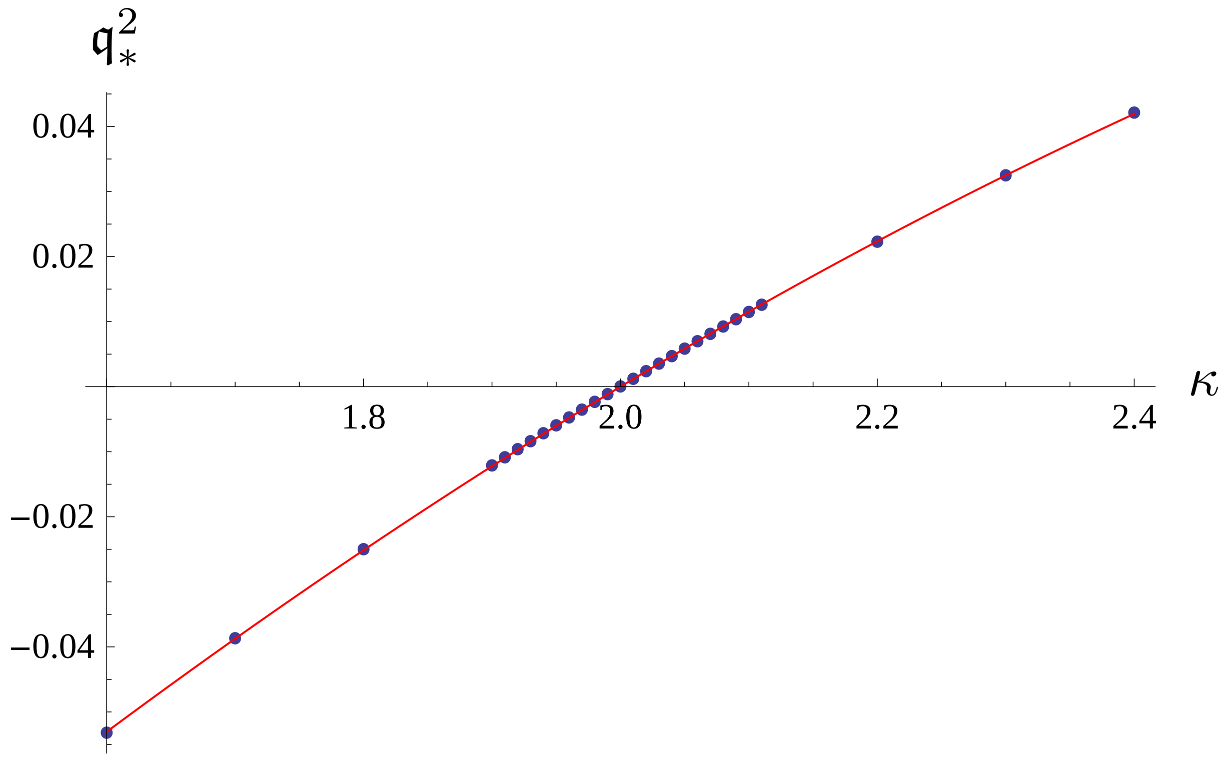

Figure 2: Poles of the static susceptibility in the vicinity of the critical point: $\chi^{-1}_{\mathfrak{w}=0,\mathfrak{q}=\mathfrak{q}_*} = 0$.

$$(2\pi T_c\,\xi)^2 \;\propto\; \mathfrak{q}_*^{-2} \;\propto\; \frac{1}{\kappa-\kappa_c} \;\propto +t^{-1/2}\,, \qquad 0<\kappa_c-\kappa \ll k_c$$

$$\xi \;\propto\; t^{-\nu} \;\propto\; t^{-1/4} \qquad \Rightarrow \qquad \nu = \frac{1}{4}$$

Given that the static critical exponent $\alpha = \frac{1}{2}$, above implies that the hyperscaling relation is violated

$$2 - \alpha \;\neq p\,\nu$$

where $p = 3$ stands for the number of spatial dimensions of the system.

