# Normalizations for computing decay rates and cross sections

$|\psi_i\rangle \to |\psi_f\rangle$

$N_i$: # of initial particles (1 or 2) $\quad p_{i=1\ldots N_i}, s_i$ momenta and pol. of initial particles

$N_f$: # " final " (any) $\quad p_{f=1\ldots N_f}, s_f$ final particles.

$$\langle\psi_f| e^{-2iHT} |\psi_i\rangle \quad : \quad -T \to T$$

evolution $T\to\infty$

$$H = H_0 + \lambda \hat{V} \qquad e^{-2iHT} = e^{-2iH_0T}\,\hat{T}\left\{ e^{-i\int_{-T}^{T}\lambda V_I(t')dt'} \right\}$$

We need to integrate over final states in the range we want to measure (all states, states going into detectors etc.)

$|\psi_i\rangle$ : normalized to 1 to compute probabilities.

$$P_{i\to f} \sim |\langle\psi_f| e^{-2iHT} |\psi_i\rangle|^2$$

$$\sum_{|\psi_f\rangle} P_{i\to f} = 1 \quad ; \quad \sum_{|\psi_f\rangle} \langle\psi_i| e^{2iHT} \underbrace{|\psi_f\rangle\langle\psi_f|}_{\mathbb{1}} e^{-2iHT} |\psi_i\rangle$$

$$= \langle\psi_i|\psi_i\rangle = 1.$$

We need $\mathbb{1}$.

Single particle states.

$$|p,s\rangle = \sqrt{2\omega_p}\, a^\dagger_{p,s}|0\rangle \quad ; \quad [a_{p,s}, a^\dagger_{q,r}]_\pm = (2\pi)^3 \delta^{(3)}(p-q)\delta^{rs}$$

$$\langle q,r|p,s\rangle = \sqrt{2\omega_p}\sqrt{2\omega_q}\,\langle 0|a_{q,r}a^\dagger_{p,s}|0\rangle = 2\omega_p(2\pi)^3\delta^{(3)}(p-q)\,\delta^{rs}$$

$$\int \frac{d^3p}{2\omega_p}\frac{1}{(2\pi)^3}|p,r\rangle\langle p,r| = \mathbb{1}_{\text{single particle}}.$$

multi-particle id $\Rightarrow$ product of single particle id.

$$\sum_{\{N_f\}} \prod_{f=1}^{N_f} \int \frac{d^3p_f}{(2\pi)^3}\frac{1}{2\omega_f}\,|\langle \psi_f|e^{-2iHT}|\psi_i\rangle|^2 = \langle\psi_i|\psi_i\rangle = 1.$$

To normalize $|\psi_i\rangle$ we introduce a Large Box $V = L^3$

$\vec{p} = \frac{2\pi}{L}(n_1, n_2, n_3)$ : discrete spectrum.

Single particle: $\delta^{(3)}(p) = \frac{1}{(2\pi)^3}\int_V d^3x\, e^{ipx}$ ; $\delta^{(3)}(0) = \frac{V}{(2\pi)^3}$

$$\langle p,r|p,r\rangle = 2\omega_p\cancel{(2\pi)^3}\frac{V}{\cancel{(2\pi)^3}} \qquad |p,r\rangle^{\text{Box}} = \frac{1}{\sqrt{V}\sqrt{2\omega_p}}|p,r\rangle$$

$$|\psi_i\rangle = \prod_{i=1}^{N_i} \frac{1}{\sqrt{V}\sqrt{2\omega_i}} |p_i, r_i\rangle = \frac{1}{V^{N_i/2}} \prod_{i=1}^{N_i} \frac{1}{\sqrt{2\omega_i}} |p_i, r_i\rangle$$

$$\underline{\langle\psi_i|\psi_i\rangle = 1}$$

$$\langle\psi_f| e^{-iHT} |\psi_i\rangle = S_{fi} \qquad \text{S-matrix}$$

$$S_{fi} = \underbrace{\delta_{fi}}_{\text{nothing happens}} + i T_{fi}$$

$\neq 0$ only if we include interactions

Furthermore:

$$iT_{fi} = i(2\pi)^4 \delta^{(4)}(P_f - P_i) \underbrace{\mathcal{M}_{fi}}$$

what we computed with Feynman diagrams.

