Dirac (spin 1/2) field

$$\mathcal{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\,\psi = \psi^\dagger \gamma_0 \,(i\gamma^\mu \partial_\mu - m)\,\psi$$

$$\psi = \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}$$ 4 complex column vector $(\tfrac{1}{2}, 0) \oplus (0, \tfrac{1}{2})$

$$\Pi_\psi = i\psi^\dagger \quad ; \quad \mathcal{L} = i\psi^\dagger \dot{\psi} + i\psi^\dagger \gamma_0 \gamma^j \partial_j \psi - m\psi^\dagger \gamma_0 \psi$$

$$H = -i\psi^\dagger \gamma_0 \gamma^j \partial_j \psi + m\psi^\dagger \gamma_0 \psi$$

We can use $i\psi^\dagger$ instead of Π. (constraint)

$$\{\psi_a^\dagger(\vec{x}), \psi_b(y)\} = \delta^{(3)}(\vec{x} - \vec{y})\,\delta_{ab}$$

<u>e.o.m.</u> for $\psi^\dagger$: $\quad i\dot{\psi} + i\gamma_0\gamma^j \partial_j \psi - m\gamma_0 \psi = 0$

$$\boxed{i\gamma_\mu \partial^\mu \psi - m\psi = 0}$$ <u>Dirac equation</u>

$$\cancel{i\partial\psi}\ (i\partial\!\!\!/ - m)\,\psi = 0.$$

Notice:

$$(i\partial\!\!\!/+m)(i\partial\!\!\!/-m) = -\partial\!\!\!/\partial\!\!\!/ \cancel{-i\partial\!\!\!/ m} \cancel{+im\partial\!\!\!/} - m^2$$

$$= (-\partial^2 - m^2)\psi = 0 \qquad \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu =$$

$$= \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu = \partial^2$$

$(\partial^2+m^2)\psi = 0$ each component satisfies the Klein-Gordon eqn.

$$\Rightarrow \psi_a = \xi_a(k)\, e^{-ikx} \;;\quad k_0^2 - \vec{k}^2 = m^2 \;;\quad (k\!\!\!/-m)\xi = 0$$

one possibility take $\psi = (i\partial\!\!\!/+m)\tilde{\psi} = (k\!\!\!/+m)\xi\, e^{-ikx}$

$\tilde{\psi} = \xi e^{-ikx}$ arbitrary constant.

More in line w/ Lorentz symmetry:

take $\underline{\vec{k}=0}$ $k_0 = m$

$$\cancel{(i\gamma_0\partial_0 - m)\psi = 0} \qquad (k\!\!\!/-m)\xi = 0 \Rightarrow \boxed{\gamma_0\xi = \xi}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_2 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \Rightarrow \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$u^1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}\sqrt{m} \qquad u^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\sqrt{m} \quad ; \quad u^{(r)} = \begin{pmatrix} \xi^{(r)} \\ \xi^{(r)} \end{pmatrix}$$

convention. $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

For other $\vec{k}$ we use a boost

$$\Lambda_{1/2}(\beta) = \begin{pmatrix} e^{\beta \frac{\hat{k}\sigma}{2}} & 0 \\ 0 & e^{-\beta\frac{\hat{k}\sigma}{2}} \end{pmatrix} ;$$

Notice $\Lambda_{1/2}^{+} = \Lambda_{1/2}$ for <u>boosts</u>

$$\boxed{ch\beta = k_0/m} \qquad \boxed{sh\beta = -|\vec{k}|/m}$$

$$e^{\frac{\beta}{2}\hat{k}\sigma} = 1 + \frac{\beta}{2}(\hat{k}.\sigma) + \frac{1}{2!}\left(\frac{\beta}{2}\right)^2 \underbrace{(\hat{k}\sigma)^2}_{1} + -$$

$$= ch\beta/2 + sh\beta/2\,(\hat{k}.\sigma)$$

$$ch\beta = 2ch^2\beta/2 - 1 = 1 + 2sh^2\beta/2$$

$$ch\beta/2 = \sqrt{\frac{1+k_0/m}{2}} = \frac{1}{\sqrt{2m}}\sqrt{m+k_0}\,; \quad sh\frac{\beta}{2} = \frac{1}{\sqrt{2m}}\sqrt{k_0-m}$$