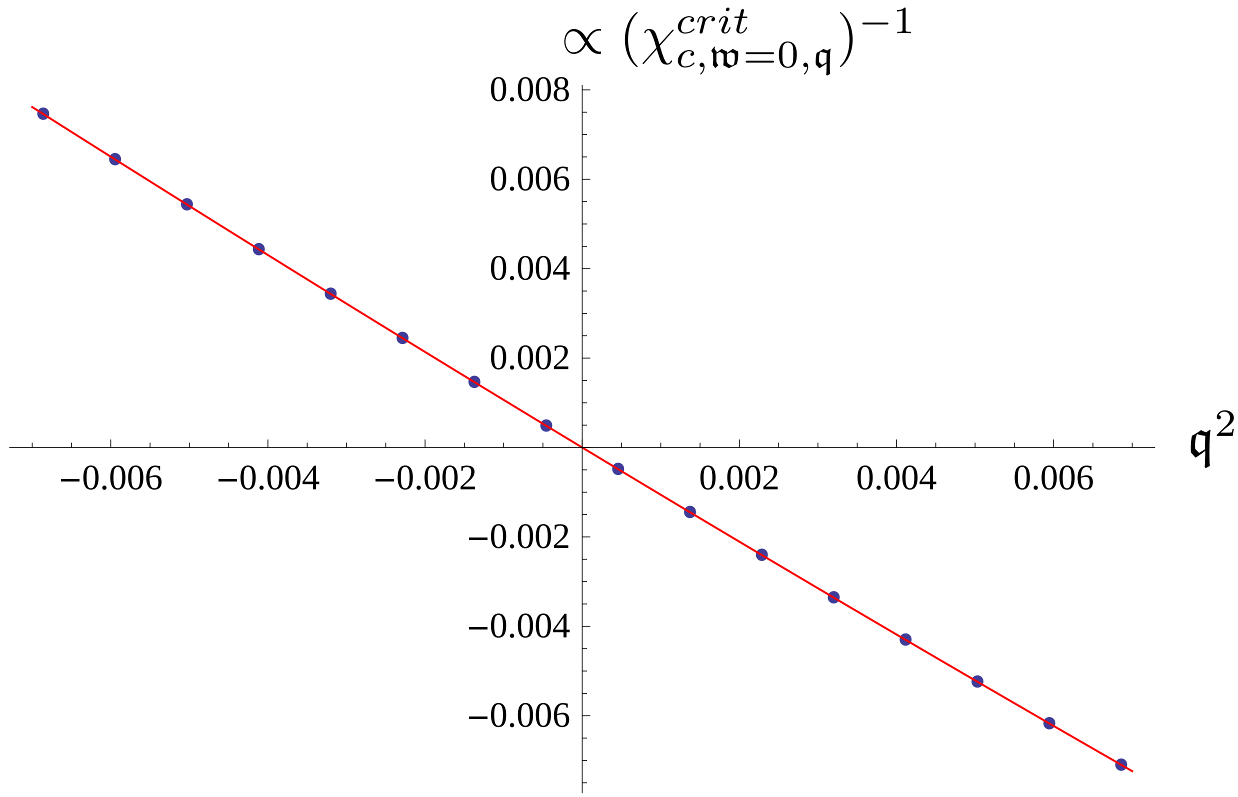

Figure 3: The scaling (blue dots) of the inverse of the static susceptibility $\chi^{crit}_{\mathfrak{w}=0,\mathfrak{q}}$ at the critical point, $\kappa = 2$. The solid red line is a quadratic fit to the data.

The red line intersects the $\mathfrak{q}^2$ axis at $\mathfrak{q}_c^2 = -1.57468 \cdot 10^{-8}$ in excellent agreement with the expected value $\mathfrak{q}_c^2 = 0$. The data implies

$$\chi^{crit}_{\mathfrak{w}=0,\mathfrak{q}} \propto \mathfrak{q}^{-2} \quad \Longleftrightarrow \quad \chi^{crit}_{\mathfrak{w}=0,\mathfrak{q}} \propto \mathfrak{q}^{-2+\eta} \quad \Longleftrightarrow \quad \eta = 0$$

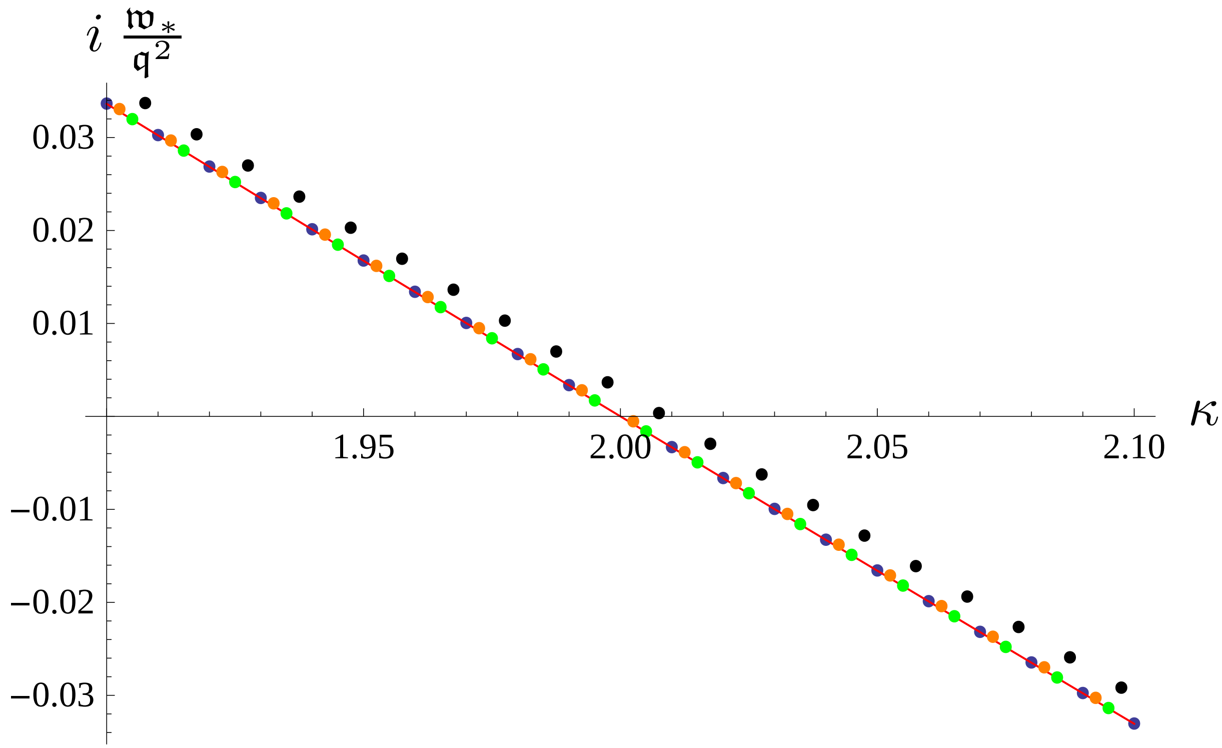

Figure 4: Poles of the dynamic susceptibility in the vicinity of the critical point, $\chi^{-1}_{\mathfrak{w}=\mathfrak{w}_*,\mathfrak{q}} = 0$ for a set of momenta values $\mathfrak{q}^2 =: 10^{-6}$ (blue dots) , 10^{-5} (green dots), 10^{-4} (orange dots) and 10^{-3} (black dots). The solid red line is a quadratic fit to $i\frac{\mathfrak{w}_*}{\mathfrak{q}^2}$ at $\mathfrak{q}^2 = 10^{-6}$.

$$\lim_{\mathfrak{q}\to 0}\; i\,\frac{\mathfrak{w}_*}{\mathfrak{q}^2} = 2.79163\;\cdot 10^{-6} - 0.333392(\kappa-2) + 0.0278087(\kappa-2)^2 + \mathcal{O}((\kappa-2)^3)$$

$$(2\pi T_c\,\tau)^{-1} \equiv i\mathfrak{w}_* \;\propto\; \mathfrak{q}^2 \cdot (\kappa - \kappa_c) \;\propto\; (2\pi T_c\,\mathfrak{q}\xi)^2 \cdot (2\pi T_c\,\xi)^{-4} \;\propto\; (2\pi T_c\,\xi)^{-4}$$

$$\tau \;\propto\; \xi^z \;\propto\; \xi^4 \qquad \Rightarrow \qquad z = 4$$

$\Rightarrow$ Thus:

- incorrect universality class of $\mathcal{N} = 4$ plasma:

$$(\alpha, \beta, \gamma, \delta, \nu, \eta) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, \frac{1}{2}, 1\right)$$

- correct universality class of $\mathcal{N} = 4$ plasma:

$$(\alpha, \beta, \gamma, \delta, \nu, \eta\,;\;\; z) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, \frac{1}{4}, 0\,;\;\; 4\right)$$

Note: $z \neq 1$ even though the holographic dual does not have Lifshitz-like scaling.

