Lecture 2

Lagrangian approach. Classical field theory:

Given fields $\phi_a(\vec{x},t)$ we define

$$S = \int d^4x \; \mathcal{L}(\phi_a, \partial_\mu \phi_a)$$

minimal action principle:

$$\phi_a + \delta\phi_a \quad ; \partial_\mu \phi_a \to \partial_\mu \phi_a + \partial_\mu \delta\phi_a$$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a(x) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \partial_\mu \phi_a \right) \underset{\text{extremal.}}{=} 0$$

$$= \int d^4x \; \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a \right) +$$

$$+ \int d^4x \; \delta\phi_a(x) \left(\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0$$

For any $\delta\phi_a(x)$ (up to b.c)

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0$$

$$\delta S = \oint dS^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta\phi_a$$

usually we take $\delta\phi_a = 0$ at ∞ in space and at $t \to \pm\infty$.
$\to$ Dirichlet

We can also take $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0$. (Neumann).

equations of motion: $\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} = 0.$

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Hamiltonian ~~approach~~:

$$\cancel{\Pi_a(x) =} \quad L = \int d^3x \, \overbrace{\mathcal{L}(\phi_a, \partial_\mu \phi_a)}^{\text{Lagrangian density}}$$

$$\Pi_a(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a} \quad ; \quad \cancel{\phi} \; \cancel{\phi}$$

Canonical momentum: $P_a = \int \Pi_a(\vec{x}) \, d\vec{x}$.

$$H = \int d^3x \, \mathcal{H}(\vec{x}) = \int d^3x \, \Pi_a(\vec{x}) \dot{\phi}_a(x) - \int \mathcal{L} \, d^3x$$

Noether's theorem:
Symmetries are associated with conserved quantities. In QFT conserved quantities are locally conserved.

$$\partial_\mu j^\mu = 0 \qquad Q = \int d^3x \, j^0(\vec{x})$$

$$\partial_0 Q = \int d^3x \, \partial_0 j^0 = - \int d^3x \, \partial_i j^i = - \oint dn^i j^i \underset{\text{bc.c.}}{=} 0$$

a symmetry is a variation of the fields that does not change the action;

$$\phi_a \to \delta\phi_a$$

$$\delta\mathcal{L} = \partial_\mu J^\mu \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a + \frac{\delta \mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu\delta\phi_a - \partial_\mu J^\mu = 0$$

On e.o.m.

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - J^\mu\right) = 0$$

$$\boxed{j^\mu = \frac{\partial \mathcal{L}}{\partial \phi_a}\delta\phi_a - J^\mu}$$

Translation invariance.

$$x_\mu \to x_\mu + a^\mu$$

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} \partial_\mu \phi_a + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_a} \partial_{\mu\nu} \phi_a$$

$$= \partial_\nu \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_a} \partial_\mu \phi_a + \frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_a} \partial_{\mu\nu} \phi_a$$

$$\partial_\nu \eta^{\mu\nu} \mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_a} \partial_\mu \phi_a \right)$$

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial \partial_\nu \phi_a} \partial_\mu \phi_a - \eta^{\mu\nu} \mathcal{L} \right) = 0$$

$$\boxed{T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi_a)} \partial_\mu \phi_a - \eta^{\mu\nu} \mathcal{L}}$$

$$\boxed{\partial_\mu T^{\mu\nu} = 0}$$

$$\boxed{\begin{aligned} E &= \int d^3x \, T_{00} \\ \vec{P} &= \int d^3x \, T_{0i} \end{aligned}}$$

Example : scalar field :

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

e.o.m. $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial_\mu\phi}\right) - \frac{\partial \mathcal{L}}{\partial\phi} = 0$ $\quad \boxed{\partial_\mu\partial^\mu\phi - m^2\phi = 0}$

$m^2 = 0 \Rightarrow$ wave equation.

$\mathcal{H}$, $\quad \pi(x) = \partial_0\phi$

$$\mathcal{H} = \partial_0\phi\,\partial_0\phi - \frac{1}{2}\partial_0\phi\,\partial_0\phi + \frac{1}{2}\nabla\phi\nabla\phi + \frac{1}{2}m^2\phi^2$$

$$\boxed{\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2}$$

$$\boxed{T_{\mu\nu} = \partial_\nu\phi\,\partial_\mu\phi - \eta^{\mu\nu}\mathcal{L}.}$$

$$T_{00} = \dot\phi^2 - \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2$$

$$\boxed{T_{00} = \frac{1}{2}\dot\phi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2}$$

in fact $T_{00} = \underbrace{\dot\phi\frac{\partial \mathcal{L}}{\partial\dot\phi}}_{\pi\dot\phi} - \mathcal{L}$ $\quad$ ✓.

$$T_{0i} = \dot\phi\,\phi_{,i} = \pi\,\partial_i\phi$$

$$\boxed{\vec{P} = \int \pi\,\vec\nabla\phi}$$

Quantization

Try decoupling modes

$$\phi(\vec{x}) = \int d^3\vec{k}\,\left(f_k\, e^{i\vec{k}\vec{x}}\, a_{\vec{k}} + f_k\, e^{-i\vec{k}\vec{x}}\, a_k^\dagger \right)$$

we take $f_k > 0$ real by redefining a_k if necessary

some constant

$$[a_k, a_{k'}^\dagger]_\pm = C\,\delta(\vec{k}-\vec{k}')$$

either boson or fermions.

cannot redefine a_k anymore.

$$\Pi(\vec{x}) = \int d^3\vec{k}\,\left(g_k\, e^{i\vec{k}\vec{x}}\, a_{\vec{k}} + g_k^*\, e^{-i\vec{k}\vec{x}}\, a_k^\dagger \right)$$

$$\nabla\phi = \int d^3k\,\left(i\vec{k} f_k e^{ikx} a_k - i\vec{k} f_k e^{-ikx} a_k^\dagger \right)$$

$$H = \frac{1}{2}\int d^3\vec{x}\,\left(\Pi^2(\vec{x}) + (\nabla\phi)^2 + m^2\phi^2 \right)$$

$$\int d^3x\, \Pi^2(x) = \int d^3k \int d^3k' \int d^3x\, \left(g_k e^{i\vec{k}\vec{x}} a_k + g_k^* e^{-ikx} a_k^\dagger\right)\left(g_{k'} e^{ik'x} a_{k'} + g_{k'}^* e^{-ik'x} a_{k'}^\dagger\right)$$

$$= \int d^3k\,(2\pi)^3 \left(g_k g_{-k}\, a_k a_{-k} + g_k g_k^*\, a_k a_k^\dagger + g_k^* g_k\, a_k^\dagger a_k + g_k^* g_{-k}^*\, a_k^\dagger a_{-k}^\dagger \right)$$

$$\int d^3x\, m^2\phi^2 = \int d^3k\,(2\pi)^3 m^2 \left(f_k f_{-k}\, a_k a_{-k} + f_k f_k\, a_k a_k^\dagger + f_k f_k\, a_k^\dagger a_k + f_k f_{-k}\, a_k^\dagger a_{-k}^\dagger \right)$$

$$\int d^3x\, (\nabla\phi)^2 = \int d^3k\,(2\pi)^3 \left(\vec{k}^2 f_k f_{-k}\, a_k a_{-k} + k^2 f_k f_k\, a_k a_k^\dagger + k^2 f_k f_k\, a_k^\dagger a_k + k^2 f_k f_{-k}\, a_k^\dagger a_{-k}^\dagger \right.$$

So:

$$H = \frac{1}{2}(2\pi)^3 \int d^3k \Big[(g_k g_{-k} + f_k f_{-k} (m^2+\vec{k}^2))\, a_k a_{-k} + (g_k^* g_{-k}^* + (k^2+m^2) f_k f_{-k}) a_k^\dagger a_{-k}^\dagger$$

$$+ (|g_k|^2 + (k^2+m^2) f_k^2)(a_k a_k^\dagger + a_k^\dagger a_k) \Big]$$

Define $\boxed{\omega_k^2 = m^2 + \vec{k}^2}$ $\omega_k > 0$

we want $\boxed{g_k g_{-k} + f_k f_{-k} \omega_k^2 = 0}$ to eliminate $a_k a_{-k}$ and also $a_k^\dagger a_{-k}^\dagger$

We then have.

$$H = \frac{(2\pi)^3}{2} \int d^3k \, (|g_k|^2 + \omega_k^2 f_k^2)(a_k a_k^\dagger + a_k^\dagger a_k)$$

If we quantize as fermions $\{a_k, a_{k'}^\dagger\} = \delta^{(3)}(k-k')$

We get H is a number, indep. of the state $\Rightarrow$ $\boxed{\text{not good}}$

we need bosons $[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}')$; $\tilde{a}_k = \frac{a_k}{(2\pi)^{3/2}}$

Then

$$H = \int \frac{d^3k}{(2\pi)^3} \epsilon_k a_k^\dagger a_k + \text{infinite constant}$$

, infinite energy density

$$\epsilon_k = (2\pi)^6 (|g_k|^2 + \omega_k^2 f_k^2)$$

$$\underbrace{\int \frac{d^3k}{(2\pi)^3} \frac{\epsilon_k}{2}}_{\text{Zero point energy}} \Big[\underbrace{\delta^{(3)}(0)\,(2\pi)^3}_{\text{volume}} \Big]$$

$$\delta(\vec{k}=0) = \int \frac{d^3x\, e^{i\vec{k}\vec{x}}}{(2\pi)^3} = \frac{V}{(2\pi)^3}$$

We can still choose g_k, f_k in many different ways.