Assuming $|\psi_f\rangle \neq |\psi_i\rangle$

$$P_{i\to f} = \frac{1}{V^{N_i}} \prod_{i=1}^{N_i} \frac{1}{2\omega_i} \prod_{f=1}^{N_f} \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2\omega_f} (2\pi)^8 \left(\delta^{(4)}(P_f - P_i)\right)^2 |\mathcal{M}_{fi}|^2$$

$$\left(\delta^{(4)}(P_f - P_i)\right)^2 = \underbrace{\delta^{(4)}(P_f - P_i)}_{\text{energy and momentum conservation}} \delta^{(3)}(0) \cdot \overbrace{\delta^{(t)}(0)}^{\text{regularise } -T \to T} = \delta^{(4)}(P_f - P_i) \frac{V(2T)}{(2\pi)^4}$$

$$P_{i\to f} = (2T)\, V^{1-N_i} (2\pi)^4 \prod_{i=1}^{N_i} \frac{1}{2\omega_i} \prod_{f=1}^{N_f} \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2\omega_f}$$

$$\cdot\, \delta^{(4)}(p_f - p_i)\, |\mathcal{M}_{fi}|^2$$

<u>$N_i = 1$</u> Decay rate.

$$\Gamma_{i\to f} = \frac{P_{i\to f}}{2T} = \frac{(2\pi)^4}{2M_i} \prod_{f=1}^{N_f} \int \frac{d^3 p_f}{(2\pi)^3} \frac{1}{2\omega_f}\, \delta^{(4)}(p_f - p_i)\, |\mathcal{M}_{fi}|^2$$

<u>$N_i = 2$</u> cross section.

$$\frac{P_{i\to f}}{2T} = \frac{1}{V} (2\pi)^4 \prod_{i=1}^{2} \frac{1}{2\omega_i} \prod_{f=1}^{N_f} \int \frac{d^3 p_f}{(2\pi)^3} \frac{\delta^{(4)}(p_f - p_i)}{2\omega_f} |\mathcal{M}_{fi}|^2$$

flux: $\rho \cdot v$ : $\frac{1}{V} |v_A - v_B|$

$\uparrow$ relative velocity

$$\frac{P_{i\to f}}{2T\ \text{flux}} = \frac{(2\pi)^4}{|v_A - v_B|} \prod_{i=1}^{2} \frac{1}{2\omega_i} \prod_{f=1}^{N_f} \int \frac{d^3 p_f}{(2\pi)^3} \frac{\delta^{(4)}(p_f - p_i)}{2\omega_f} |\mathcal{M}_{fi}|^2$$

$$\sigma_{i\to f} = \frac{(2\pi)^4}{|v_A - v_B|} \frac{1}{2\omega_A \, 2\omega_B} \left( \prod_{f=1}^{N_f} \int \frac{d^3 p_f}{(2\pi)^3 \, 2\omega_f} \right) \delta^{(4)}(p_f - p_A - p_B) |\mathcal{M}_{f0}|^2$$

$A + B \to |\psi_f\rangle$

Example : Higgs decay into two fermions

$$\phi_H \; ; \; \bar{\psi}_f \, , \, \psi_f$$

$$V = g_f \int d^3x \; \bar{\psi}_f \psi_f \phi_H$$

$$\langle \underbrace{p_1, s_1}_{f} ; \underbrace{p_2\, s_2}_{\bar{f}} | -i g_f \int d^4x \; \bar{\psi}_f \psi_f \phi_H | p_i \rangle =$$

$$= -i g_f \int d^4x \; e^{-i p_i \cdot x} \; \bar{u}^{s_1}_{p_1} \; v^{s_2}_{p_2} \; e^{i p_1 x + i p_2 x}$$

$$= -i g_f \; (2\pi)^4 \; \delta^{(4)}(p_1 + p_2 - p_i) \; \bar{u}^{s_1}_{p_1} \; v^{s_2}_{p_2}$$

$$\mathcal{M}_{fi} = - g_f \; \bar{u}^{s_1}_{p_1} \; v^{s_2}_{p_2}$$

$$\Gamma = \frac{(2\pi)^4}{2 m_H} g_f^2 \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} \; \frac{1}{2\omega_1 2\omega_2} \; \delta^{(4)}(p_1 + p_2 - p_i) \; \left| \bar{u}^{s_1}_{p_1} v^{s_2}_{p_2} \right|^2$$

$$\left(u^{\dagger s_1}_{p_1}\gamma^0 v^{s_2}_{p_2}\right)^{\dagger} \bar{u}^{s_1}_{p_1} v^{s_2}_{p_2} = v^{\dagger s_2}_{p_2}\gamma^0 u^{s_1}_{p_1} \bar{u}^{s_1}_{p_1} v^{s_2}_{p_2}$$

$$= \bar{v}^{s_2}_{p_2} u^{s_1}_{p_1} \bar{u}^{s_1}_{p_1} v^{s_2}_{p_2} = \mathrm{Tr}\left(v^{s_2}_{p_2}\bar{v}^{s_2}_{p_2} u^{s_1}_{p_1}\bar{u}^{s_1}_{p_1}\right)$$