Hydrodynamics and models of bulk viscosity at criticality

$$T^{\mu\nu} = T^{\mu\nu}_{equilibrium} + T^{\mu\nu}_{non-equilibrium}$$

$$T^{\mu\nu}_{eq} = \epsilon\, u^{\mu}u^{\nu} + P\Delta^{\mu\nu}\,, \qquad T^{\mu\nu}_{non-eq} = -\eta\, \sigma^{\mu\nu} - \zeta(\nabla u)$$

$$u^{\mu}u_{\mu} = -1\,, \qquad \Delta^{\mu\nu} = \eta^{\mu\nu} + u^{\mu}u^{\nu}$$

where η, ζ are the shear and the bulk viscosities and $\sigma^{\mu\nu}$ is a shear tensor (which is traceless):

$$\eta_{\mu\nu}\sigma^{\mu\nu} = 0$$

In CFT $\qquad T^{\mu}_{\mu} = 0 \qquad \Rightarrow$

$$-\epsilon + 3P - 3\zeta(\nabla u) = 0 \qquad \Rightarrow \qquad \epsilon = 3P\Big|_{CFT}\,, \qquad \zeta\Big|_{CFT} = 0$$

$\Rightarrow$ so in order to see $\zeta \neq 0$ we need to look @ non-conformal theories

- Naively, second-order phase transitions imply scale invariance $\Rightarrow$

$$\zeta \to 0 \qquad \text{or} \qquad \zeta \to \infty$$

  Not true: $\zeta = 0$ necessitates the full *space-time* scale invariance, while at criticality we have only *spatial* scale-invariance.

- Even though a CFT has $\zeta = 0$, it might still have a non-trivial z as determined from the dynamical susceptibility

What is bulk viscosity at criticality?

- Experiments: typically,

$$\frac{\zeta}{\eta} \lesssim 1$$

however, for $^3\mathrm{He}$ in the vicinity of liquid-vapor critical point

$$\frac{\zeta}{\eta} \gtrsim 10^6$$

- Phenomenology: QCD first order confinement/deconfinement curve (in (T, μ) plane) ends at a critical point of the 3d Ising model universality class. Son-Stephanov (hep-ph/0401052) argued that the dynamical universality class of QCD is that of the liquid-vapor point. For the liquid-vapor critical point Onuki computed:

$$z \approx 3$$

Some theoretical models

- KKT model (A):

$$\zeta_{singular} \propto c_v \propto |t|^{-\alpha}$$

- Quasi-particle models (B):

$$\zeta_{singular} \propto |t|^{\alpha+4\beta-1}$$

- Onuki's dynamical model (C):

$$\zeta_{singular} \propto \xi^{z-\alpha/\nu} \propto |t|^{-z\nu+\alpha}$$

$\Rightarrow$ above scalings are p-independent

$\Rightarrow$ vastly different results!!

$\Rightarrow$ holography to the rescue

What to compute and how?

One of the on-shell modes of

$$0 = \nabla_\mu T^{\mu\nu}$$

is a sound wave:

$$\omega = \pm c_s\, q - i\, \Gamma\, q^2 + \mathcal{O}(q^3)$$

where

$$c_s^2 = \frac{\partial P}{\partial \epsilon}\,, \qquad T \cdot \Gamma = \frac{\eta}{s} \left(\frac{p-1}{p} + \frac{\zeta}{2\eta} \right)$$

It appears as a pole in the two-point correlation function of the stress-energy tensor.

$\Rightarrow$ In a dual holographic description the sound wave arises as one of the quasinormal modes of the black hole describing the thermal equilibrium state of a strongly coupled gauge theory plasma (Kovtun-Starinets, hep-th/0506184).

Thus the strategy is to:

- construct the gravitational description of the gauge theory plasma undergoing second-order phase transition; compute the static critical exponents;
- compute the dispersion relation of the 'sound' quasinormal mode;
- extract the critical exponent of the bulk viscosity;
- interpret the result in available framework of the dynamical critical phenomena

Our holographic playground:

- mass-deformed $\mathcal{N}=4$ plasma;
- $\mathcal{N}=2^*$ gauge theory $\Leftrightarrow$ mass-deformed $\mathcal{N}=4$ $SU(N)$ SYM in $d=4$;
- $\mathcal{N}=1$ $SU(N+M)\times SU(N)$ cascading gauge theory in $d=4$;

Mass deformed $\mathcal{N} = 4^*$ plasma

$\Rightarrow$ Gravity:

$$S_5 = \frac{1}{16\pi G_5} \int_{\mathcal{M}_5} d^5\xi \sqrt{-g} \left(R - \frac{1}{4}\phi^{4/3} F^2 - \frac{1}{3}\phi^{-2} \left(\partial\phi\right)^2 + 4\phi^{2/3} + 8\phi^{-1/3} + \delta\mathcal{L} \right)$$

where $\delta\mathcal{L}$ is a mass deformation

$$\delta\mathcal{L} = -\frac{1}{2} \left(\partial\chi\right)^2 - \frac{m^2}{2}\chi^2 + \mathcal{O}\left(\chi^4\right) , \qquad \Delta(\Delta - 4) = m^2$$

$\Rightarrow$ QFT:

$$\mathcal{L}_{CFT} \to \mathcal{L}_{CFT} - M\mathcal{O}_3 , \qquad M \propto \lambda$$

where λ is a coefficient of the non-normalizable mode of χ near the asymptotic AdS_5 boundary

Note: the gauge/gravity relation is expected to hold only to $\mathcal{O}(M^2)$. We can always achieve this provided $M \ll T_c$.