$$e^{\frac{\beta}{2}\hat{k}.\sigma} = \frac{1}{\sqrt{2m}}\left(\sqrt{k_0+m} - \sqrt{k_0-m}\,(\hat{k}.\sigma)\right)$$

$$u^{(r)}(\vec{k}) = \begin{pmatrix} \sqrt{\frac{k_0+m}{2}} - \sqrt{\frac{k_0-m}{2}}\,(\hat{k}.\sigma) & 0 \\ 0 & \sqrt{\frac{k_0+m}{2}} + \sqrt{\frac{k_0-m}{2}}\,(\hat{k}\sigma) \end{pmatrix} \begin{pmatrix} \xi^{(r)} \\ \xi^{(r)} \end{pmatrix}$$

where $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\sum_r u^r_{\vec{k}} \bar{u}^r_{\vec{k}} = m \Lambda_{1/2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Lambda^{-1}_{1/2} =$$

$$= m \Lambda_{1/2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Lambda^{-1}_{1/2} = m \Lambda_{1/2} (1 + \gamma_0) \Lambda^{-1}_{1/2} =$$

$$= m + \underbrace{\Lambda_{1/2} m \gamma_0 \Lambda^{-1}_{1/2}}_{\cancel{k_0}\ \underset{\uparrow \atop \mathring{k}^0}{m} \Lambda^0_{\mu} \gamma^\mu = k_\mu \gamma^\mu} = m + \not{k}$$

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We need also solutions with e^{ikx}

$$\psi^{(s)}_a = v^s_a(k)\, e^{ikx} \quad ; \quad (i\not{\partial} - m)\psi = 0$$

$$\Leftrightarrow (-\not{k} - m) v = 0$$

$$(\not{k} + m) v = 0 \qquad v_{\vec{k}=0} \;/\; (1+\gamma_0) v = 0 \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \chi \end{pmatrix} = 0$$

$$v^{(s)}_0 = \sqrt{m} \begin{pmatrix} \xi^r \\ -\xi^r \end{pmatrix} \qquad \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ as before}$$

$$v^{(s)}_{\vec{k}} = \Lambda_{1/2} v^{(s)}_0 = \begin{pmatrix} \sqrt{\frac{k_0+m}{2}} - \sqrt{\frac{k_0-m}{2}} (\hat{k}\sigma) & 0 \\ 0 & \sqrt{\frac{k_0+m}{2}} + \sqrt{\frac{k_0-m}{2}} (\hat{k}\sigma) \end{pmatrix} \begin{pmatrix} \xi^r \\ -\xi^r \end{pmatrix}$$

$$\bar{v}^{r}_{\vec{k}}\, v^{s}_{\vec{k}} = \bar{v}^{r}_{0}\, v^{s}_{0} = m\,(\xi^{r\dagger}\,{-\xi^{r\dagger}})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi^{s}\\ -\xi^{s}\end{pmatrix}$$

$$= m\,(\xi^{r\dagger}, -\xi^{r\dagger})\begin{pmatrix}-\xi^{s}\\ \xi^{s}\end{pmatrix} = -2m\,\xi^{r\dagger}\xi^{s} = -2m\,\delta^{rs}$$

$$(v^{r}_{k})^{\dagger}\, v^{s}_{k} = \underset{(\xi^{r\dagger}\,-\xi^{r\dagger})}{(v^{r}_{0})^{\dagger}}\begin{pmatrix}ch\beta + sh\beta\,\vec{k}\vec{\sigma} & 0\\ 0 & ch\beta - sh\beta\,\vec{k}\vec{\sigma}\end{pmatrix}\underset{\begin{pmatrix}\xi^{r\dagger}\\ -\xi^{r\dagger}\end{pmatrix}}{v_{0}^{s}} =$$

$$= 2m\, ch\beta\,(\xi^{r})^{\dagger}\xi^{s} = 2k_0\,\delta^{rs}$$

no sign change w/ respect to $(u^{r}_{k})^{\dagger}u^{s}_{k}$; since no γ_0 involved.