Sum over final pol. $\sum_s u^s_p \bar{u}^s_p = \not{p} + m$ $\quad \sum_s v^s_p \bar{v}^s_p = \not{p} - m$

$$\sum_{s_1=1,2}\sum_{s_2=1,2} = \mathrm{Tr}\,(\not{p}_2 + m_f)(\not{p}_1 + m_f) = 4\,(p_1\cdot p_2 - m_f^2)$$

$p_2 \longleftarrow H \longrightarrow p_1$

$(\varepsilon_1, \vec{p})\ (\varepsilon_2, -\vec{p}) \Rightarrow \varepsilon_1 = \varepsilon_2 \qquad \varepsilon^2 - \vec{p}^{\,2} = m_f^2$

$\varepsilon_1 + \varepsilon_2 = m_H \qquad \varepsilon_1 = \varepsilon_2 = \frac{m_H}{2}$

$$p_1 p_2 = \varepsilon^2 + \vec{p}^{\,2} = \varepsilon^2 + \varepsilon^2 - m_f^2 = 2\,\frac{m_H^2}{4} - m_f^2 = \frac{m_H^2}{2} - m_f^2$$

$$\sum_{\substack{s_1=1,2\\ s_2=1,2}} = 4\left(\frac{m_H^2}{2} - 2m_f^2\right) = 2\,(m_H^2 - 4m_f^2) > 0$$

should be positive.

$$\Gamma = \frac{(2\pi)^4}{2m_H} g_f^2 \cdot 2\,(m_H^2 - 4m_f^2)\underbrace{\int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2\omega_1 2\omega_2}\delta(\omega_1+\omega_2-m_H)\,\delta^{(3)}(\vec{p}_1+\vec{p}_2)}$$

$$\frac{1}{(2\pi)^3}\int\frac{d^3p_1}{(2\pi)^3}\frac{1}{4\omega_1^2}\delta(2\omega - m_H)$$

$$= \frac{4\pi}{\cancel{4}(2\pi)^6}\int_0^\infty \frac{p_1^2\,dp_1\,\delta\left(2\sqrt{m_f^2+p^2} - m_H\right)}{m_H^2/\cancel{4}}$$

$$p = \sqrt{\varepsilon^2 - m_f^2} = \sqrt{\frac{m_H^2}{4} - m_f^2} = \frac{1}{2}\sqrt{m_H^2 - 4m_f^2}$$

$$\delta\left(2\sqrt{p^2+m_f^2} - m_H\right) = \frac{\delta\left(p - \frac{1}{2}\sqrt{m_H^2 - 4m_f^2}\right)}{\cancel{2}\,\frac{2p}{\cancel{2}\sqrt{p^2+m_f^2}}} = \frac{\varepsilon}{2p}\,\delta(p - \bar{p})$$

$$= \frac{4\pi}{(2\pi)^6}\,\frac{1}{m_H^2}\,\frac{\varepsilon}{2\cancel{p}}\,p^{\cancel{2}} = \frac{4\pi}{2^6\pi^6}\,\frac{1}{2}\,\frac{1}{m_H^2}\,\frac{\cancel{m_H}}{2}\,\frac{1}{2}\sqrt{m_H^2 - 4m_f^2}$$

$$\Gamma = \frac{(2\pi)^4}{2m_H}\,g_f^2 \cdot 2\left(m_H^2 - 4m_f^2\right)\frac{4\pi}{\pi^6 2^9}\,\frac{\sqrt{m_H^2 - 4m_f^2}}{m_H}$$

$$\Gamma = \frac{g_f^2}{8\pi}\,\frac{\left(m_H^2 - 4m_f^2\right)^{3/2}}{m_H^2}$$

$$\boxed{m_W^2 = \frac{\pi\alpha}{\sqrt{2}\,G_F\,s^2\theta_W}}$$

$$g_f = -\frac{m_f}{v} \qquad \boxed{\frac{1}{v^2} = \sqrt{2}\,G_F}$$

$$\boxed{\Gamma = \frac{m_f^2}{8\pi}\sqrt{2}\,G_F\,m_H\left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2}}$$

or $$\Gamma = \left(\frac{\alpha m_H}{8 s^2\theta_W}\right)\frac{m_f^2}{m_W^2}\underbrace{\left(1 - \frac{4m_f^2}{m_H^2}\right)^{3/2}}_{\approx 1}$$

$$\Gamma = \frac{m_f^2}{8\pi}\sqrt{2}\, G_F\, m_H \left(1 - \frac{4 m_f^2}{m_H^2}\right)^{3/2} \times \overset{\propto N_c}{3}$$

$$\boxed{m_b = 4.2\ \text{GeV}} \qquad m_H = 125\ \text{GeV} \qquad G_F = 1.166\times 10^{-5}\ \text{GeV}^{-2}$$

$$\left(1 - \frac{4 m_f^2}{m_H^2}\right)^{3/2} \simeq 1.$$

$$\Gamma = 4.34\times 10^{-3}\ \text{GeV} \qquad\qquad \Gamma = 4.2\times 10^{-3}\ \text{GeV}$$