Repeating the thermodynamic analysis we find:

$$\Omega_\pm(\mu,t) = -\frac{27N^2\mu^4}{32\pi^2}\left(1+s_t^0\,\frac{M^2}{\mu^2}\right)\left(1\pm s_t^1\,\frac{M^2}{\mu^2}\,t^{1/2}+\frac{8}{3}\left(1+s_t^2\,\frac{M^2}{\mu^2}\right)\,t\right.$$
$$\left.\mp\frac{16\sqrt{2}}{27}\left(1+s_t^3\,\frac{M^2}{\mu^2}\right)\,t^{3/2}+\cdots+\mathcal{O}\left(\frac{M^4}{\mu^4}\right)\right)$$

where s_t^i denote the deformations from the CFT thermodynamics near the criticality; in the above expression we already took into account the fact that T_c got shifted by order M^2/μ^2 correction

$\Rightarrow$ Unless

$$s_t^1 = 0$$

the static critical exponents are modified: $C \propto \pm s_t^1\,t^{-3/2}$, instead of $\propto t^{-1/2}$

$\Rightarrow$ it is possible to show that the first law of thermodynamics (which numerically is valid in the deformed model $\sim 10^{-10}$) guarantees $s_t^1 = 0$

$\Rightarrow$ with a bit more work it can be shown that the universality classes (static+dynamic) of $\mathcal{N}=4$ SYM plasma are robust against mass deformation

Sound waves in mass deformed $\mathcal{N}=4$ plasma

$\Rightarrow$ Hydrodynamics is more complicated since besides $T^{\mu\nu}$ we have conserved $U(1)_R$ current J^μ:

$$J^\mu = \rho u^\mu + \nu^\mu$$

where ν^μ is the dissipative part satisfying $u^\mu \nu_\mu = 0$:

$$\nu^\mu = \sigma_Q \Delta^{\mu\nu} \left(-\partial_\nu \mu + \frac{\mu}{T} \partial_\nu T \right)$$

σ_Q is a new transport coefficient, the conductivity

$\Rightarrow$ We can parametrize the dispersion relation for the sound waves as before

$$\omega = \pm c_s\, q - i\Gamma\, q^2 + \mathcal{O}(q^3)$$

$$c_s^2 = \left((\epsilon + P)\, \frac{\partial(P,\rho)}{\partial(T,\mu)} + \rho\, \frac{\partial(\epsilon,P)}{\partial(T,\mu)} \right) \left((\epsilon + P)\, \frac{\partial(\epsilon,\rho)}{\partial(T,\mu)} \right)^{-1}$$

$$\Gamma = \frac{2\eta}{3(\epsilon+P)} \left(1 + \frac{3\zeta}{4\eta}\right) - \frac{\sigma_Q}{2T} \left(\frac{\partial P}{\partial \rho}\right)_\epsilon \left((\epsilon + P)\, \frac{\partial(P,\rho)}{\partial(T,\mu)} + \rho \frac{\partial(\epsilon,P)}{\partial(T,\mu)} \right)^{-1} \times$$
$$\times \left((\epsilon+P) \left(\left(\frac{\partial \rho}{\partial \ln \mu}\right)_T + \left(\frac{\partial \rho}{\partial \ln T}\right)_\mu \right) - \rho \left(\left(\frac{\partial \epsilon}{\partial \ln \mu}\right)_T + \left(\frac{\partial \epsilon}{\partial \ln T}\right)_\mu \right) \right)$$

- In a CFT , i.e, using the equation of state $\epsilon = 3P$, we recover the usual results

$$c_s^2 = \frac{1}{3}\,, \qquad \Gamma = \frac{2\eta}{3sT}\, \frac{sT}{sT - \mu\rho} = \frac{1}{6\pi T}\, \frac{sT}{sT - \mu\rho}$$

- Note

$$\Gamma = \cdots + \sigma_Q \times \mathcal{O}\left(\frac{M^4}{T^4}\right)$$

thus we do not need to worry about σ_Q.