$$\sum_r v^{r}_{\vec{k}}\,\bar{v}^{r}_{\vec{k}} = m\,\Lambda_{1/2}\sum_r\begin{pmatrix}\xi^{r}\\ -\xi^{r}\end{pmatrix}(\xi^{\dagger r}\, -\xi^{\dagger r})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Lambda_{1/2}^{-1} =$$

$$= m\,\Lambda_{1/2}\begin{pmatrix}\mathbb{1} & -\mathbb{1}\\ -\mathbb{1} & \mathbb{1}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\Lambda_{1/2}^{-1} = m\,\Lambda_{1/2}\begin{pmatrix}-\mathbb{1} & \mathbb{1}\\ \mathbb{1} & -\mathbb{1}\end{pmatrix}\Lambda_{1/2}^{-1} =$$

$$= \cancel{m}\,\Lambda_{1/2}\,(-m + m\gamma_0)\,\Lambda_{1/2}^{-1} = \not{k} - m$$

$$(u^{r}_{\vec{k}})^{\dagger}\, v^{s}_{-\vec{k}} = (u^{r}_{0})^{\dagger}\,\Lambda_{1/2}(\vec{k})\,\underbrace{\overset{\text{just opposite boost}}{\Lambda_{1/2}(-\vec{k})}}_{\Lambda_{1/2}^{-1}(\vec{k})}\, v^{s}_{0} = (u^{r}_{0})^{\dagger}\, v^{s}_{0}$$

$$= (\xi^{r\dagger}\;\xi^{r\dagger})\begin{pmatrix}\xi^{s}\\ -\xi^{s}\end{pmatrix} = (\xi^{r})^{\dagger}\xi^{s} - (\xi^{r})^{\dagger}\xi^{s} = 0$$

$$(\bar{u}^{r}_{\vec{k}})\, v^{s}_{\vec{k}} = \bar{u}^{r}_{0}\, v^{s}_{0} = (\xi^{r\dagger}\;\xi^{r\dagger})\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\xi^{s}\\ -\xi^{s}\end{pmatrix} = \xi^{r\dagger}\xi^{s} - \xi^{r\dagger}\xi^{s} = 0.$$

Summary

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\Lambda_{1/2}(\vec{k}) = \begin{pmatrix} e^{\frac{\beta}{2}\hat{k}\sigma} & 0 \\ 0 & e^{-\frac{\beta}{2}\hat{k}\sigma} \end{pmatrix} \quad ; \quad \begin{array}{l} ch\beta = k_0/m \\ sh\beta = -|\vec{k}|/m \end{array}$$

$$u^r_{\vec{k}} = \Lambda_{1/2}\, u^r_0 \quad ; \quad u^r_0 = \sqrt{m}\begin{pmatrix} \xi^r \\ \xi^r \end{pmatrix} \quad ; \quad \xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v^r_{\vec{k}} = \Lambda_{1/2}\, v^r_0 \quad ; \quad v^r_0 = \sqrt{m}\begin{pmatrix} \xi^r \\ -\xi^r \end{pmatrix}$$

$$(\not{k} - m)\, u^r_{\vec{k}} = 0 \qquad\qquad (\not{k} + m)\, v^r_{\vec{k}} = 0$$

$$\bar{u}^r_{\vec{k}}\, u^s_{\vec{k}} = 2m\,\delta^{rs} \qquad\qquad \bar{v}^r_{\vec{k}}\, v^s_{\vec{k}} = -2m\,\delta^{rs}$$

$$(u^r_{\vec{k}})^\dagger u^s_{\vec{k}} = 2k_0\,\delta^{rs} \qquad\qquad (v^r_{\vec{k}})^\dagger v^s_{\vec{k}} = 2k_0\,\delta^{rs}$$

$$\sum_r u^r_{\vec{k}}\, \bar{u}^r_{\vec{k}} = \not{k} + m \qquad\qquad \sum_r v^r_{\vec{k}}\, \bar{v}^r_{\vec{k}} = \not{k} - m$$

$$(u^r_{\vec{k}})^\dagger v^s_{-\vec{k}} = 0 \qquad\qquad (v^r_{\vec{k}})^\dagger u^s_{-\vec{k}} = 0$$

$$\bar{u}^r_{\vec{k}}\, v^s_{\vec{k}} = 0 \qquad\qquad \bar{v}^r_{\vec{k}}\, u^s_{\vec{k}} = 0$$

