662, Homework I, (3 problems)

Problem 1

For a harmonic oscillator in usual single-particle quantum mechanics, use the Schroedinger formalism to compute the propagator:

$$K(x_f, t_f; x_i, t_i) = \langle x_f | e^{-\frac{i}{\hbar} H(t_f - t_i)} | x_i \rangle, \qquad H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \tag{0.1}$$

In order to do that recall that

$$\langle x | n \rangle = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x \right) \tag{0.2}$$

where $H_n(y)$ are Hermite polynomials that satisfy the identity ($|\mathrm{Re}\alpha| < \frac{1}{2}$)

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} H_n(x) H_n(y) = \frac{1}{2} \frac{1}{\sqrt{\frac{1}{4} - \alpha^2}} e^{x^2 + y^2 - \frac{(x+y)^2}{2(1+2\alpha)} - \frac{(x-y)^2}{2(1-2\alpha)}} \tag{0.3}$$

Compare the propagator with the exponential of the classical action. Can you explain the result using the path integral formalism?

Problem 2

Consider N scalar fields $\phi_{a=1\ldots N}$ of the same mass m. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a \tag{0.4}$$

where repeated indices are summed over. The Lagrangian has a symmetry $\phi_a \to R_{ab} \phi_b$ for any constant orthogonal matrix R_{ab}.

a) Write the equations of motion, Hamiltonian and spatial momentum $\vec{P}$.

b) Find the infinitesimal form of the symmetry described above $\phi_a \to R_{ab} \phi_b$ to find a conserved Noether current and conserved charges Q_{ab} (antisymmetric in ab). Use the equations of motion to check that it is indeed conserved.

c) Quantize the scalar fields, that is, write appropriate expressions for $\phi_a(\vec{x})$ and $\Pi_a(\vec{x})$ in terms of momentum creation and annihilation operators and check the canonical commutation relations. Compared to the complex scalar field done in class, notice that you now need only one set of annihilation (and creation) operators, for each real field (they are their own antiparticles).

d) Write the Hamiltonian H, momentum $\vec{P}$ and conserved charge Q_{ab} in terms of annihilation and creation operators and show that Q_{ab} is conserved (namely commutes with H). Also show that Q_{ab} generates infinitesimal symmetry transformations of the fields.

Problem 3

Consider the Feynman propagator for a massive scalar field in momentum space:

$$\Delta_F(p) = -\frac{i}{p^2 - m^2 + i\epsilon} \tag{0.5}$$

a) Perform a Fourier transform and write it explicitly in space-time coordinates in terms of Bessel functions discussing separately its form outside and inside the light cone.

b) Discuss its behavior in the limit of large space-like separation, time-like separation and near the light-cone.