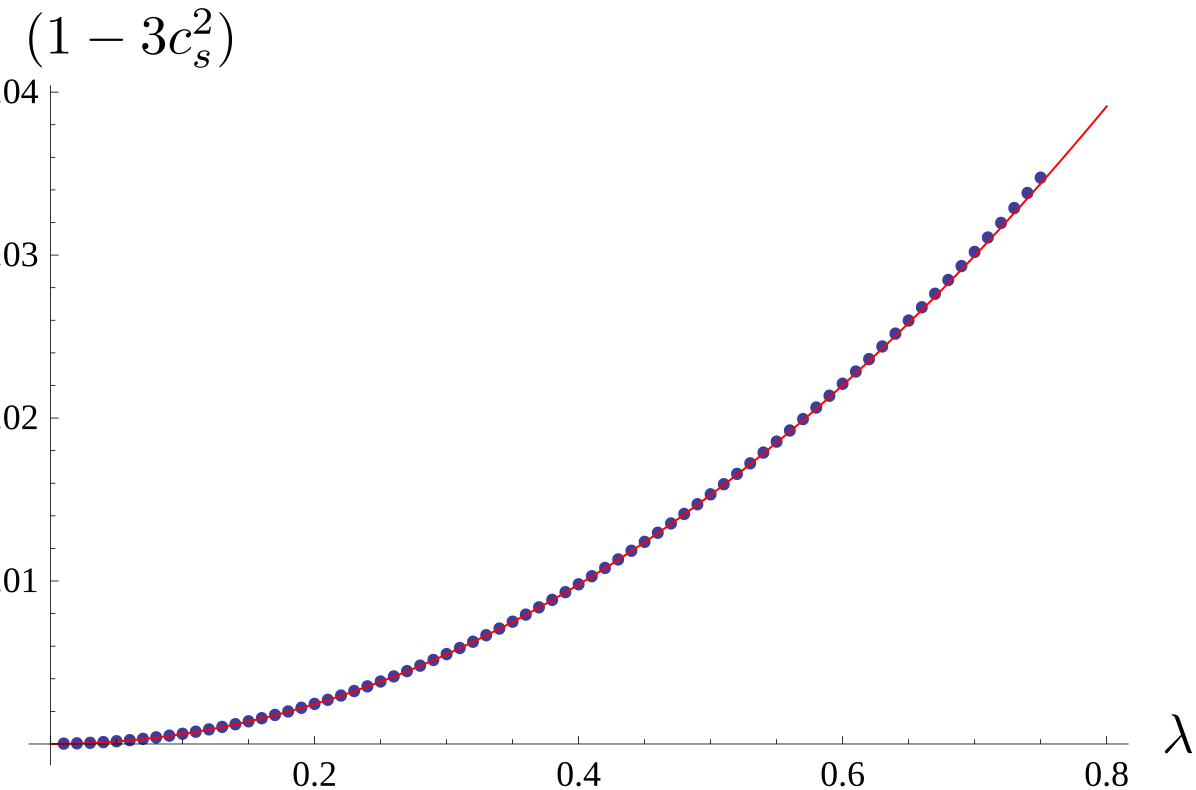

Figure 5: Deviation of the speed of sound $(1-3c_s^2)$, in mass-deformed RN plasma from its conformal value as a function of the mass-deformation parameter λ at $\kappa = 2$. The blue dots represents data obtained from the spectrum of quasinormal modes, and the solid red line represents thermodynamic prediction. Agreement is $\sim 10^{-6}$.

$\Rightarrow$ We find that the bulk viscosity is finite at the critical point with

$$\frac{\zeta}{\eta} = 3.0488(5)\ \left(\frac{1}{3} - c_s^2\right) + \mathcal{O}\left(\left(\frac{1}{3} - c_s^2\right)^2\right)$$

- it satisfies the bulk viscosity bound in strongly coupled plasma

$$\frac{\zeta}{\eta} \geq 2\left(\frac{1}{3} - c_s^2\right)$$

- with regard to a critical behavior:

$$\frac{\zeta}{\eta} \propto |t|^0$$

Recall:

- KKT model (A):

$$\zeta_{singular} \propto c_v \propto |t|^{-\alpha}$$

- Quasi-particle models (B):

$$\zeta_{singular} \propto |t|^{\alpha+4\beta-1}$$

- Onuki's dynamical model (C):

$$\zeta_{singular} \propto \xi^{z-\alpha/\nu} \propto |t|^{-z\nu+\alpha}$$

Thus:

- Model A is inconsistent with holographic analysis as it predicts divergent bulk viscosity, $\zeta \propto |t|^{-1/2}$;
- Model B does not contradict our holographic analysis as it predicts that $\zeta_{singular} \propto |t|^{3/2}$;
- Model C is inconsistent with holographic analysis as it predicts divergent bulk viscosity, $\zeta \propto |t|^{-1/2}$;

Actually:

Model B is not applicable as the relaxation time is divergent:

$$\tau \propto \xi^4 \rightarrow \infty \qquad \text{at the transition}$$

$\mathcal{N} = 2^*$ gauge theory (a QFT story)

$\Longrightarrow$ Start with $\mathcal{N} = 4\; SU(N)$ SYM. In $\mathcal{N} = 1$ 4d susy language, it is a gauge theory of a vector multiplet V, an adjoint chiral superfield Φ (related by $\mathcal{N} = 2$ susy to V) and an adjoint pair $\{Q, \tilde{Q}\}$ of chiral multiplets, forming an $\mathcal{N} = 2$ hypermultiplet. The theory has a superpotential:

$$W = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr}\left(\left[Q, \tilde{Q}\right] \Phi\right)$$

We can break susy down to $\mathcal{N} = 2$, by giving a mass for $\mathcal{N} = 2$ hypermultiplet:

$$W = \frac{2\sqrt{2}}{g_{YM}^2} \mathrm{Tr}\left(\left[Q, \tilde{Q}\right] \Phi\right) + \frac{m}{g_{YM}^2}\left(\mathrm{Tr}Q^2 + \mathrm{Tr}\tilde{Q}^2\right)$$

This theory is known as $\mathcal{N} = 2^*$ gauge theory

When $m \neq 0$, the mass deformation lifts the $\{Q, \tilde{Q}\}$ hypermultiplet moduli directions; we are left with the $(N-1)$ complex dimensional Coulomb branch, parametrized by

$$\Phi = \mathrm{diag}\left(a_1, a_2, \cdots, a_N\right) \,, \qquad \sum_i a_i = 0$$

We will study $\mathcal{N} = 2^*$ gauge theory at a particular point on the Coulomb branch moduli space:

$$a_i \in [-a_0, a_0] \,, \qquad a_0^2 = \frac{m^2 g_{YM}^2 N}{\pi}$$

with the (continuous in the large N-limit) linear number density

$$\rho(a) = \frac{2}{m^2 g_{YM}^2} \sqrt{a_0^2 - a^2} \,, \qquad \int_{-a_0}^{a_0} da \; \rho(a) = N$$

Reason: we understand the dual supergravity solution only at this point on the moduli space.

$\Rightarrow$ We are going to study $\mathcal{N} = 2^*$ plasma at finite temperature T, thus breaking SUSY anyway

$\Rightarrow$ The gravitational description at $T \neq 0$ allows for an additional parameter in the deformation: the masses of the bosonic and the fermionic components of a hypermultiplet $\{Q, \tilde{Q}\}$ can be different

$$m_b \neq m_f \,, \qquad \mathcal{N} = 2 \ \ \text{SUSY} : \qquad m_b = m_f = m$$

Some facts about $\mathcal{N} = 2^*$ thermodynamics in (T, m_b, m_f) parameter space:

- for the range of parameters studies (up to $\frac{m}{T} \sim 10$) the theory is in deconfined phase
- whenever $m_f^2 < m_b^2$ the theory undergoes a phase transition with the vanishing speed of sound;
- at the transition,

$$T_c = T_c \left(m_f^2 / m_b^2 \right)$$

-

$$m_f = 0 : \qquad m_b / T_c \approx 2.32591$$

$\Rightarrow$ We focus on thermodynamics of $\mathcal{N} = 2^*$ plasma with $m_f = 0\,, m_b \neq 0$.