To get f_k, g_k use canonical com. rel.

$$\left.\begin{aligned}[\phi(\vec{x}), \phi(\vec{y})] &= 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\pi(\vec{x}), \phi(\vec{y})] &= -i\,\delta^{(3)}(\vec{x}-\vec{y})\end{aligned}\right\}\ \text{bosonic}$$

$$[\phi(\vec{x}), \phi(\vec{y})] = (2\pi)^3 \int d^3k \; f_k^2\, e^{i\vec{k}(\vec{x}-\vec{y})} - f_k^2\, e^{-i\vec{k}(\vec{x}-\vec{y})}$$

$$= (2\pi)^3 \int d^3k \; (f_k^2 - f_{-k}^2)\, e^{i\vec{k}(\vec{x}-\vec{y})} = 0$$

$$\boxed{f_k = f_{-k}} > 0$$

$$[\pi(\vec{x}), \pi(\vec{y})] = (2\pi)^3 \int d^3k \; (|g_k|^2 - |g_{-k}|^2)\, e^{i\vec{k}(\vec{x}-\vec{y})} = 0$$

$$|g_k| = |g_{-k}|$$

$$[\pi(\vec{x}), \phi(\vec{y})] = (2\pi)^3 \int d^3k \; g_k f_k\, e^{i\vec{k}(\vec{x}-\vec{y})} - g_k^* f_k\, e^{-i\vec{k}(\vec{x}-\vec{y})}$$

$$= (2\pi)^3 \int d^3k \; (g_k f_k - g_{-k}^* f_{-k})\, e^{i\vec{k}(\vec{x}-\vec{y})}$$

$$= -i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\boxed{g_k f_k - g_{-k}^* f_{-k} = -\frac{i}{(2\pi)^6}} \qquad \boxed{g_k - g_{-k}^* = -\frac{i}{(2\pi)^6 f_k}}$$

$$g_k g_{-k} = -\omega_k^2 f_k^2$$

$$|g_k| = |g_{-k}|$$

$$g_k - g_{-k}^* = -\frac{i}{(2\pi)^6 f_k}.$$

$$g_k = e^{i\varphi_k} |g_k|$$

$$g_{-k} = e^{i\varphi_{-k}} |g_k|$$

$$e^{i(\varphi_k + \varphi_{-k})} = -1.$$

Assuming $\varphi_k = \varphi_{-k}$ by rot. invariance $\varphi_k = \varphi_{-k} = \pm\frac{\pi}{2}$

$$g_k - g_{-k}^* = -\frac{i}{(2\pi)^6 f_k} \Rightarrow g_k = -\frac{i}{2 f_k (2\pi)^6}$$

$$-\frac{1}{4 f_k^2 (2\pi)^{12}} = -\omega_k^2 f_k^2 \rightarrow f_k^4 = \frac{1}{4\omega_k^2 (2\pi)^{12}}$$

$$f_k = \frac{1}{\sqrt{2\omega_k}\,(2\pi)^3} \quad ; \quad g_k = -\frac{i}{2(2\pi)^6}\sqrt{2\omega_k}\,(2\pi)^3$$

$$g_k = -i\sqrt{\frac{\omega_k}{2}}\frac{1}{(2\pi)^3}$$

$$\epsilon_k = \cancel{(2\pi)^6}\left(\frac{\omega_k}{2}\frac{1}{\cancel{(2\pi)^6}} + \frac{\omega_k^{\cancel{2}}}{2\cancel{\omega_k}\,\cancel{(2\pi)^6}}\right) = \omega_k.$$

$$\boxed{\begin{aligned}\phi(\vec{x}) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(a_{\vec{k}}\, e^{i\vec{k}\vec{x}} + a_k^\dagger\, e^{-i\vec{k}\vec{x}} \right) \\ \Pi(x) &= -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left(a_{\vec{k}}\, e^{i\vec{k}\vec{x}} - a_k^\dagger\, e^{-i\vec{k}\vec{x}} \right)\end{aligned}}$$

$$H = \int \frac{d^3k}{(2\pi)^3}\, \omega_k\, a_k^\dagger a_k$$

$$\sqrt{\vec{k}^2 + m^2}$$

depends on follows from canonical quantization

agrees with $\boxed{\omega_k^2 - \vec{k}^2 = m^2}$ relativistic particle.

Also momentum:

$$\vec{P}^i = -\int d^3x\, \Pi . \nabla^i \phi = \frac{1}{2} \int \frac{d^3\vec{k}}{(2\pi)^3} \vec{k}^i \big(\overbrace{a_k a_{-k} + a_k^\dagger a_{-k}^\dagger}^{0 \text{ by } k \leftrightarrow -k} \big)$$

$$+ \frac{1}{2} \int d^3\vec{k}\, \vec{k}^i \left(a_k a_k^\dagger + a_k^\dagger a_k \right) = \int \frac{d^3k}{(2\pi)^3} \vec{k} \left(a_k^\dagger a_k + \cancel{\tfrac{1}{2}} \right)$$

by rotation Sym.

$$= \int \frac{d^3k}{(2\pi)^3} n_{\vec{k}}\, \vec{k}$$

$$[\vec{P}, \phi(\vec{x})] = -\int d^3x' \, (-i) \, \vec{\nabla}\phi \; \delta(x-x') = i\nabla\phi$$

$$\boxed{\phi(\vec{x}) = e^{-i\vec{P}\vec{x}} \, \phi(\vec{x}=0) \, e^{i\vec{P}\vec{x}}}$$

indeed $\quad \nabla\phi = -iP\phi + i\phi P = -i[P,\phi]$

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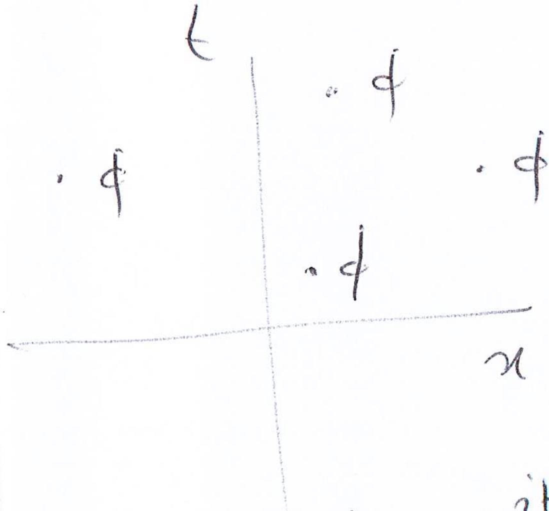
To get a better field theory picture:



we define $\phi(\vec{x},t)$

Heisenberg picture.

$$\phi(\vec{x},t) = e^{iHt}\phi(\vec{x})e^{-iHt}$$

we only need $\quad e^{iHt} a_k e^{-iHt} = a_k(t)$

$$\partial_t a_k(t) = i[H, a_k(t)] = i\int \frac{d^3k'}{(2\pi)^3} \omega_{k'} \left[a^\dagger_{k'} a_{k'}, a_k(t)\right]$$

$$[H, a_k(t=0)] = -\omega_k a_k \quad \Rightarrow \quad a_k(t) = f(t) \, a_k$$

$$\dot{f} a_k = -i\omega_k f a_k \qquad f = e^{-i\omega_k t}$$

⑫

$$a_k(t) = e^{-i\omega_k t}\, a_k$$

$$a_k^\dagger(t) = e^{i\omega_k t}\, a_k^\dagger \qquad \phi(x) = \phi^{(-)}(x) + \phi^{(+)}(x)$$

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(e^{-ik_\mu x^\mu} a_{\vec{k}} + e^{ik_\mu x^\mu} a_{\vec{k}}^\dagger \right) \\ \Pi(x) &= \dot{\phi} = -i \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{\omega_k}{2}} \left(e^{-ik_\mu x^\mu} a_{\vec{k}} - e^{ik_\mu x^\mu} a_{\vec{k}}^\dagger \right) \end{aligned}}$$

We have defined an operator at each point of space-time. This is the "field theory" point of view. These operators are represented on the Hilbert space of particles. $|n_{k_1} \ldots n_{k_2} \ldots\rangle$.

