Problem 1

A hydrogen atom in the ground state is put in an electric field in the z-direction
\[ \vec{E} = E_0 \hat{z} \]  

a) Compute the first correction to the ground state \( |\psi^{(1)}\rangle \).

b) In the corrected state, compute, at first order, the mean value of the electric dipole moment \( \vec{P} = -e\vec{r} \).

c) From the formula \( \vec{P} = \alpha \vec{E} \), determine an expression for the polarizability \( \alpha \) of the atom. Find upper and lower bounds for \( \alpha \) giving explicit expressions for both.

Problem 2

Consider a harmonic oscillator with Hamiltonian
\[ H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \]  

If a perturbation \( V = \lambda x^4 \) is added, namely
\[ H = H_0 + V = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \lambda x^4 \]

a) Compute the first order correction to the energies of the first three states (ground state and two excited states).

b) Consider a trial wave-function given by a Gaussian
\[ \psi(x) = A e^{-\frac{1}{2}ax^2} \]  

Find the normalization constant \( A \) (so that \( \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \)) and determine \( a \) such that \( \langle \psi | H | \psi \rangle \) is a minimum. Consider the resulting \( \psi(x) \) as the new approximate ground state.

c) Compare the results of a) and b) for the energy of the ground state. Should they agree? Discuss the results.