Finally

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=1,2} \left(u^\sigma_{\vec{k}}\, e^{-ikx} c_{k,\sigma} + v^\sigma_{\vec{k}}\, d^\dagger_{k,\sigma} e^{ikx} \right)$$

$$\bar\psi(x) = \psi^\dagger \gamma_0 = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=1,2} \left(\bar u^\sigma_{\vec{k}}\, e^{ikx} c^\dagger_{k,\sigma} + \bar v^\sigma_{\vec{k}}\, d_{k,\sigma} e^{-ikx} \right)$$

$$(\not\partial - m)\psi = 0$$

$$H = \int d^3x \left(-i \bar\psi \gamma^j \partial_j \psi + m \bar\psi \psi \right)$$

$$= \int d^3x \left(\underbrace{-i \bar\psi \not\partial \psi + m \bar\psi \psi}_{0 \text{ by Dirac eqn.}} + i \bar\psi \gamma^0 \partial_0 \psi \right)$$

$$\boxed{H = i \int d^3x\, \psi^\dagger \dot\psi}$$

$$= i \int d^3x \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}\sqrt{2\omega_{k'}}} \sum_{\sigma\sigma'} \left((u^\sigma_{\vec{k}})^\dagger e^{ikx} c^\dagger_{k,\sigma} + (v^\sigma_{\vec{k}})^\dagger d_{k,\sigma} e^{-ikx} \right)$$

$$\left(-i\omega_k\, u^{\sigma'}_{\vec{k}'} e^{-ik'x} c_{\vec{k}',\sigma} + i\omega_k\, v^{\sigma'}_{\vec{k}'} d^\dagger_{k',\sigma} e^{ik'x} \right)$$

$$= i \int \frac{d^3k}{(2\pi)^3} \frac{i\omega_k}{2\omega_k} \Big(-\sum_{\sigma\sigma'} \underbrace{(u^\sigma_{\vec{k}})^\dagger u^{\sigma'}_{\vec{k}}}_{2\omega_k \delta^{\sigma\sigma'}} c^\dagger_{k,\sigma} c_{k,\sigma} + \sum_{\sigma\sigma'} \underbrace{(u^\sigma_{\vec{k}})^\dagger v^{\sigma'}_{-\vec{k}}}_{0} e^{2i\omega_k t} c^\dagger_{k}\, d^\dagger_{-k,\sigma}$$

$$- \sum_{\sigma\sigma'} \underbrace{(v^\sigma_{\vec{k}})^\dagger u^{\sigma'}_{-\vec{k}}}_{0} d_{k,\sigma} c_{-k,\sigma} e^{-2i\omega_k t} + \underbrace{(v^\sigma_{\vec{k}})^\dagger (v^{\sigma'}_{\vec{k}})}_{2\omega_k \delta^{\sigma\sigma'}} d_{k,\sigma} d^\dagger_{k,\sigma} \Big)$$

$$= -\int \frac{d^3k}{(2\pi)^3} \omega_k \left(-c^\dagger_{\vec{k},\sigma} c_{\vec{k}\sigma} + d_{\vec{k},\sigma} d^\dagger_{\vec{k},\sigma} \right)$$

$$H = \int \frac{d^3k}{(2\pi)^3}\, \omega_k \left(c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma} - d_{\vec{k},\sigma} d^\dagger_{\vec{k},\sigma} \right)$$

If $d_{k,\sigma}$ represents bosons then $d_{k,\sigma} d^\dagger_{k,\sigma} = d^\dagger_{k,\sigma} d_{k,\sigma} + \delta(k-k') \delta_{\sigma\sigma'}$

and energy is not bounded from below.

we need <u>fermions</u>

$$\{ d_{k,\sigma}, d^\dagger_{k',\sigma'} \} = \delta_{\sigma\sigma'}\, \delta^{(3)}(\vec{k}-\vec{k}') . (2\pi)^3$$

$$H = \int \frac{d^3k}{(2\pi)^3}\, \omega_k \left(c^\dagger_{\vec{k},\sigma} c_{\vec{k},\sigma} + d^\dagger_{\vec{k},\sigma} d_{\vec{k},\sigma} \right) - 2 \int \frac{d^3k}{(2\pi)^3} (2\pi)^3 \underbrace{\delta^{(3)}(0)}_{V/(2\pi)^3} \omega_k$$