We can do the same in curved space, for conformal theories, etc.

Commutators:

$$[\phi(x),\phi(y)] = \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\left[e^{-ik(x-y)} - e^{ik(x-y)}\right]$$

$$= \Delta_c(x-y) \quad \text{c-number.}$$

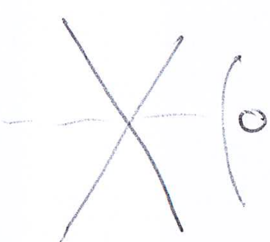
commuting number

for free-fields $\rightarrow$ $[\phi(x),\phi(y)] = \langle 0|[\phi(x),\phi(y)]|0\rangle$

$$[\phi(x),\phi(y)]_{\text{e.T.}} = 0$$

Causality:

$$[\phi(x),\pi(y)]_{\text{e.T.}} = 0 \text{ if } x \neq y$$



by Lorentz inv.

$$[\mathcal{O}(x),\mathcal{O}(y)] = 0$$

for $(x-y)^2 < 0$. $\boxed{x \neq y}$

$$U_\Lambda \phi(x) U_\Lambda^{-1} = \phi(\Lambda x)$$

$$U[\mathcal{O}(x),\mathcal{O}(y)]U_\Lambda^{-1} = [\mathcal{O}(\Lambda x),\mathcal{O}(\Lambda y)]$$

if 0 then

zero.

Wightman function

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\, e^{-ik(x-y)}$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{e^{-i\omega_k t}}{2\omega_k} = \int dk_0 \frac{d^3k}{(2\pi)^3}\frac{\delta(k_0-\omega_k)}{2k_0}\, e^{-ik_0 t}$$

$$\boxed{\delta(k_0^2-\omega_k^2) = \frac{1}{2\omega_k}\left[\delta(k_0-\omega_k)+\delta(k_0+\omega_k)\right]}$$

$$= \int \frac{d^4k}{(2\pi)^3}\,\delta(k_0^2-\omega_k^2)\,\Theta(k_0)\, e^{-ik_0 t}$$

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^3}\,\delta(k_0^2-\vec{k}^2-m^2)\,\Theta(k_0)\, e^{-ik(x-y)}$$

$$= \underbrace{\int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\,\Theta(k_0)}_{\text{Lorentz invariant}}\, e^{-ik(x-y)}$$

$$\langle 0|\phi(y)\phi(x)|0\rangle = \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2-m^2)\,\Theta(k_0)\, e^{ik(x-y)}$$

Commutator again

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\Big[e^{-i\omega_k(t_x-t_y)+i\vec k(\vec x-\vec y)} - \underbrace{e^{i\omega_k(t_x-t_y)-i\vec k(\vec x-\vec y)}}_{\vec k\to-\vec k}\Big]$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\left[e^{-i\omega_k(t_x-t_y)} - e^{i\omega_k(t_x-t_y)}\right]e^{i\vec k(\vec x-\vec y)}$$

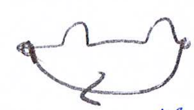
Assume $t_x > t_y$ $\quad x^0 > y^0$

$$\frac{1}{2\pi i}\int dk_0 \frac{e^{-ik_0(t_x-t_y)}}{k_0^2-\omega_k^2} = -\frac{2\pi i}{2\pi i}\left(\frac{e^{-i\omega_k(t_x-t_y)}}{2\omega_k} - \frac{e^{i\omega_k(t_x-t_y)}}{2\omega_k}\right)$$

$\frac{1}{(k_0-\omega_k)(k_0+\omega_k)}$ $\quad \lfloor k_0 \quad$ $k_0 + i\eta$

$e^{\eta(t_x-t_y)}$

$t_x - t_y > 0 \to$



$$= -\frac{1}{2\pi i}\int\frac{d^4k}{(2\pi)^3}\frac{e^{-ik(x-y)}}{k^2-m^2 \pm i\epsilon\,\mathrm{sg}(k_0)}$$

$k_0^2 = m^2 + \vec k^2 + i\epsilon \quad$ $k_0^2 = \omega_k^2 + i\epsilon$

$k_0 = \pm\omega_k \pm i\epsilon$

$t_x < t_y$

$k_0 = \pm\sqrt{\quad}$

$= \pm\omega_k \pm i\epsilon$

$\omega_k - i\epsilon$

$-\omega_k - i\epsilon$

$\pm\omega_k \pm i\epsilon\,\mathrm{sg}(k_0)$

$$[\phi(x),\phi(y)] = \frac{i}{\cancel{2\pi}}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2-m^2-i\epsilon\,\mathrm{sg}(k_0)}\,\theta(t_x-t_y)$$

$$-i\int\frac{d^4k}{(2\pi)^4}\frac{e^{+ik(x-y)}}{k^2-m^2-i\epsilon\,\mathrm{sg}(k_0)}\,\theta(t_y-t_x)$$

Retarded Green function $= \theta(x^0-y^0)\,\langle 0|[\phi(x),\phi(y)]|0\rangle$

$$D_R(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2-i\varepsilon\, sg(k_0)}\,e^{-ik(x-y)}$$

$$D_A(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon\, sg(k_0)}\,e^{-ik(x-y)}$$

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i}{k^2-m^2+i\varepsilon}\,e^{-ik(x-y)}$$

$\left(=D_F(y-x)\right)$

^ can be analytically continued to Euclidean space. Wick rotation.

$$[\phi(x),\phi(y)] = D_R(x-y) - D_A(x-y)$$

they satisfy:

$$(\Box+m^2)\,D_{R,A,F}(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\frac{i(-k^2+m^2)}{k^2-m^2-i\varepsilon\, sg(k_0)}\,e^{-ik(x-y)}$$

$$= -i\,\delta^{(4)}(x-y)$$

What if Feynman prop.

$$t_x > t_y \qquad \int \frac{d^3k}{(2\pi)^3} \frac{(+1)}{2\omega_k} e^{-i\omega_k(t_x - t_y) + i\vec{k}(\vec{x}-\vec{y})}$$

$$= \langle 0| \phi(x) \phi(y) |0\rangle$$

(interchange (x, y))

$$t_y > t_x \qquad = \langle 0| \phi(y) \phi(x) |0\rangle$$

Lorentz invariant because operator commute at space-like separation where time-ordering is frame dependent

$$\boxed{D_F(x-y) = \langle 0| \hat{T}\{\phi(x) \phi(y)\} |0\rangle}$$

e.g. check $(\partial^2 + m^2) D_F = -i\delta(x-y)$ eqn.

$$\partial_0^x D_F = \langle 0| \hat{T}\{\dot{\phi}(x) \phi(y)\} |0\rangle + \delta(x^0 - y^0) \underbrace{\langle 0| [\phi(x), \phi(y)] |0\rangle_{E.T.}}_{0}$$

$$\partial_0^x D_F = \langle 0| \hat{T}\{\pi(x) \phi(y)\} |0\rangle$$

$$\partial_0^{2x} D_F = \langle 0| \hat{T}\{\dot{\pi}(x) \phi(y)\} |0\rangle + \delta(x^0 - y^0) \langle 0| [\pi(x), \phi(y)] |0\rangle_{E.T.}$$

$$= \nabla_x^2 \langle 0| \hat{T}\{\phi(x) \phi(y)\} |0\rangle - i\delta^{(4)}(x-y)$$

$$\boxed{(\partial_{0x}^2 - \nabla_x^2) D_F(x-y) = -i\delta^{(4)}(x-y)}$$

Lorentz symmetry

Rotations: $t' = t$

$$\vec{x}' = A.\vec{x} \quad ; \quad A \in SO(3) \, ; \; A^t = A^{-1}$$

Boosts:

along x
$$\begin{cases} t' = \dfrac{t - vx}{\sqrt{1-v^2}} = \mathrm{ch}\beta\, t - \mathrm{sh}\beta\, x \\[2ex] x' = \dfrac{x - vt}{\sqrt{1-v^2}} = \mathrm{ch}\beta\, x - \mathrm{sh}\beta\, t \\[2ex] y' = y, \quad z' = z \end{cases}$$

In general we represent Lorentz transformations by Λ : 4×4 matrices

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ or $x' = \Lambda x$

such that $x'^t \eta x' = x^t \eta x$; $\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ $\quad \eta^2 = \mathbb{1}$

$x^t \Lambda^t \eta \Lambda x = x^t \eta x \quad \forall x \Rightarrow \Lambda^t \eta \Lambda = \eta$ or $\boxed{\Lambda^t \eta = \eta \Lambda^{-1}}$

or $\eta \Lambda^t \eta = \Lambda^{-1}$

preserves η metric: $SO(3,1)$ group.