$\Rightarrow$ We now identify above phase transition as a second-order phase transition with the **naive** static critical exponents

$$(\alpha, \beta, \gamma, \delta, \nu, \eta) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2, \frac{1}{2}, 1\right)$$

Note: the phase transition is appears to be not the mean-field one, as anomalous critical exponent η is nonzero $\Rightarrow$ a more careful (direct) analysis show that $\eta = 0$ and $\nu = \frac{1}{4}$

$\Rightarrow$ Some of thermodynamic plots presents data as a function of the gravitational parameter ρ_{11}. It is possible to establish precisely the relation

$$\rho_{11} \qquad \Leftrightarrow \qquad \frac{m_b^2}{T^2}$$

$\Rightarrow$ this relation is complicated at low temperatures, but fairly simple at high-T:

$$\rho_{11} = \frac{\sqrt{2}}{24\pi^2} \left(\frac{m_b}{T}\right)^2 + \mathcal{O}\left(\frac{m_b^4}{T^4}\right)$$

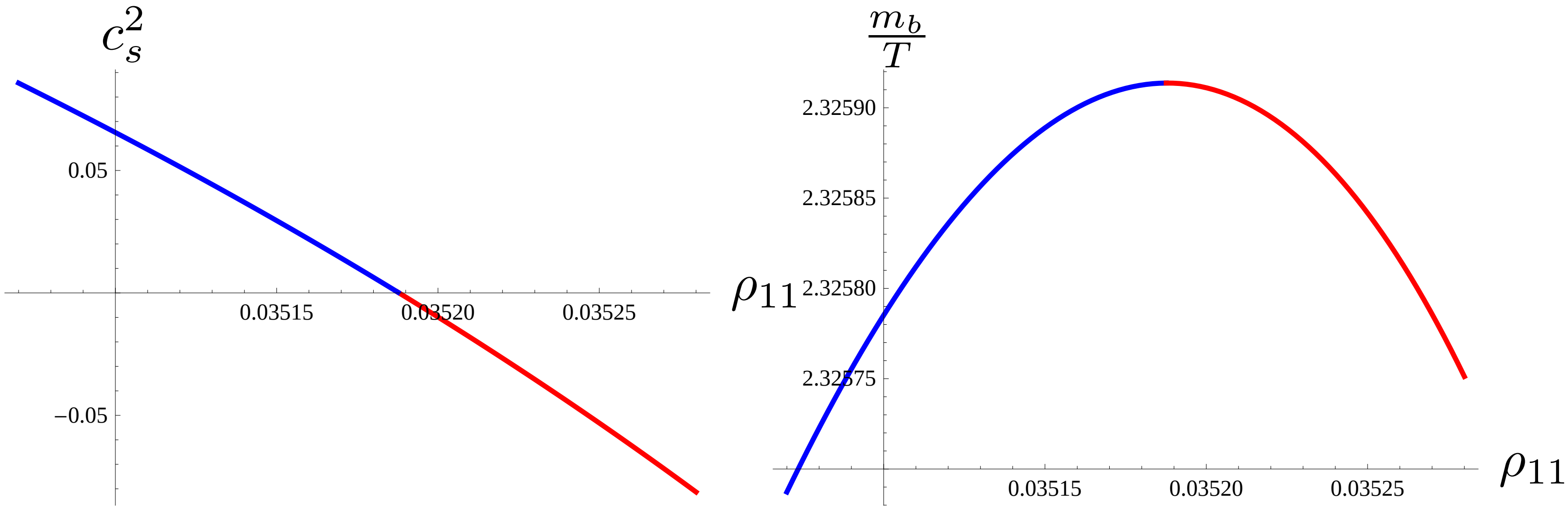

Figure 6: The speed of sound c_s^2 (left plot) and the reduced temperature $\frac{m_b}{T}$ (right plot) of the strongly coupled $\mathcal{N} = 2^*$ plasma with $m_f = 0$ and $m_b \neq 0$ as a function of the dual gravitation parameter ρ_{11}.

Introduce

$$\Delta\rho_{11} = \rho_{11} - \rho_{11}^c \qquad \Rightarrow$$

$$t \propto (\Delta\rho_{11})^2 , \qquad c_s^2\Big|_{blue} \propto \left(-c_s^2\right)\Big|_{red} \propto |\Delta\rho_{11}| \propto t^{1/2}$$