Zero point energy (vacuum energy)

$$E_0 = -2 \int \frac{d^3k}{(2\pi)^3}\, \omega_k \,.\, V$$

volume.

negative and equivalent to 4 bosons. $\left(E_{\text{boson}} = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \omega_k V \right)$

The factor of 4 is because there are 4 particles

$e^-\uparrow\downarrow$ $\quad e^+\uparrow\downarrow$ $\quad$ So each fermion contributes minus one boson to vacuum energy.

(if they have the same mass).

<u>Compute now:</u>

$$\{\psi(x), \bar{\psi}(y)\}_{E.T.} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_{\sigma} \left[u^{\sigma}_{\vec{k}} \bar{u}^{\sigma}_{\vec{k}} e^{ik(y-x)} + v^{\sigma}_{\vec{k}} \bar{v}^{\sigma}_{\vec{k}} e^{-ik(y-x)} \right]$$

equal time

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \left[(\not{k}+m) e^{ik(y-x)} + (\not{k}-m) e^{-ik(y-x)} \right]$$

equal time $y^0 - x^0 = 0$. $e^{-i\vec{k}(\vec{y}-\vec{x})}$ $e^{i\vec{k}(\vec{y}-\vec{x})}$

we can change integration variable $\vec{k} \to -\vec{k}$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{y}-\vec{x})} \underbrace{\left(k_0\gamma^0 + \cancel{k_j\gamma^j} \cancel{+m} + k_0\gamma^0 \cancel{-k_j\gamma^j} \cancel{-m} \right)}_{2k_0\gamma^0 = 2\omega_k\gamma^0}$$

$$= \gamma^0 \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}(\vec{y}-\vec{x})} = \gamma^0 \delta(\vec{y}-\vec{x})$$

multiplying by γ^0

$$\{\psi_a(x), \psi^{\dagger}_b(y)\} = \delta_{ab}\, \delta(\vec{y}-\vec{x})$$

Feynman propagator

$$\langle 0|T\{\psi_a(x)\bar\psi_b(y)\}|0\rangle = \begin{cases} \langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle & x^0>y^0 \\ -\langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle & x^0<y^0 \end{cases}$$

notice minus sign.

$$\langle 0|\psi_a(x)\bar\psi_b(y)|0\rangle = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\underbrace{\sum_\sigma\left(u^\sigma_{\vec k}\bar u^\sigma_{\vec k}\right)_{ab}}_{(\not k+m)_{ab}}e^{ik(y-x)}$$

$\partial_x \to -ik$

$$= (i\not\partial_x+m)_{ab}\int\frac{d^3k}{(2\pi)^3}\frac{e^{-ik(x-y)}}{2\omega_k}$$

~~boson~~ boson propagator

$$\langle 0|\bar\psi_b(y)\psi_a(x)|0\rangle = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\underbrace{\sum_\sigma\left(v^\sigma_{\vec k}\bar v^\sigma_{\vec k}\right)_{ab}}_{(\not k+m)}e^{ik(x-y)}$$

$$= (-i\not\partial_x-m)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{+ik(x-y)}$$

$$-\langle 0|\bar\psi_b(x)\psi_a(x)|0\rangle = (i\not\partial_x+m)\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{+ik(x-y)}$$

$$S_F(x-y) = (i\not\partial_x+m)\,\Delta_F(x-y) = \int\frac{d^4k}{(2\pi)^4}\frac{i(\not k+m)}{(k^2-m^2+i\epsilon)}e^{-ik(x-y)}$$

$$\frac{\not{k}+m}{k^2-m^2+i\varepsilon} \quad ; \quad (\not{k}-m)\frac{(\not{k}+m)}{k^2-m^2} = \frac{k^2-m^2}{k^2-m^2} = 1.$$

So sometimes it is written as

$$S_F(k) = \frac{i}{\not{k}-m} \quad \text{understanding} \quad \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}$$

correct prescription