$\Lambda = e^{-iM}$ exponential form. $\eta \underbrace{e^{-iM^t}}_{\Lambda^t} \eta = \underbrace{e^{iM}}_{\Lambda^{-1}}$

$$e^{-i\eta M^t \eta} = e^{iM} \Rightarrow \boxed{\eta M^t \eta = -M}$$

M: purely imaginary

$$\eta_{\mu\nu} M^t_{\nu\alpha} \eta_{\alpha\beta} = -M_{\mu\beta}$$

$$\begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}\begin{pmatrix} m_{00}&m_{01}&m_{02}&m_{03}\\ m_{10}&m_{11}&m_{12}&m_{13}\\ m_{20}&m_{21}&m_{22}&m_{23}\\ m_{30}&m_{31}&m_{32}&m_{33} \end{pmatrix}\begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix} = \begin{pmatrix} m_{00}&-m_{01}&-m_{02}&-m_{03}\\ -m_{10}&m_{11}&m_{12}&m_{13}\\ -m_{20}&m_{21}&m_{22}&m_{23}\\ -m_{30}&m_{31}&m_{32}&m_{33} \end{pmatrix} = \begin{pmatrix} -m_{00}&-m_{10}&-m_{20}&-m_{30}\\ -m_{01}&-m_{11}&-m_{21}&-m_{31}\\ -m_{02}&-m_{12}&-m_{22}&-m_{32}\\ -m_{03}&-m_{13}&-m_{23}&-m_{33} \end{pmatrix}$$

$$M = \begin{pmatrix} 0&m_{01}&m_{02}&m_{03}\\ m_{01}&0&m_{12}&m_{13}\\ m_{02}&-m_{12}&0&m_{23}\\ m_{03}&-m_{13}&-m_{23}&0 \end{pmatrix}$$

generators: $\quad M^{01} = i\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \quad M^{02} = i\begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix} \quad M^{03} = i\begin{pmatrix} 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix}$

Boosts

Rot. $\rightarrow$ $M^{12} = i\begin{pmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&+1&0&0\\ 0&0&0&0 \end{pmatrix} \quad M^{13} = i\begin{pmatrix} 0&0&0&0\\ 0&0&0&-1\\ 0&0&0&0\\ 0&+1&0&0 \end{pmatrix} \quad M^{23} = i\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&+1&0 \end{pmatrix}$

6 generators
$$(M^{\alpha\beta})^{\mu}{}_{\nu} = i\left(\eta^{\alpha\mu}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\delta^{\alpha}_{\nu}\right)$$

labels generator $\qquad$ matrix indices

$$(M^{01})^0{}_1 \; ; \; (M^{01})^1{}_0 \; ; \; (M^{12})^1{}_2 \qquad (M^{12})^2{}_1$$

$$1 \qquad 1 \qquad -1 \qquad +1$$

$$\boxed{\Lambda = e^{-i\theta_{\alpha\beta}M^{\alpha\beta}}}$$

$\mu \leftrightarrow \alpha$
$\nu \leftrightarrow \beta$

$$[M^{\alpha\beta}, M^{\mu\nu}]^{\rho}{}_{\sigma} = i i\left(\eta^{\alpha\rho}\delta^{\beta}_{\delta} - \eta^{\beta\rho}\delta^{\alpha}_{\delta}\right)\left(\eta^{\mu\delta}\delta^{\nu}_{\sigma} - \eta^{\nu\delta}\delta^{\mu}_{\sigma}\right) - (\mu\nu \leftrightarrow \alpha\beta)$$

$$= i i\Big[\eta^{\mu\beta}\left(\eta^{\alpha\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\beta}\delta^{\alpha}_{\sigma}\right) - \eta^{\beta\nu}\left(\eta^{\alpha\rho}\delta^{\mu}_{\sigma} - \eta^{\mu\rho}\delta^{\alpha}_{\sigma}\right) - \eta^{\alpha\mu}\left(\eta^{\beta\rho}\delta^{\nu}_{\sigma} - \eta^{\nu\rho}\delta^{\beta}_{\sigma}\right) +$$
$$+ \eta^{\alpha\nu}\left(\eta^{\beta\rho}\delta^{\mu}_{\sigma} - \eta^{\mu\rho}\delta^{\beta}_{\sigma}\right)\Big]$$

$(\mu\leftrightarrow\alpha)(\nu\leftrightarrow\beta)$

$$[M^{\alpha\beta}, M^{\mu\nu}] = i\left(\eta^{\mu\beta} M^{\alpha\nu} - \eta^{\beta\nu} M^{\alpha\mu} - \eta^{\alpha\mu} M^{\beta\nu} + \eta^{\alpha\nu} M^{\beta\mu}\right)$$: Lorentz algebra.

$$[M^{01}, M^{23}] = 0 \qquad [M^{02}, M^{13}] = 0 \qquad [M^{03}, M^{12}] = 0$$

Definition

$$I_1 = \frac{1}{2}(M^{23} + iM^{01}) \quad ; \quad J_1 = \frac{1}{2}(M^{23} - iM^{01})$$

$$I_2 = \frac{1}{2}(M^{31} + iM^{02}) \quad ; \quad J_2 = \frac{1}{2}(M^{31} - iM^{02})$$

$$I_3 = \frac{1}{2}(M^{12} + iM^{03}) \quad ; \quad J_3 = \frac{1}{2}(M^{12} - iM^{03})$$

$$\left[\frac{1}{2}(M^{23} \pm iM^{01}), \frac{1}{2}(M^{31} \pm iM^{02})\right] = \frac{i}{4}\left(-M^{21} \pm i(-)M^{30} \pm iM^{03} \pm \pm(+)M^{12}\right)$$

$$= \frac{i}{4}\left(\pm iM^{03} \pm iM^{03} + M^{12} \pm \pm M^{12}\right)$$

$$[I_1, J_2] = 0 = [I_2, J_1] \qquad [I_1, I_2] = \frac{i}{2}(iM^{03} + M^{12}) = iI_3$$

$$[J_1, J_2] = \frac{i}{2}(-iM^{03} + M^{12}) = iJ_3$$

$$\left[\frac{1}{2}(M^{23} \pm iM^{01}), \frac{1}{2}(M^{12} \pm iM^{03})\right] = \frac{i}{4}\left(-M^{31} \pm iM^{20} \pm i(-)M^{02} \pm \pm(+)M^{13}\right)$$

$$= \frac{i}{4}\left(-i(\pm M^{02} \pm M^{02}) + M^{13} \pm \pm M^{13}\right) \Rightarrow [I_1, J_3] = [I_3, J_1] = 0$$

$$[I_1, I_3] = \frac{i}{2}(-iM^{02} + M^{13}) = -iI_2 \quad ; \quad [J_1, J_3] = \frac{i}{2}(iM^{02} + M^{13}) = -iJ_2$$

$$\left[\frac{1}{2}(M^{31} \pm i M^{02}), \frac{1}{2}(M^{12} \pm i M^{03})\right] = \frac{i}{4}\left(-M^{32} \pm i(-)M^{10} \pm i M^{01} \pm (+) M^{23}\right)$$

$$= \frac{i}{4}\left(M^{23} \pm M^{23} \pm i M^{01} \pm i M^{01}\right)$$

$$[I_1, J_3] = [I_3, J_1] = 0$$

$$[I_1, I_3] = \frac{i}{2}\left(M^{23} + i M^{01}\right) = i I_1 \qquad [J_1, J_3] = i J_1$$

$$[I_1, J_1] = 0 \qquad [I_2, J_2] = 0 \qquad [I_3, J_3] = 0$$

<u>Lorentz Algebra:</u>

$$I_{a=1,2,3} \qquad J_{a=1,2,3}$$

$$[I_a, I_b] = i\,\varepsilon_{abc} I_c \;; \qquad [J_a, J_b] = i\,\varepsilon_{abc} J_c$$

$$[I_a, J_b] = 0$$

Representations: map between group elements and matrices

$$g_1 \cdot g_2 = g_3$$

$$\boxed{g \to \Lambda(g)} \qquad \Lambda(g_1) \cdot \Lambda(g_2) = \Lambda(g_3)$$

We can use $SU(2)$ rep. j. as

(j_1, j_2) give a representation for the algebra.

example $\left(\frac{1}{2}, 0\right)$ $\begin{cases} I_a = \frac{\sigma_a}{2} \\ J_a = 0 \end{cases}$ Pauli matrices.

$\left(0, \frac{1}{2}\right)$ $\begin{cases} I_a = 0 \\ J_a = \frac{\sigma_a}{2} \end{cases}$

but representation is not unitary!

Matrices $M^{\mu\nu}$ are not hermitian.

I_a, J_a are hermitian but $M^{01} = -i(J_1 - J_1)$ is not

rotations $M^{23} = I_1 + J_1$ are hermitian but boosts are not.