↑ too high!

renormalization of mass

$$\overline{m}_b(Q^2) = \left(\frac{\ln(m_b/\Lambda)}{\ln(Q/\Lambda)}\right)^{4/b_0} m_b \quad ; \quad b_0 = 11 - \frac{2}{3} n_f$$

$$n_f = 5 \rightarrow b_0 = 23/3$$

$$\boxed{\Lambda_{QCD} = 150\ \text{MeV}}$$

$$\overline{m}_b(m_H^2) = \left(\frac{\ln(m_b/\Lambda)}{\ln(m_H/\Lambda)}\right)^{\frac{12}{23}} m_b = 2.9\ \text{GeV}$$

$$\boxed{\Gamma = 2.07\times 10^{-3}\ \text{GeV} \ll M_H}$$ ($\simeq 50\%$) better (actually 60%)

$$\tau = \frac{\hbar}{\Gamma} \quad ; \quad \frac{\hbar c}{c} = \frac{200\ \text{MeV fm}}{3\times 10^8\ \text{m/s}} = \frac{0.2\ \text{GeV}\times 10^{-15-8}}{3}\ \text{s} = 6.67\times 10^{-25}\ \text{GeV s}$$

$$\boxed{\tau \sim 3.22\times 0^{-22}\ \text{s}}$$

## Scattering

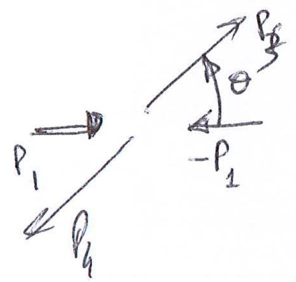
$$\langle p_3, p_4 | (-i\lambda) \int d^4x \, \phi^4(x) \, | p_1, p_2 \rangle =$$

$$= (-i\lambda) \int d^4x \; e^{ip_3x + ip_4x - ip_1x - ip_2x}$$

$$= (-i\lambda)\,(2\pi)^4 \, \delta^{(4)}(p_1 + p_2 - p_3 - p_4)$$

$$\boxed{\mathcal{M}_{fi} = -\lambda}$$

$$\sigma_{i\to f} = \frac{(2\pi)^4}{|v_1 - v_2|} \; \frac{1}{2\omega_1 2\omega_2} \int \frac{d^3p_3}{(2\pi)^3} \, \frac{d^3p_4}{(2\pi)^3} \, \frac{1}{2\omega_3 2\omega_4} \, \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \, \lambda^2$$




$$\vec{p}_2 = -\vec{p}_1 \qquad \vec{p} = \frac{mv}{\sqrt{1-v^2}} \qquad \varepsilon = \frac{m}{\sqrt{1-v^2}}$$

$$v = |\vec{p}|/\varepsilon$$

$$\sigma_{i\to f} = \frac{(2\pi)^4}{2|\vec{p}_i|/\cancel{\varepsilon_i}} \; \frac{\lambda^2}{4\,\varepsilon_i^{\cancel{2}}} \int \frac{d^3p_3}{(2\pi)^3} \, \frac{d^3p_4}{(2\pi)^3} \, \frac{\delta^{(3)}(\vec{p}_3 + \vec{p}_4)\,\delta(\varepsilon_3 + \varepsilon_4 - 2\varepsilon_i)}{4\,\varepsilon_3\varepsilon_4}$$

$$= \frac{2^4\pi^4}{2^{11}\pi^6} \; \frac{\lambda^2}{\varepsilon_i |\vec{p}_i|} \int \frac{d^3p_3}{\varepsilon_3^2} \, \delta(2\varepsilon_3 + 2\varepsilon_i)$$

$$\mathcal{E} = \sqrt{p^2+m^2}$$

$$d\mathcal{E} = \frac{\cancel{2}p\,dp}{\cancel{2}\sqrt{p^2+m^2}} = \frac{p\,dp}{\mathcal{E}}$$

$$\sigma_{i\to f} = \frac{1}{2^7\pi^2}\frac{\lambda^2}{\mathcal{E}_i|\vec{p}_i|}\int d\Omega \underbrace{\int \frac{p_f^2\,dp_f}{\mathcal{E}_f^2}\frac{1}{2}\delta(\mathcal{E}_f-\mathcal{E}_i)}$$

$$\int_0^\infty p_f \frac{\cancel{\mathcal{E}_f}\,d\mathcal{E}_f}{\mathcal{E}_f^2}\frac{1}{2}\delta(\mathcal{E}_f-\mathcal{E}_i)$$