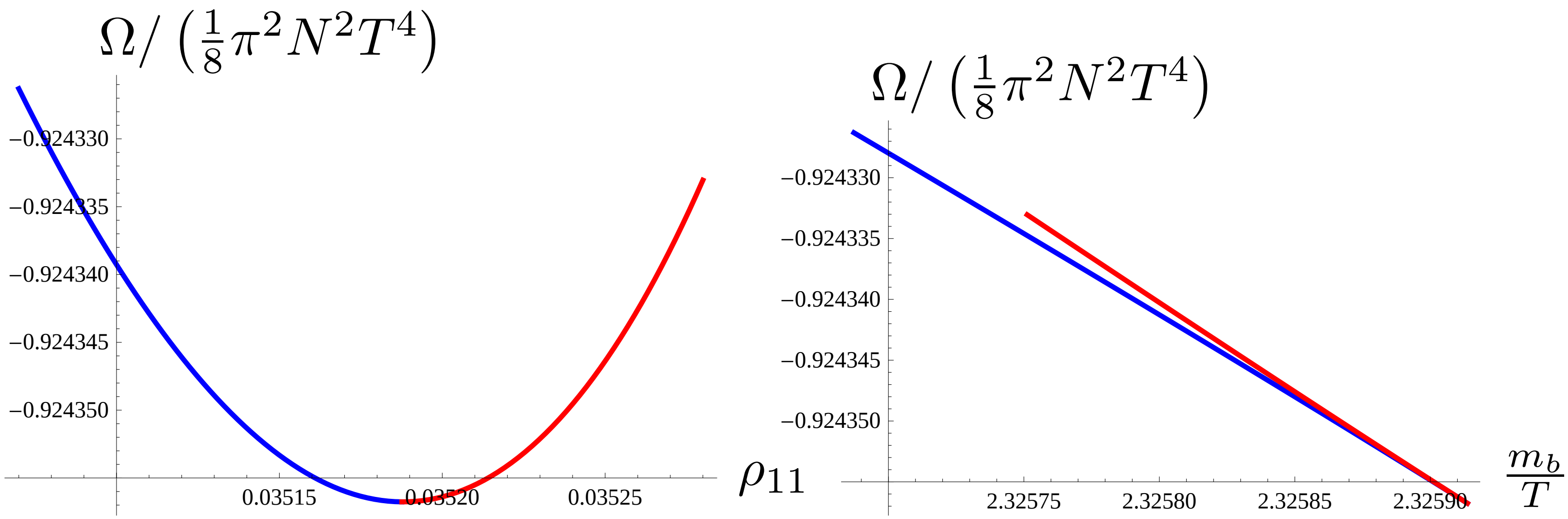

Figure 7: Free energy densities Ω_o of the “ordered” phase (blue curves) and Ω_d of the “disordered” phase (red curves) as a function of ρ_{11} (left plot) and $\frac{m_b}{T}$ (right plot) of the $\mathcal{N} = 2^*$ plasma with $m_f = 0$.

$\Rightarrow$ We identify the free energy $\mathcal{W}$ of the effective ferromagnet as

$$\mathcal{W} = \Omega_o - \Omega_d = \Omega^{blue} - \Omega^{red} < 0$$

$\Rightarrow$

$$c_{\mathcal{H}} = -T\left(\frac{\partial^2 \mathcal{W}}{\partial T^2}\right) = \frac{s}{c_s^2}\Bigg|_{red}^{blue} \propto c_s^{-2}\Bigg|_{red}^{blue} \propto t^{-1/2} \qquad \Rightarrow \qquad \alpha = \frac{1}{2}$$

$\Rightarrow$ To determine the critical exponent β we need to identify the control parameter corresponding to the external magnetic field $\mathcal{H}$ of the effective ferromagnet. We propose to identify

$$\mathcal{H} = m_b$$

Since $T_c \propto m_b \propto \mathcal{H}$, and $t \propto (\Delta\rho_{11})^2$,

$$\partial_{\mathcal{H}} \;\propto\; -\partial_t \;\propto\; -\frac{1}{\Delta\rho_{11}}\,\partial_{\Delta\rho_{11}}$$

From the best fit to the free energy difference:

$$\mathcal{W} \;\propto -|\Delta\rho_{11}|^3$$

$\Rightarrow$

$$\mathcal{M} = -\left(\frac{\partial\mathcal{W}}{\partial\mathcal{H}}\right) \;\propto\; \frac{1}{\Delta\rho_{11}}\,\partial_{\Delta\rho_{11}}\mathcal{W} \;\propto\; -|\Delta\rho_{11}| \;\propto\; -t^{1/2} \qquad \Rightarrow \qquad \beta = \frac{1}{2}$$

$\Rightarrow$ the rest of the critical exponents is determined from the scaling relations

Cascading gauge theory (a QFT story)

$\Rightarrow$ Consider $\mathcal{N} = 1$ $SU(K+P) \times SU(K)$ gauge theory with 2 chiral superfields A_1, A_2 in $(K+P, \bar{K})$ representation and 2 chiral superfields B_1, B_2 in $(\overline{K+P}, K)$ representation with a quartic superpotential:

$$W \sim \mathrm{Tr}(A_i B_j A_k B_\ell)\epsilon^{ik}\epsilon^{jl}$$

- when $P = 0$ the theory flows in the IR to a strongly coupled SCFT
- when $P \neq 0$, the scale invariance is broken. Perturbatively, the theory has two gauge

couplings $g_i(\mu)$ and

$$\frac{d}{d\ln\mu}\left(\frac{4\pi}{g_1^2(\mu)} + \frac{4\pi}{g_2^2(\mu)}\right) = 0$$

$$\frac{4\pi}{g_2^2(\mu)} - \frac{4\pi}{g_1^2(\mu)} \sim P\ \ln\frac{\mu}{\Lambda}$$

$\Rightarrow$ Λ is the strong coupling scale of the theory

Some facts about cascading plasma thermodynamics in (T, Λ) parameter space:

- for

$$T > T_{confinement} = 0.6141111(3)\Lambda$$

cascading gauge theory is deconfined; has an unbroken $U(1)$ chiral symmetry

- at $T = T_{confinement}$ cascading plasma undergoes a first-order phase transition to a confined phase with spontaneously broken chiral symmetry

- Although non-perturbatively unstable due to the nucleation of bubbles of the confined phase, the deconfined $U(1)$ symmetric phase can be extended to temperatures lower than $T_{confinement}$ — this phase remains (perturbatively) thermodynamically and dynamically stable down to T_c:

$$T_c = 0.8749(0) \times T_{confinement} \; < T \; < T_{confinement}$$

$\Rightarrow$ At $T = T_c$ cascading plasma undergoes a second-order phase transition identical to the one in $\mathcal{N} = 2^*$ plasma

$\Rightarrow$ To compute the critical exponent β we identify the 'effective external magnetic field' as $\mathcal{H} = \Lambda$