$\left(\frac{1}{2}, 0\right)$: $\Lambda_{1/2} = e^{-i\theta_{\alpha\beta}M^{\alpha\beta}} = e^{-i\theta_a \frac{\sigma_a}{2}}$

any complex number

$\Lambda_{1/2}$ 2x2 complex matrix. $\Rightarrow$ 8 real parameters. (only 6 actually)
in principle.

$Tr\, \sigma_a = 0 \Rightarrow \det \Lambda_{1/2} = 1$ det : complex $\Rightarrow$ 2 real cond.

$\Rightarrow$ 6 real parameters OK!

$\det(e^A) = e^{\overbrace{a_1 + a_2 - - + a_n}} = e^{TrA}$

eigenvalues.

2x2 complex matrices det = 1

$\boxed{SL(2, \mathbb{C}) \text{ group}}$

$$SL(2,\mathbb{C}) \equiv SO(3,1)$$

$\uparrow$ $2 \leftrightarrow 1$ (covering. 2π rotation)

Vector representation

$(\frac{1}{2},\frac{1}{2})$ 4 components. $\quad I_a = \sigma_a/2 \quad J_a = \sigma_a/2$

acting on different indices

$$x^\mu \to x^{A}{}_{\alpha\dot\alpha}$$

α: $(\frac{1}{2},0)$, $\dot\alpha$: $(0,\frac{1}{2})$

Parity ~~inter~~ changes sign of 1,2,3 but not 0. $\Rightarrow I_a \leftrightarrow J_a$.

$(\frac{1}{2},\frac{1}{2})$ parity invariant.

Also: $\quad (\frac{1}{2},0)\oplus(0,\frac{1}{2}) \quad$ Dirac representation.

$$\left[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})\right]\otimes\left[(0,\tfrac{1}{2})\oplus(\tfrac{1}{2},0)\right] = \left[(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0)\right]\oplus\left[(\tfrac{1}{2},0)\otimes(0,\tfrac{1}{2})\right]\oplus$$

$$\oplus\left[(0,\tfrac{1}{2})\otimes(\tfrac{1}{2},0)\right]\oplus\left[(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})\right] =$$

$$= (0,0)\oplus(1,0)\oplus(\tfrac{1}{2},\tfrac{1}{2})\oplus(\tfrac{1}{2},\tfrac{1}{2})\oplus(0,0)\oplus(0,1)$$

scalar $\qquad$ vector $\qquad$ scalar

antisymmetric tensor..

$1 \quad 3 \quad 4 \quad 4 \quad 1 \quad 3 \to \underline{16 = 4\times 4}$ ✓

with parity

Scalar, pseudoscalar	1+1	2
Vector, pseudovector	4+4	8
antisymmetric self dual tensor + anti self dual tensor	3+3	6
		16 ✓

$$F^{\mu\nu} = \pm \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$F^{\mu\nu} = \alpha\, \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

$$F^{\rho\sigma} = -\frac{1}{4\alpha} \epsilon^{\rho\sigma\mu\nu} F_{\mu\nu} \to \alpha = \pm i/2 \qquad \boxed{\vec{E} = \pm i \vec{B}}$$

Dirac rep. $\quad M^{23} = I_1 + J_1 = \epsilon^{ijk}(I_k + J_k)$

$$\left(M^{ij}\right)_{\text{Dirac}} = \frac{1}{2}\begin{pmatrix} \epsilon^{ijk}\sigma_k & 0 \\ 0 & \epsilon^{ijk}\sigma_k \end{pmatrix} \begin{matrix} \}\,(\tfrac{1}{2},0) \\ \}\,(0,\tfrac{1}{2}) \end{matrix} \quad \equiv S^{ij}$$

$$M^{01} = -i(I_1 + J_1) \quad ; \quad M^{0j} = -i(I_j - J_j)$$

$$\left(M^{0j}\right)_{\text{Dirac}} = -\frac{i}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \begin{matrix} \}\,(\tfrac{1}{2},0) \\ \}\,(0,\tfrac{1}{2}) \end{matrix} \quad \equiv S^{0j}$$

; (j_1, j_2) contains unitary reps of rotations w/ spin $|j_1 - j_2| \to |j_1 + j_2|$

⇒ ~~[illegible]~~

$(1/2, 0) \to$ Spin 1/2

$(0, 1/2) \to$

<u>Dirac method</u> Construct $\gamma^\mu \to S^{\mu\nu}$

Find matrices γ^μ / $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

In particular $(\gamma^0)^2 = 1$ $(\gamma^i)^2 = -1$

Define $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$; $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu}$

$$S^{\mu\nu} = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{2}\gamma^\mu\gamma^\nu - \frac{i}{2}\eta^{\mu\nu}\cdot\mathbb{1}$$

$$[S^{\mu\nu}, \gamma^\alpha] = \frac{i}{2}(\gamma^\mu\gamma^\nu\gamma^\alpha - \gamma^\alpha\gamma^\mu\gamma^\nu)$$

$$= \frac{i}{2}(-\gamma^\mu\gamma^\alpha\gamma^\nu + 2\eta^{\nu\alpha}\gamma^\mu - \gamma^\alpha\gamma^\mu\gamma^\nu)$$

$$+\frac{i}{2}(\cancel{\gamma^\alpha\gamma^\mu}\gamma^\nu - 2\eta^{\mu\alpha}\gamma^\nu + 2\eta^{\nu\alpha}\gamma^\mu - \cancel{\gamma^\alpha\gamma^\mu\gamma^\nu})$$

$$= i(\eta^{\nu\alpha}\gamma^\mu - \eta^{\mu\alpha}\gamma^\nu) \leftarrow \gamma^\mu \text{ rotates as a vector}$$

$$[S^{\mu\nu}, S^{\alpha\beta}] = \frac{i}{2}[S^{\mu\nu}, \gamma^\alpha\gamma^\beta] = \frac{ii}{2}\gamma^\alpha(\eta^{\nu\beta}\gamma^\mu - \eta^{\mu\beta}\gamma^\nu)$$

$$+\frac{ii}{2}(\eta^{\nu\alpha}\gamma^\mu - \eta^{\mu\alpha}\gamma^\nu)\gamma^\beta =$$

$$= i\eta^{\nu\beta}(S^{\alpha\mu} + \cancel{\tfrac{i}{2}\eta^{\alpha\mu}}) - i\eta^{\mu\beta}(S^{\alpha\nu} + \cancel{\tfrac{i}{2}\eta^{\alpha\nu}}) + i\eta^{\nu\alpha}(S^{\mu\beta} + \cancel{\tfrac{i}{2}\eta^{\mu\beta}})$$

$$-i\eta^{\mu\alpha}(S^{\nu\beta} + \cancel{\tfrac{i}{2}\eta^{\nu\beta}}) = i(\eta^{\nu\beta}S^{\alpha\mu} - \eta^{\mu\beta}S^{\alpha\nu} + \eta^{\nu\alpha}S^{\mu\beta} - \eta^{\mu\alpha}S^{\nu\beta})$$

$S^{\mu\nu}$: is a representation of Lorentz group generators, Lorentz algebra.

minimum size 4x4.

Chiral representation: $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

$\sigma_1 \otimes \mathbb{1}$ $\qquad$ $i\sigma_2 \otimes \sigma^i$

$$S^{0i} = \frac{i}{2} \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{i}{2} \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \epsilon^{ijk}\sigma^k & 0 \\ 0 & \epsilon^{ijk}\sigma^k \end{pmatrix}$$

<u>Same as before!</u>

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recall $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

<u>Properties</u> $\gamma_0 \gamma^\mu \gamma_0 = \begin{cases} \gamma_0 & \mu = 0 \\ -\gamma^i \gamma_0 \gamma_0 = -\gamma^i & \mu = i \end{cases}$

$$(\gamma^\mu)^\dagger = \begin{cases} \gamma^0 & \mu = 0 \\ -\gamma^i & \mu = i \end{cases} \Rightarrow \boxed{(\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0}$$

Aside

Other representations of Dirac matrices.

Fermion method:

Define $c_1 = \frac{1}{2}(\gamma_0 + \gamma_3)$ $\quad$ $c_2 = \frac{i}{2}(\gamma_1 + i\gamma_2)$

$c_1^+ = \frac{1}{2}(\gamma_0 - \gamma_3)$ $\quad$ $c_2^+ = \frac{i}{2}(\gamma_1 - i\gamma_2)$

Only non-vanishing anti-commutators:

$\{c_1, c_1^+\} = 1 = \{c_2, c_2^+\}$ $\quad$ ← same as fermionic algebra for 2 states.

e.g. $c_1^2 = \frac{1}{4}(\gamma_0^2 + \gamma_3^2) = 0.$ $\quad$ basis $|00\rangle |01\rangle\ |10\rangle\ |11\rangle$

$(\gamma_0\gamma_3 + \gamma_3\gamma_0) = 0$

$$c_1 = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 0 & 1 & 0 \\ 01 & 0 & 0 & 0 & 1 \\ 10 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 \end{array} \quad c_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad c_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad c_2^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

fermionic - sign

leads to $\gamma_0 = \sigma_1 \otimes \mathbb{1}$, $\gamma_1 = i\sigma_3 \otimes \sigma_1$; $\gamma_2 = i\sigma_3 \otimes \sigma_2$; $\gamma_3 = i\sigma_2 \otimes \mathbb{1}$

Same idea / lead to Majorana representation: purely imag.