$$\frac{p_f}{2\mathcal{E}_f} \qquad |p_f| = |p_i|$$

$$= \frac{\lambda^2\ \cancel{|p_f|}}{2^7\pi^2\cancel{|\vec{p}_i|}2\mathcal{E}_i^2}\int d\Omega$$

$$s = (p_1+p_2)^2 = (E_{CM})^2$$

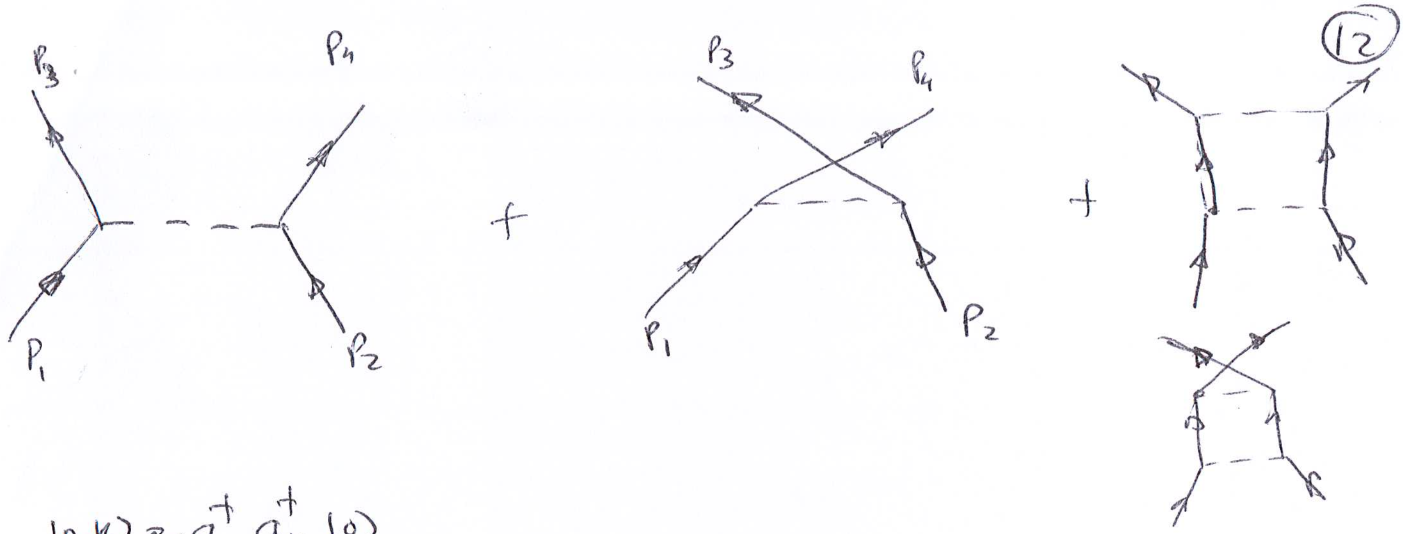
$$E_{CM} = 2\mathcal{E}_i$$

$$\frac{d\sigma_i}{d\Omega} = \frac{\lambda^2}{2^6\pi^2E_{CM}^2} = \frac{\lambda^2}{2^6\pi^2 s} = \frac{\lambda^2}{64\pi^2 s}$$

$$\sigma_T = \frac{\lambda^2}{2\times16\pi s} = \frac{\lambda^2}{32\pi s}$$

No corrections to external legs.

↑ bosons. half-integral




$|p,k\rangle \approx a_p^\dagger a_k^\dagger |0\rangle$

$\langle p,k| = (|p,k\rangle)^\dagger = \langle 0| a_k a_p$ $\qquad \langle p k | p k\rangle \sim 1$

$\langle p_3 p_4 | \bar\psi \psi \bar\psi \psi | p_1 p_2\rangle \sim \langle 0| a_{p_4} a_{p_3} \bar\psi \psi \bar\psi \psi \, a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle$

$+$ $\quad$ $+$ $\quad$ $-$

~~$\langle p_3 p_4 | \bar\psi \psi \bar\psi \psi | p_1 p_2 \rangle \sim \langle$~~

$$(-ig)^2 \left[ \bar u^{s_3}_{p_3} u^{s_1}_{p_1} \bar u^{s_4}_{p_4} u^{s_2}_{p_2} \frac{i}{(p_3-p_1)^2 - m^2 + i\varepsilon} - (\bar u^{s_3}_{p_3} u^{s_2}_{p_2})(\bar u^{s_4}_{p_4} u^{s_1}_{p_1}) \frac{i}{(p_4-p_1)^2 - m^2 + i\varepsilon} \right]$$

non-relativistic

$$+ig^2 \left( 2m\delta^{s_3 s_1} \, 2m\delta^{s_4 s_2} \frac{1}{(\vec p_1 - \vec p_3)^2 + m^2} - \frac{2m\delta^{s_2 s_3} \, 2m\delta^{s_1 s_4}}{(\vec p_4 - \vec p_1) + m^2} \right.$$