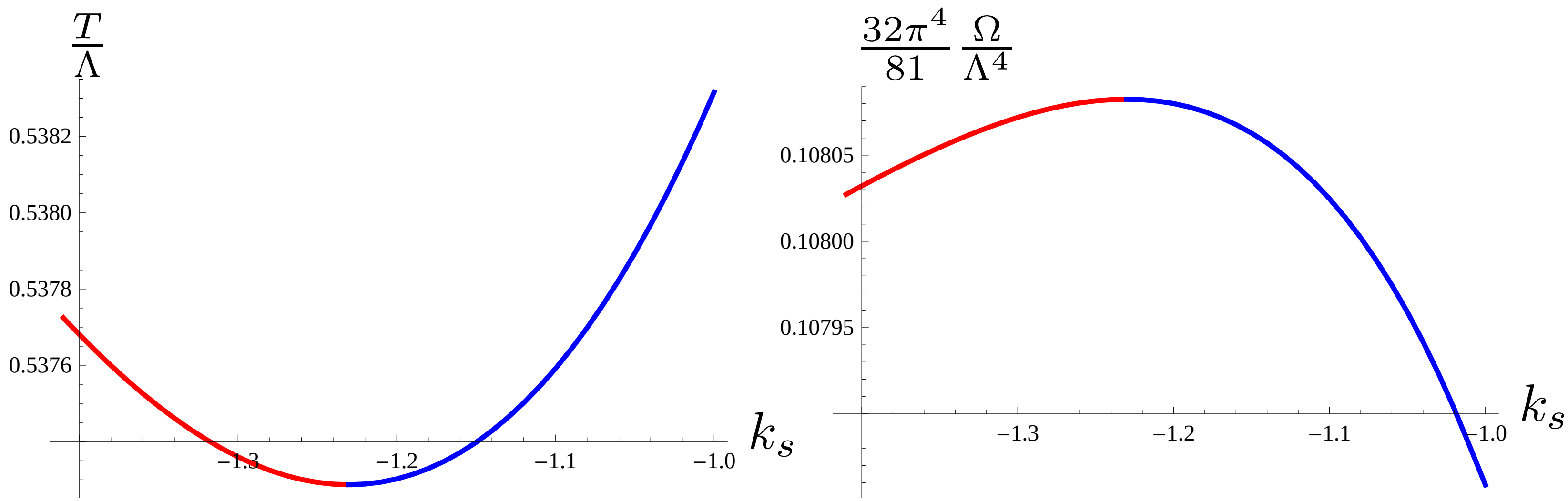

Figure 8: The reduced temperature $\frac{T}{\Lambda}$ (left plot) and the free energy densities Ω_o of the "ordered" phase (blue curve, right plot) and Ω_d of the "disordered" phase (red curve, right plot), of the strongly coupled cascading plasma as a function of the dual gravitational parameter k_s.

$\Rightarrow$ The dual gravitational parameter is uniquely related to $\frac{T}{\Lambda}$; for $T \gg \Lambda$:

$$k_s = 2\ln\frac{T}{\Lambda} + \mathcal{O}\left(\ln\left[\ln\frac{T}{\Lambda}\right]\right)$$

Holographic bulk viscosity

- $\mathcal{N}=2^*$ plasma at $m_f=0\,,m_b\neq 0$:

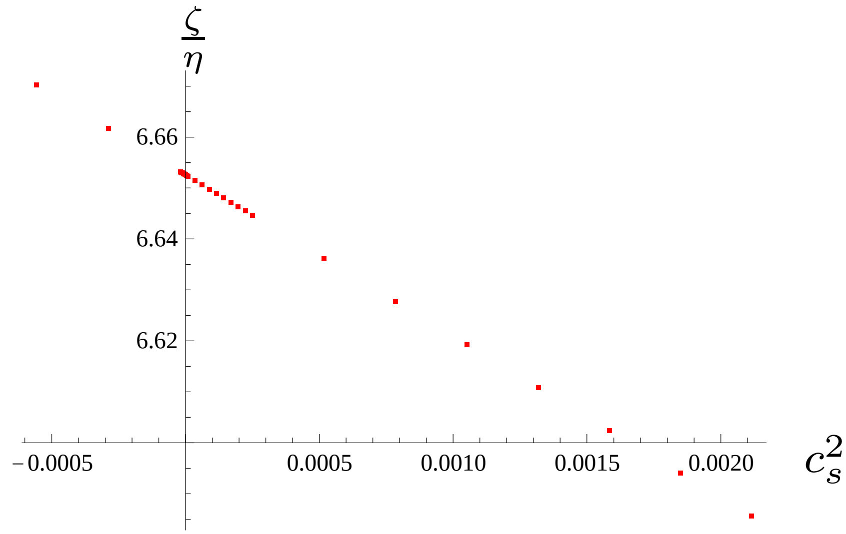

Figure 9: Ratio of viscosities $\frac{\zeta}{\eta}$ in $\mathcal{N}=2^*$ gauge theory plasma near the critical point. Note that the critical point corresponds to $c_s^2=0$.

$$\frac{\zeta}{\eta}\propto\left(c_s^2\right)^0\propto|t|^0$$

Recall:

- KKT model (A):

$$\zeta_{singular} \propto c_v \propto |t|^{-\alpha}$$

- Quasi-particle models (B):

$$\zeta_{singular} \propto |t|^{\alpha+4\beta-1}$$

- Onuki's dynamical model (C):

$$\zeta_{singular} \propto \xi^{z-\alpha/\nu} \propto |t|^{-z\nu+\alpha}$$

Thus:

- Model A is inconsistent with holographic analysis as it predicts divergent bulk viscosity,

$\zeta \ \propto \ |t|^{-1/2}$;

- Model B does not contradict our holographic analysis as it predicts that

$\zeta_{singular} \ \propto \ |t|^{3/2}$;

- Model C agrees with holographic analysis, provided the dynamical exponent z is

$$z \le 1$$

Note: A direct computations (to appear) show that $z = 0$ in $\mathcal{N} = 2^*$ plasma.

$\Rightarrow$ Identical results apply to cascading gauge theory

Conclusions

We argued that gauge/gravity correspondence is useful in understanding the dynamical critical phenomena of continuous phase transitions. Its utility lies in the motion of ’universality classes’ $\Rightarrow$ once we identify a gravitation model in a particular universality class, that model can essentially solve for the critical behavior of the full class. Might lead to some real experimental predictions!

Future directions

- Further understanding critical phenomena in the presence of chemical potentials. Here, we need to distinguish 2 cases: spontaneous breaking of discrete *or* continuous symmetries
- Can we understand (derive?) TDLG from holography?
- CFT's might have nontrivial z's $\Rightarrow$ would infinitesimal deformation of a CFT by a relevant operator near the transition produce a bulk viscosity governed by the same dynamical critical exponent?
- related. . . will dynamical susceptibility determine the same z as the bulk viscosity? (Note: the former is defined even for CFT's)

$\Rightarrow$ Need to study more models!