$\gamma_0 = i(\epsilon \otimes \sigma_1)$ $\quad$ $\gamma_1 = i(\sigma_1 \otimes \mathbb{1})$ $\quad$ $\gamma_2 = i(\sigma_3 \otimes \mathbb{1})$ $\quad$ $\gamma_3 = i(\epsilon \otimes \epsilon)$

$\epsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Dirac spinor

$$\psi \xrightarrow[\Lambda]{} \psi' = \Lambda_{1/2}\psi = e^{-\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}}\psi$$

$\uparrow$ four component complex column vector $\begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$

$$\psi^\dagger \to \psi^\dagger \Lambda_{1/2}^\dagger = \psi^\dagger \left(e^{+\frac{i}{2}\theta_{\mu\nu}(S^{\mu\nu})^\dagger}\right)$$

$$(S^{\mu\nu})^\dagger = \gamma_0 S^{\mu\nu}\gamma_0$$

because $\gamma_0 S^{\mu\nu}\gamma_0 = \frac{i}{4}\gamma_0(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\gamma_0 = \frac{i}{4}\left[(\gamma^\mu)^\dagger(\gamma^\nu)^\dagger - (\gamma^\nu)^\dagger(\gamma^\mu)^\dagger\right]$

$$= -\frac{i}{4}[\gamma^\mu, \gamma^\nu]^\dagger = (S^{\mu\nu})^\dagger.$$

$$\psi^\dagger\gamma_0 \to \psi^\dagger\Lambda_{1/2}^\dagger\gamma_0 = \psi^\dagger\gamma_0\gamma_0\Lambda_{1/2}^\dagger\gamma_0 = \psi^\dagger\gamma_0 e^{\frac{i}{2}\theta_{\mu\nu}S^{\mu\nu}}$$

$$= \psi^\dagger\gamma_0\Lambda_{1/2}^{-1}$$

$\psi^\dagger\gamma_0 = \bar\psi$, $\bar\psi \to \bar\psi\Lambda_{1/2}^{-1}$

$\rightarrow \bar\psi\psi$ <u>scalar</u>.

Property :

$$\boxed{\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu}$$

to prove it define $\gamma^\mu(t) = e^{\frac{i}{\cancel{4}} t \theta_{\alpha\beta} S^{\alpha\beta}} \gamma^\mu e^{-\frac{i}{\cancel{4}} t \theta_{\alpha\beta} S^{\alpha\beta}}$

and $\tilde{\gamma}^\mu(t) = \left(e^{-i\theta_{\alpha\beta} M^{\alpha\beta} t}\right)^\mu{}_\nu \gamma^\nu$

$$\boxed{\gamma^\mu(0) = \tilde{\gamma}^\mu(0)}$$

$$\partial_t \gamma^\mu(t) = \frac{i}{\cancel{4}} \Lambda_{1/2}^{-1} \left[\theta_{\alpha\beta} S^{\alpha\beta}, \gamma^\mu\right] \Lambda_{1/2}$$

$$= \frac{i}{\cancel{4}} \theta_{\alpha\beta} \Lambda_{1/2}^{-1} \left(i\eta^{\beta\mu}\gamma^\alpha - i\eta^{\alpha\mu}\gamma^\beta\right) \Lambda_{1/2}$$

$$= -\frac{1}{\cancel{2}} \theta_{\alpha\mu} \gamma^\alpha(t) + \frac{1}{2} \theta_{\mu\beta} \gamma^\beta(t)$$

$$= 2\, \theta_{\mu\alpha} \gamma^\alpha(t)$$

$$\partial_t \tilde{\gamma}^\mu(0) = -i \left(\theta_{\alpha\beta} M^{\alpha\beta}\right)^\mu{}_\nu \tilde{\gamma}^\nu(t) = +\theta_{\alpha\beta}\left(\eta^{\alpha\mu}\delta^\beta_\nu - \eta^{\beta\mu}\delta^\alpha_\nu\right)\tilde{\gamma}^\nu$$

$$= +\theta_{\mu\beta}\tilde{\gamma}^\beta + \theta_{\alpha\mu}\tilde{\gamma}^\alpha = 2\,\theta_{\mu\beta}\tilde{\gamma}^\beta(t)$$

$$\Rightarrow \gamma^\mu(t) = \tilde{\gamma}^\mu(t) \Rightarrow \gamma^\mu(t=1) = \tilde{\gamma}^\mu(t=1).$$

then

$$\underbrace{\bar\psi \gamma^\mu \psi}_{\text{vector.}} \to \bar\psi \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} \psi = \Lambda^\mu{}_\nu \bar\psi \gamma^\nu \psi$$

Define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \{\gamma^5, \gamma^\mu\} = 0 \qquad (\gamma^5)^2 = 1$$

$$\Lambda_{1/2}^{-1}\gamma^5\Lambda_{1/2} = \gamma^5$$

extra dimension

γ-matrix $(i\gamma_5)$ spacial.

$(i\gamma_5)^2 = -1.$

$\bar\psi \gamma_5 \psi$ pseudo-scalar

$\bar\psi \gamma_5 \gamma^\mu \psi$ pseudo-vector.

$\bar\psi \sigma^{\mu\nu} \psi$: tensor. anti-sym.

$\Downarrow$

$S^{\mu\nu}$

Lorentz symmetry acting on states

Method of induced representations

Massive particle $\rightarrow$ $K^\mu \rightarrow \underbrace{(m, 0, 0, 0)}_{\text{frame where it is at rest.}}$

Boost $\vec{k} = 0 \xrightarrow{\Lambda_{\vec{k}}} \vec{k}$; $\Lambda(\vec{k})$

$$K_0' = \mathrm{ch}\beta\, K_0 = m\, \mathrm{ch}\beta$$

$$\vec{K}' = -\mathrm{sh}\beta\, K_0 = -m\, \mathrm{sh}\beta$$

$$\mathrm{ch}\beta = K_0/m$$

$$\mathrm{sh}\beta = -\frac{\vec{k}}{m} \qquad \mathrm{th}\beta = v = -\frac{\vec{k}}{K_0}$$

rotations, We start from a unitary rep of the "little group"

At rest a particle has $2s+1$ states where s = spin.

$$\boxed{|\vec{K}, \sigma\rangle = \sqrt{2\omega_{\vec{k}}}\, a^{\dagger}_{k,\sigma}|0\rangle} \qquad \sigma = -s \ldots s.$$

$$\langle \vec{K}', \sigma' | K, \sigma\rangle = 2\omega_k (2\pi)^3 \delta^{(3)}(\vec{k}' - \vec{k})\, \delta_{\sigma'\sigma}$$

Define

$$|\vec{K}, \sigma\rangle = U_{\Lambda(\vec{k})} |\overset{\vec{k}=0}{0}, \sigma\rangle$$

Another possibility eigenstates of $\epsilon^{ijk} M_{ij} k_l = \vec{J}.\vec{k}$

We know how to rotate:

$$U_R |0,\sigma\rangle = R_{\sigma\sigma'} |0,\sigma'\rangle$$

Euler angles.

↑ usual Wigner matrix $D^{j}_{\sigma\sigma'}(\theta_a)$

$$U_\Lambda |\vec{k},\sigma\rangle = U_\Lambda U_{\Lambda(\vec{k})} |0,\sigma\rangle =$$

any Λ

$$= U_{\Lambda(\Lambda\vec{k})} \underbrace{\underbrace{U^{-1}_{\Lambda(\Lambda\vec{k})}}_{\substack{\text{inverse boost}\\ \Lambda\vec{k}\to 0}} \underbrace{U_\Lambda}_{\text{any } \Lambda} \underbrace{U_{\Lambda(\vec{k})}}_{\text{boost } 0\to\vec{k}}}_{0\to 0 \text{ is a rotation.}} |0,\sigma\rangle$$

$$= U_{\Lambda(\Lambda\vec{k})} \cdot R_{\sigma\sigma'}(\Lambda,\vec{k}) |0,\sigma'\rangle$$

$$= R_{\sigma\sigma'}(\Lambda,\vec{k})\, U_{\Lambda(\Lambda\vec{k})} |0,\sigma'\rangle$$

$$\boxed{U_\Lambda |\vec{k},\sigma\rangle = R_{\sigma\sigma'}(\Lambda,\vec{k}) |\Lambda\vec{k},\sigma'\rangle}$$

we know how U_Λ acts on any state $|\vec{k},\sigma\rangle$.