$p_1 \approx (m, \vec p_1)$

$p_3 \sim (m, \vec p_3)$

$$i\mathcal{M} = i\,4m^2 g^2 \left( \frac{\delta^{s_1 s_3} \delta^{s_2 s_4}}{(\vec p_1 - \vec p_3)^2 + m^2} - \frac{\delta^{s_2 s_3} \delta^{s_1 s_4}}{(\vec p_4 - \vec p_1) + m^2} \right)$$

Spin is conserved at low energies.

momentum transfer

# Born approximation

$$\langle p' | iT | p \rangle = -i \tilde{V}(q) (2\pi) \delta(E_{p'} - E_p) \qquad q = p' - p$$

$$\tilde{V}(q) = \frac{-g^2}{q^2 + m_\phi^2}$$

$$V(\vec{x}) = \int \frac{d^3 q}{(2\pi)^3} \frac{-g^2}{q^2 + m_\phi^2} e^{iq\cdot x} = -\frac{g^2}{4\pi} \frac{1}{r} e^{-m_\phi r}$$

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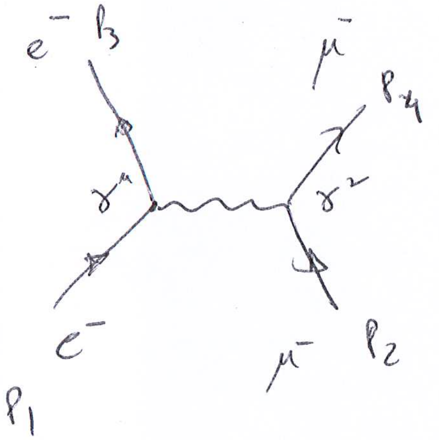



$e^- + \mu^- \to e^- + \mu^-$

$$i\mathcal{M} = (-ie)^2 \left(\bar{u}_{p_3}^{s_3} \gamma^\mu u_{p_1}^{s_1}\right)_{e^-} \frac{(-i\eta_{\mu\nu})}{(p_1 - p_3)^2 + i\varepsilon} \left(\bar{u}_{p_4}^{s_4} \gamma^\nu u_{p_2}^{s_2}\right)_{\mu^-}$$

$$|\mathcal{M}|^2 = \frac{e^4 \eta_{\mu\nu} \eta_{\mu'\nu'}}{(p_1 - p_3)^2 + i\varepsilon} \left(\bar{u}_{p_3}^{s_3} \gamma^\mu u_{p_1}^{s_1}\right) \left(\bar{u}_{p_3}^{s_3} \gamma^{\mu'} u_{p_1}^{s_1}\right)^\dagger \left(\bar{u}_{p_4}^{s_4} \gamma^\nu u_{p_3}^{s_3}\right)$$
$$\left(\bar{u}_{p_4}^{s_4} \gamma^{\nu'} u_{p_3}^{s_3}\right)^\dagger$$

$$(\bar{u}_3 \gamma^\mu u_1)^\dagger = u_1^\dagger (\gamma^\mu)^\dagger \underbrace{(\gamma^0)^\dagger}_{\gamma^0} u_3 = \bar{u}_1 \gamma^\mu u_3$$

$$(\gamma^0)^\dagger = \gamma^0 \qquad (\gamma^i)^\dagger = -\gamma^i = \gamma_0 \gamma^i \gamma_0$$

$$(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

$\frac{1}{2}\sum_{s_1}$ average over initial pol.

$\sum_{s_3}$ sum over final.

$$\frac{1}{4}\sum_{s_{1,2,3,4}} |\mathcal{M}|^2$$

$$\mathrm{Tr}\left[\bar{u}^{s_3}_{p_3} \gamma^\mu u^{s_1}_{p_1} \bar{u}^{s_1}_{p_1} \gamma^{\mu'} u^{s_3}_{p_3}\right] = \mathrm{Tr}\left((\not{p}_3 + m)\gamma^\mu (\not{p}_1 + m)\gamma^{\mu'}\right)$$

$$= \mathrm{Tr}(\not{p}_3 \gamma^\mu \not{p}_1 \gamma^{\mu'}) + m^2\, 4\eta^{\mu\mu'} = 4\,\cancel{(p_1 p_3)}$$

$$= 4 p_3^\mu p_1^{\mu'} - 4(p_1 p_3)\eta^{\mu\mu'} + 4 p_3^{\mu'} p_1^\mu + 4 m^2 \eta^{\mu\mu'}$$

