662, Homework II, (2 problems)

Problem 1

The Lagrangian (density) for a Dirac fermion

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{0.1}$$

is invariant under the transformation

$$\psi(x) \to e^{iq\alpha}\psi(x) \tag{0.2}$$

a) Compute the corresponding conserved current (electromagnetic current).

b) Write the current in terms of oscillators and check that electron and positron have opposite charges.

c) Consider the transformation $\psi(x) \to e^{i\alpha\gamma^5}\psi(x)$. Check that this is a symmetry of the Lagrangian when $m = 0$ but not if $m \neq 0$. Write the corresponding conserved current and compute its divergence in the $m \neq 0$ case using the Dirac equation.

Problem 2

A Dirac field transforms in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of the Lorentz group. In the chiral representation[1] the Dirac spinor can be written as

$$\psi = \begin{pmatrix} \xi_L \\ \xi_R \end{pmatrix} \tag{0.3}$$

where $\xi_{L,R}$ are two component spinors such that ξ_L is in the $(\frac{1}{2}, 0)$ and ξ_R in the $(0, \frac{1}{2})$.

a) Show that the Dirac equation mixes both components.

b) Show that of $m = 0$ one can set *e.g.* $\xi_R = 0$ and still satisfy the Dirac equation (Weyl spinor).

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[1] This is the representation used in class where $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$.

c) Show that ξ_L^* (conjugate) transforms (after an appropriate change of basis) in the $(0, \frac{1}{2})$ representation and therefore can be identified with ξ_R.

d) Using the result of **c)** write a massive Dirac equation for just ξ_L (Majorana fermion). Show that this equation is not invariant under the charge symmetry $\xi_L \to e^{iq\alpha}\xi_L$ of problem 1) and therefore this fermion has no charge ($q = 0$).