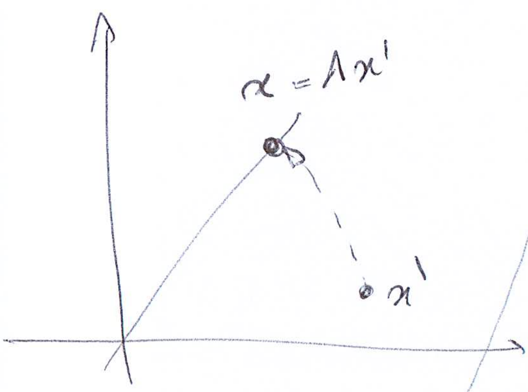
On creation operators:

$$\boxed{U_\Lambda\, a^\dagger_{k,\sigma}\, U_\Lambda^{-1} = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}}\; R_{\sigma\sigma'}(\Lambda,\vec{k})\; a^\dagger_{\Lambda\vec{k},\sigma}}$$

How to relate them?

On fields

$$\boxed{U_\Lambda\, \psi_a(x)\, U_\Lambda^{-1} = \Lambda^{-1}_{ab}\, \psi_b(\Lambda x)}$$



$$\psi(\Lambda x') = \Lambda \cdot \tilde{\psi}(x')$$

$$\tilde{\psi}(x') = \Lambda^{-1}\psi(\Lambda x')$$

$$\boxed{U_\Lambda\, \psi(x)\, U_\Lambda^{-1} = \Lambda^{-1}\psi(\Lambda x)}$$

We need a finite dim rep. that contains the same spin s that we used to construct the rep. of the particles. (spin of the particles).

$$\psi_a(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_\sigma \left(u_a^\sigma(\vec{k})\, a_{\vec{k},\sigma}\, e^{-ikx} + \right.$$

$$\left. + \, v_a^\sigma(\vec{k})\, b^\dagger_{\vec{k},\sigma}\, e^{ikx} \right)$$

Spin → rotations

$$u_a^\sigma(\vec{k})$$

momentum

non-unitary finite dim. rep. of Lorentz group.

What do we need?

invariant measure.

$$U_\Lambda \psi_a(x) U_\Lambda^{-1} = \int \frac{d^3k}{(2\pi)^3} \frac{\cancel{\sqrt{2\omega_k}}}{\sqrt{2\omega_k}} \sqrt{\frac{2\omega_{\Lambda k}}{\cancel{2\omega_k}}} \sum_{\sigma\sigma'} \underbrace{R^{(\Lambda,\vec{k})}_{\sigma\sigma'} u_a^\sigma(k)}_{\Lambda^{-1}_{aa'} u^{\sigma'}_{a'}(\Lambda k)\, a_{\Lambda k,\sigma'}} a_{\Lambda k,\sigma'} e^{-ikx} + \ldots$$

$$= \Lambda^{-1}_{aa'} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sqrt{2\omega_{\Lambda k}} \sum_{\sigma'} u^{\sigma'}_{a'}(\Lambda k)\, a_{\Lambda k,\sigma'}\, e^{-i(\Lambda k)\eta^{\dagger}\Lambda x} + \ldots$$

$$\tilde{k} = \Lambda k$$

$$= \Lambda^{-1}_{aa'} \int \frac{d^3\tilde{k}}{(2\pi)^3} \frac{\sqrt{2\omega_{\tilde{k}}}}{(2\omega_{\tilde{k}})} \sum_{\sigma'} u^{\sigma'}_{a'}(\tilde{k})\, a_{\tilde{k},\sigma'}\, e^{-i\tilde{k}\cdot\Lambda x} + \ldots$$

$$= \Lambda^{-1}_{aa'} \psi_{a'}(\Lambda x) \qquad \underline{\text{works!}}$$

We need

$$R_{\sigma\sigma'}(\Lambda, \vec{k})\, u_a^{\sigma}(k) = \Lambda^{-1}_{aa'}\, u_{a'}^{\sigma'}(\Lambda k)$$

$$\boxed{\Lambda_{aa'}\, R_{\sigma\sigma'}(\Lambda, k)\, u_{a'}^{\sigma}(k) = u_a^{\sigma'}(\Lambda k)}$$

Define $u_a^{\sigma}(\vec{k}) = \Lambda_{aa'}^{(\vec{k})}\, u_{a'}(0)$

$$\Lambda \cdot \Lambda(\vec{k})\, u^{\sigma}(0)\, R_{\sigma\sigma'}(\Lambda, k) = \Lambda_{\Lambda k}\, u^{\sigma'}(0)$$

$$\underbrace{\Lambda^{-1}_{(\Lambda k)}\, \Lambda \cdot \Lambda(\vec{k})}_{R(\Lambda, k)}\, u^{\sigma}\, R_{\sigma\sigma'}(\Lambda, k) = u^{\sigma'}(0)$$

$$R(\Lambda, k)_{aa'}\, u_{a'}^{\sigma}(0)\, \underset{\tilde{R}^{-1}}{R_{\sigma\sigma'}(\Lambda, k)} = u_a^{\sigma'}(0)$$

So $u_a^{\sigma}(0)$ should be invariant under rotations of both indices. We need some representation, namely spin. $\Rightarrow$ Dirac equation

Dirac (spin 1/2) field

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi = \psi^\dagger \gamma_0 \,(i\gamma^\mu \partial_\mu - m)\,\psi$$

$$\psi = \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}$$ 4 complex column vector $(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})$

$$\Pi_\psi = i\psi^\dagger \quad ; \quad \mathcal{L} = i\psi^\dagger\dot{\psi} + i\psi^\dagger\gamma_0\gamma^i\partial_i\psi - m\psi^\dagger\gamma_0\psi$$

$$H = -i\psi^\dagger\gamma_0\gamma^i\partial_i\psi + m\psi^\dagger\gamma_0\psi$$

We can use $i\psi^\dagger$ instead of Π. (constraint)

$$\{\psi^\dagger_a(\vec{x}),\ \psi_b(y)\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

<u>e.o.m.</u> for $\psi^\dagger$: $i\dot{\psi} + i\gamma_0\gamma^i\partial_i\psi - m\gamma_0\psi = 0$

$$\boxed{i\gamma_\mu\partial^\mu\psi - m\psi = 0}$$ <u>Dirac equation</u>

$$\cancel{i\partial\!\!\!/\psi}\ (i\partial\!\!\!/ - m)\,\psi = 0.$$

Notice:

$$(i\not\partial + m)(i\not\partial - m) = -\not\partial\not\partial \cancel{- i\not\partial m} \cancel{+ im\not\partial} - m^2$$

$$= (-\partial^2 - m^2)\psi = 0 \qquad \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu =$$

$$= \frac{1}{2}\{\gamma^\mu\gamma^\nu\}\partial_\mu\partial_\nu = \partial^2$$

$(\partial^2 + m^2)\psi = 0$ each component satisfies the Klein-Gordon eqn.

$$\Rightarrow \psi_a = \xi_a(k)e^{-ikx} \quad ; \quad k_0^2 - \vec{k}^2 = m^2 \quad ; \quad (\not k - m)\xi = 0$$

one possibility take $\psi = (i\not\partial + m)\tilde{\psi} = (\not k + m)\xi e^{-ikx}$

$\tilde{\psi} = \xi e^{-ikx}$; ξ arbitrary constant.