$$\frac{1}{4}\sum_{s_i} |\mathcal{M}|^2 = \frac{16}{4} \frac{e^4 \eta_{\mu\nu}\eta_{\mu'\nu'}}{((p_1 - p_3)^2)^2} \left(p_1^{\mu'} p_3^\mu + p_1^\mu p_3^{\mu'} + (m^2 - p_1 p_3)\eta^{\mu\mu'}\right) \cdot$$

$$\cdot \left(p_2^\nu p_4^{\nu'} + p_4^\nu p_2^{\nu'} + (M^2 - p_2 p_4)\eta^{\nu\nu'}\right)$$

$$= \frac{4e^4}{t^2}\Big[(p_2 p_3)(p_1 p_4) + (p_1 p_2)(p_3 p_4) + (M^2 - p_2 p_4)(p_1 p_3) +$$

$$+ (p_1 p_2)(p_3 p_4) + (p_1 p_4)(p_2 p_3) + (M^2 - p_2 p_4)(p_1 p_3) + (m^2 - p_1 p_3)\big[(p_2 p_4) + (p_2 p_4) + 4(M^2 - (p_2 p_4))\big]\Big]$$

unpolarized.

$$\frac{1}{4}\sum_{s_i}|M|^2 = \frac{4e^4}{t^2}\Big[2(p_2p_3)(p_1p_4) + 2(p_1p_2)(p_3p_4) + 2M^2(p_1p_3) -$$

$$-2\cancel{(p_1p_3)(p_2p_4)} + 2m^2(p_2p_4) + 4m^2M^2 - 4m^2(p_2p_4)$$

$$-2\cancel{(p_1p_3)(p_2p_4)} - 4M^2(p_1p_3) + 4\cancel{(p_1p_3)(p_2p_4)}\Big]$$

$$= \frac{8e^4}{t^2}\Big[(p_2p_3)(p_1p_4) + (p_1p_2)(p_3p_4) - M^2(p_1p_3) - m^2(p_2p_4) + 2m^2M^2\Big]$$

$$s = (p_1+p_2)^2 = m^2 + M^2 + 2p_1p_2$$

$$t = (p_3-p_1)^2 = 2m^2 - 2p_1p_3 = 2M^2 - 2p_2p_4$$

$$u = (p_4-p_1)^2 = m^2 + M^2 - 2p_1p_4 = m^2 + M^2 - 2p_2p_3$$

$$= \frac{8e^4}{t^2}\Big[\frac{1}{2}(m^2+M^2-u)\,\frac{1}{2}(m^2+M^2-u) + \frac{1}{4}(s-m^2-M^2)^2 -$$

$$-\frac{1}{2}M^2(2m^2-t) - \frac{m^2}{2}(2M^2-t) + 2m^2M^2\Big]$$

$$= \frac{8e^4}{t^2}\Big[\frac{1}{4}\Big[m^4 + M^4 + 2\cancel{m^2M^2} + u^2 - 2um^2 - 2uM^2 + s^2 + m^4 + M^4 -$$

$$-2sm^2 - 2sM^2 + 2\cancel{m^2M^2} - \cancel{4m^2M^2} + 2M^2t - 4m^2M^2 + 2m^2t + 8m^2M^2\Big]$$

$$= \frac{2e^4}{t^2}\Big[2m^4 + 2M^4 + 4m^2M^2 + s^2 + u^2 + M^2(2t-2u-2s) + m^2(2t-2s-2u)\Big]$$

$$= \frac{2e^4}{t}\left[2(m^2+M^2)^2 + s^2 + u^2 + 2(m^2+M^2)(t-u-s)\right]$$

$$(p_1+p_2-p_3-p_4)^2 = 0$$

$$m^2+M^2+m^2+M^2+2p_1p_2-2p_1p_3-2p_1p_4+2p_2p_3-2p_2p_4+2p_3p_4=0$$

$$\cancel{2m^2}+\cancel{2M^2}+s-m^2-M^2+t\cancel{-2m^2}+u-m^2-M^2+u-m^2-M^2+t\cancel{-2M^2}=0$$

$$+s-m^2+M^2$$

$\cancel{8}$

$$2(s+t+u) = 4(m^2+M^2)$$

$$s+t+u = 2(m^2+M^2)$$

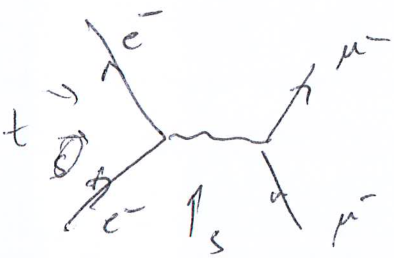
$$u+s = 2(m^2+M^2)-t$$

$$t-u-s = 2t-2(m^2+M^2)$$

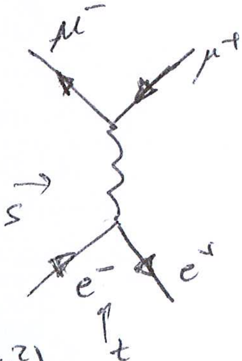
$$\frac{2e^4}{t}\left[2(m^2+M^2)^2+s^2+u^2+4(m^2+M^2)t-4(m^2+M^2)^2\right]$$

large energy.