More in line w/ Lorentz symmetry:

take $\underline{\vec{k} = 0}$ $\quad k_0 = m$

$\cancel{(i\gamma_0\partial_0 - m)\psi = 0}$ $\qquad (\not k - m)\xi = 0 \Rightarrow \boxed{\gamma_0\xi = \xi}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\sqrt{m} \qquad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\sqrt{m}$$

convention.

for other $\vec{k}$ we use a <u>boost</u>.

$$S(\beta) = \Lambda_{1/2}(\beta) = \begin{pmatrix} e^{\beta \frac{\hat{k}\cdot\sigma}{2}} & 0 \\ 0 & e^{-\beta\frac{\hat{k}\cdot\sigma}{2}} \end{pmatrix} \qquad \left(\text{Notice } \Lambda_{1/2}^{+} = \Lambda_{1/2}\right)$$

$$\boxed{ch\beta = k_0/m} \qquad sh\beta = -\frac{|\vec{k}|}{m}$$

$$e^{\frac{\beta}{2}\hat{k}\cdot\sigma} = 1 + \frac{\beta}{2}(\hat{k}\sigma) + \frac{1}{2!}\left(\frac{\beta}{2}\right)^2 \underbrace{(\hat{k}\sigma)^2}_{1.} + \cdots$$

$$= ch\,\beta/2 + sh\frac{\beta}{2}\,(\hat{k}\cdot\sigma)$$

$ch^2 - sh^2 = 1$

$ch^2 + sh^2$

$$ch\beta = 2ch^2\beta/2 - 1 = 1 + 2sh^2\beta/2$$

$$ch\beta/2 = \sqrt{\frac{1 + k_0/m}{2}} \qquad sh\beta/2 = -\sqrt{\frac{k_0/m - 1}{2}}$$

$$= \frac{1}{\sqrt{2m}}\sqrt{m+k_0} + \frac{1}{\sqrt{2m}}\sqrt{k_0 - m}\;(\hat{k}\cdot\sigma)$$

$$u^{(\alpha)}(\vec{k}) = \begin{pmatrix} \frac{1}{\sqrt{2m}}\sqrt{m+k_0} - \frac{1}{\sqrt{2m}}\sqrt{k_0-m}\,(\hat{k}\sigma) & 0 \\ 0 & \frac{1}{\sqrt{2m}}\sqrt{m+k_0} + \frac{1}{\sqrt{2m}}\sqrt{k_0-m}\,(\hat{k}\sigma) \end{pmatrix} \begin{pmatrix} \xi^{r}(0) \\ \xi \end{pmatrix}$$

Normalization

$\bar{u}^r_{\vec{k}} \, u^s_{\vec{k}} = 2m$ (Lorentz invariant)

we compute for $\vec{k}=0$

$$(u^r_k)^\dagger u^s_k = (u^r_0)^\dagger \Lambda^\dagger_{1/2} \Lambda_{1/2} u^s_0 =$$

$$= (u^r_0)^\dagger \begin{pmatrix} e^{\beta \hat{k}\sigma} & 0 \\ 0 & e^{-\beta \hat{k}\sigma} \end{pmatrix} u^s_0 = (u^r_0)^\dagger \begin{pmatrix} ch\beta + sh\beta\, \hat{k}\sigma & 0 \\ 0 & ch\beta - sh\beta\, \hat{k}\sigma \end{pmatrix} u^s_0$$

$$= m (\xi^r_0)^\dagger \begin{pmatrix} \overset{k_0/m}{ch\beta} + \frac{\vec{k}}{m}\sigma & 0 \\ 0 & ch\beta + \frac{\vec{k}}{m}\sigma \end{pmatrix} \xi^s_0$$

$$= (\xi^r_0)^\dagger \begin{pmatrix} k_0+k_3 & -k_1+ik_2 & 0 & 0 \\ -k_1-ik_2 & k_0+k_3 & k_0 1 & 0 \\ 0 & 0 & k_0+k_3 & k_1-ik_2 \\ 0 & 0 & k_1+ik_2 & k_0-k_3 \end{pmatrix} \begin{pmatrix} \xi \end{pmatrix}$$

$$(1\,0\,1\,0) \begin{pmatrix} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k_0-k_3 & -k_1+ik_2 & k_0+k_3 & k_1-ik_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$= 2k_0$ or 0

$(0\,1\,0\,1) \begin{pmatrix} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 2k_0$

$$\Rightarrow \boxed{(u^r_k)^\dagger u^s_k = 2k_0 \, \delta^{rs}}$$

Solutions with e^{ikx}

$$\psi_a^{(s)} = v_a^s(k)\, e^{ikx}$$

$$(-\not{k} + m)\, v = 0 \qquad \gamma_0 v = -v \leftarrow \vec{k} = 0$$

$$\sqrt{m}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = v^{(1)} \qquad \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\sqrt{m} = v^{(2)}$$

$$\bar{v}^{(1)} v^{(1)} = m\,(1\,0\,-1\,0)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix} = -2m$$

$$\boxed{\bar{v}^{(r)}_{\vec{k}}\, v^{(s)}_{\vec{k}} = -2m\,\delta^{rs}}$$

$$\boxed{v^r_{\vec{k}} = \Lambda_{1/2}(\beta)\, v_0^r}$$

$$(v^r_k)^\dagger (v^s_k) = 2k_0\,\delta^{rs}$$

$$(u^r_k)^\dagger (v^s_{-k}) = (u_0^r)^\dagger\, \cancel{\Lambda_{1/2}(\beta)}\, \cancel{\Lambda_{1/2}(-\beta)}\, v_0^s$$

$$= \cancel{(1\,0\,1\,0)\begin{pmatrix}0\\+1\\0\\-1\end{pmatrix}}\ (\xi^\dagger, \xi^\dagger)\begin{pmatrix}\xi\\-\xi\end{pmatrix} = \xi^\dagger\xi - \xi^\dagger\xi = 0.$$

Finally

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=1,2} \left(u^\sigma_{\vec{k}} e^{-ikx} a_{\vec{k},\sigma} + v^\sigma_{\vec{k}} d^\dagger_{k,\sigma} e^{+ikx} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=1,2} \left(\bar{u}^\sigma_{\vec{k}} e^{ikx} a^\dagger_{k,\sigma} + \bar{v}^\sigma_{\vec{k}} d_{k,\sigma} e^{-ikx} \right)$$

$$H = \int d^3x \left(-i\bar{\psi}\gamma^j\partial_j\psi + m\bar{\psi}\psi \right)$$

$$= \int d^3x \left(\cancel{-i\bar{\psi}\not{\partial}\psi} + i\bar{\psi}\gamma_0\partial_0\psi \cancel{+ m\bar{\psi}\psi} \right)$$

$$= i\int d^3x\, \psi^\dagger\dot{\psi}$$

$$= i\int d^3x \int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}\sqrt{2\omega_{k'}}}\sum_{\sigma\sigma'}\left[(u^\sigma_k)^\dagger e^{+ikx} a^\dagger_{\vec{k},\sigma} + (v^\sigma_{\vec{k}})^\dagger d_{k,\sigma} e^{-ikx}\right]$$

$$\left[-u^{\sigma'}_{k'} e^{-ik'x}\, i\omega_k\, a_{k',\sigma'} + v^{\sigma'}_{k'}\, i\omega_k\, d^\dagger_{k',\sigma'} e^{+ik'x}\right]$$

$$= i\int\frac{d^3k}{(2\pi)^3}\frac{1}{\cancel{2\omega_k}}\sum_{\sigma\sigma'}\Big(\cancel{2\omega_k}\, a^\dagger_{k,\sigma} a_{k,\sigma}\,(-i\omega_k)\, a^\dagger_{k,\sigma} a_{k,\sigma} + \cancel{2\omega_k}\, i\omega_k\, d_{k,\sigma} d^\dagger_{k,\sigma}$$

$$+ \big(\underbrace{u^\sigma_k\, v^{\sigma'}_{-k}}_{0}\, i\omega_k\, a^\dagger_{k,\sigma} d^\dagger_{-k,\sigma}\big) + i\omega_k \underbrace{(v^\sigma_k)^\dagger u^{\sigma'}_{-k}}_{0} d_{k,\sigma} a_{-k,\sigma'}\Big]$$

$$= \int \frac{d^3k}{(2\pi)^3}\, \omega_k \left(a^\dagger_{k,\sigma} a_{k,\sigma} - d_{k,\sigma} d^\dagger_{k,\sigma} \right)$$

if $d_{k,\sigma}$ represents bosons then energy not bounded below

we need <u>fermions</u>

$$\{ d_{k,\sigma}, d^\dagger_{k',\sigma'} \} = \delta_{\sigma\sigma'}\, \delta^{(3)}(\vec{k}-\vec{k}')\, (2\pi)^3$$

$$d_{k,\sigma} d^\dagger_{k,\sigma} = - d^\dagger_{k,\sigma} d_{k,\sigma} + \underset{\uparrow}{\overset{\delta_{\sigma\sigma}}{2}}\, \delta^{(3)}(0)\, (2\pi)^3$$

$$= \int \frac{d^3k}{(2\pi)^3}\, \omega_k \left(a^\dagger_{k,\sigma} a_{k,\sigma} + d^\dagger_{k,\sigma} d_{k,\sigma} \right) -$$

$$- 2 \int \frac{d^3k}{(2\pi)^3} (2\pi)^3 \delta^{(3)}(0)\, \omega_k$$

$$\delta^{(3)}(k) = \int \frac{d^3x\, e^{ikx}}{(2\pi)^3}$$

$$\delta^{(3)}(0) = V/(2\pi)^3$$

Zero point energy:

$$-2 \int \frac{d^3k}{(2\pi)^3}\, \omega_k \cdot V$$

volume

negative

as 4 bosons (bosons have a 1/2)

($e^-\uparrow\downarrow$ $e^+\uparrow\downarrow$) 4 particles.