$$\frac{1}{4}\sum_{s,v}|\mathcal{M}|^2 = \frac{2e^4}{t^2}(s^2+u^2)$$




$e^+ + e^- \to \mu^+ + \mu^-$




$s \rightleftarrows t$

$$|\mathcal{M}|^2 = \frac{2e^4}{s^2}(t^2+u^2)$$

<u>CM</u>

$v_A = -v_B = |\vec{p}|/\varepsilon$ high-energy

$$\sigma_{e^- + e^+ \to \mu^- + \mu^+} = \frac{(2\pi)^4}{2\,\frac{|p_i|}{\cancel{\varepsilon_i}}} \frac{1}{4\varepsilon_i^2} \int \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} \frac{1}{2\omega_3 2\omega_4} \cdot$$

$$\cdot\, \delta^{(3)}(p_3 + p_4 \underbrace{- p_1 - p_2}_{0})\, \delta^{\varepsilon}(\varepsilon_3 + \varepsilon_4 - 2\varepsilon_i) \cdot$$

$$\cdot\, \frac{2e^4}{s^2}(t^2 + u^2)$$

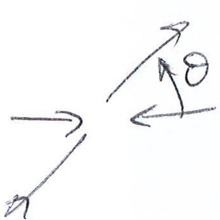
$$= \frac{(2\pi)^4}{8} \frac{1}{\varepsilon_i p_i} \frac{\cancel{2}e^4}{2^6 \pi^6} \int d^3p_3 \frac{1}{4\omega_3^2} \frac{1}{2} \delta(\varepsilon_3 - \varepsilon_i) \cdot$$

$$\int p_3^2 dp_3 \cdot \frac{t^2+u^2}{s^2}$$

$$\int p_3 \varepsilon_3 d\varepsilon_3$$

$$= \frac{2^{5-3-6-3-1}}{\cancel{\varepsilon_i} p_i \pi^2} e^4 \int d\Omega \frac{p_3 \cancel{\varepsilon_i}}{\varepsilon_i^2} \frac{t^2+u^2}{s^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{2^7 \pi^2} \frac{p_3}{p_i} \frac{1}{\varepsilon_i^2} \frac{t^2+u^2}{s^2}$$



$p_1 = (\varepsilon_i, \vec{p})$ $\quad p_3 = (\varepsilon_i, \vec{p}_3)$

$p_2 = (\varepsilon_i, -\vec{p})$ $\quad p_4 = (\varepsilon_i, -\vec{p}_3)$

$s=(2\varepsilon_i)^2$ $\qquad p^2=\varepsilon^2-m^2 \quad p_3^2=\varepsilon^2-M^2$

$$t=(\vec{p}-\vec{p}_3)^2=p^2+p_3^2-2\vec{p}\,\vec{p}_3=2\varepsilon^2-m^2-M^2-2\,p\,p_3\,c\theta$$

$$\simeq 2\varepsilon^2(1-c\theta)$$

$$u=(p_4-p_1)^2=(-\vec{p}_3-\vec{p})^2=\cancel{2pp_4\ldots}=\vec{p}^2+\vec{p}_4^2+2\vec{p}\,\vec{p}_3$$

$$=2\varepsilon^2+2\varepsilon^2 c\theta$$

$$\frac{t^2+u^2}{s^2}=\frac{4\varepsilon^4\left[(1-c\theta)^2+(1+c\theta)^2\right]}{(2\varepsilon)^4}=\frac{1}{4}\left[2+2c^2\theta\right]$$

$$\frac{t^2+u^2}{s^2}=\frac{1}{2}(1+c^2\theta)$$

$$\frac{d\sigma}{d\Omega}=\frac{e^4}{2^5\pi^2}\,\frac{1}{s}\,\frac{1}{2}(1+c^2\theta)=\frac{e^4}{64\pi^2}\,\frac{1+c^2\theta}{s}$$

$\alpha=e^2/4\pi$

$$\mu+\mu^3/3=2+2/3=8/3$$

$$\int\frac{d\sigma}{d\Omega}\,s\theta\,d\theta\,d\varphi=\frac{2\pi e^4}{64\pi^2}\,\frac{1}{s}\int_{-1}^{1}d\mu\,(1+\mu^2)=\frac{e^4}{12\pi}\,\frac{1}{s}$$

$$=\frac{16\pi^2\alpha^2}{12\pi s}=\frac{4\pi\alpha^2}{3s}$$ ~~muon~~ ~~muon production~~

$$R=\frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=3\sum_q e_q^2$$