we can define $\vec{v} = \vec{n}/g_0$

$$e^{-\frac{1}{2}\int d^d x\, (\partial_\mu \vec{v})^2}$$

$\|\vec{v}\| = 1/g_0$

g_0 = inverse radius of sphere

$g_0 \to 0$, $R \to \infty$ free

$g_0 \to \infty$, $R \to 0$ strong coupling

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$$Z = \int \mathcal{D}\vec{n}\; e^{-\frac{1}{2g_0^2}\int d^d x\, (\partial_\mu \vec{n})^2} \prod_x \delta(\vec{n}^2 - 1)$$

$$= \int \mathcal{D}\hat{n} \int \mathcal{D}\alpha\; e^{-\frac{1}{2g_0^2}\int d^d x\, (\partial_\mu \vec{n})^2 - \int \frac{i\alpha}{2g_0^2}(\vec{n}^2 - 1)}$$

$$= \int \mathcal{D}\hat{n} \int \mathcal{D}\alpha\; e^{-\frac{1}{2g_0^2}\int d^d x\, (-\vec{n}\,\partial^2 \vec{n} + i\alpha \vec{n}^2) + \int \frac{i\alpha}{2g_0^2}}$$

$$= \int \mathcal{D}\alpha\; \det{}^{-N/2}(-\partial^2 + i\alpha)\; e^{\frac{i}{2g_0^2}\int \alpha}$$

$$= \int \mathcal{D}\alpha\; e^{-\frac{N}{2}\mathrm{Tr}\ln(-\partial^2 + i\alpha) + \frac{i}{2g_0^2 N} N \int \alpha(x)\, dx}$$

$$-\frac{\cancel{N}}{2}\frac{\delta}{\delta\alpha(x)}\mathrm{Tr}\ln(-\partial^2 + i\alpha) + \frac{i\cancel{N}}{2g_0^2 \cancel{N}} = 0$$

$\alpha = -im^2$

$$-\frac{1}{\cancel{2}}\cancel{i}\frac{\delta}{\delta m^2}\mathrm{Tr}\ln(-\partial^2 + m^2) + \frac{\cancel{i}}{\cancel{2}g_0^2 N} = 0$$

$$\frac{1}{g_0^2 N} = \frac{\delta}{\delta m^2}\mathrm{Tr}\ln(-\partial^2 + m^2) = \frac{\delta}{\delta m^2}\int \frac{d^d k}{(2\pi)^d}\ln(k^2 + m^2)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m^2} = \frac{1}{g_0^2 N}$$ (gap equation)

$\boxed{d=2}$

$$\frac{2\pi}{4\pi^2} \int_0^\Lambda \frac{k\,dk}{k^2+m^2} = \frac{1}{4\pi} \ln(k^2+m^2)\Big|_0^\Lambda = \frac{1}{4\pi} \ln\left(1+\Lambda^2/m^2\right)$$

$$\Lambda \to \infty \qquad \frac{1}{4\pi} \ln\left(\frac{\Lambda^2}{m^2}\right) + \underbrace{\ln\left(1+m^2/\Lambda^2\right)}_{\substack{\to 0 \\ \Lambda\to\infty}} = \frac{1}{2\pi} \ln\frac{\Lambda}{m}$$

$$\frac{1}{g_0^2 N} = \frac{1}{2\pi} \ln\Lambda - \frac{1}{2\pi} \ln m$$

$g_0(\Lambda)$ to cancel ∞.

$$\frac{1}{g_0^2 N} = \frac{1}{g^2 N} + \frac{1}{2\pi} \ln \Lambda/\mu$$

$$\frac{1}{g^2 N} - \frac{1}{2\pi} \ln\mu = -\frac{1}{2\pi} \ln m$$

$$\boxed{\frac{1}{g^2 N} = \frac{1}{2\pi} \ln(\mu/m)}$$

$$\boxed{m = \mu\, e^{-2\pi/g^2 N}}$$ mass gap.

$$S = \frac{1}{2g_0^2} \int d^d x \, (\partial_\mu \vec{n})^2 + m^2 \vec{n}^2$$

N massive fields, no symmetry breaking.

m indep. of μ

$$0 = \frac{\partial m}{\partial \mu} = e^{-\frac{2\pi}{g^2 N}} + \mu \left(+ \frac{4\pi}{g^3 N} \right) \frac{\partial g}{\partial \mu} e^{-\frac{2\pi}{g^2 N}}$$

$$\beta = \mu \frac{\partial g}{\partial \mu} = - \frac{g^3 N}{4\pi}$$

$$\lambda = g^2 N \qquad \mu \frac{\partial \lambda}{\partial \mu} = \mu\, 2gN \left(\frac{\partial g}{\partial \mu} \right) = - 2gN \frac{g^3 N}{4\pi}$$

$$= - \frac{g^4 N^2}{2\pi} = - \frac{\lambda^2}{2\pi}$$

$$\boxed{\beta_\lambda = - \frac{\lambda^2}{2\pi}}$$

$\boxed{\beta < 0}$ asymptotically free theory.

$\boxed{d=3}$

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} = \frac{4\pi}{8\pi^3} \int_0^\Lambda \frac{k^2\,dk}{k^2+m^2} = \frac{1}{2\pi^2} \int_0^\Lambda \frac{k^2+m^2-m^2}{k^2+m^2}\,dk$$

$$= \frac{\Lambda}{2\pi^2} - \frac{m^2}{2\pi^2} \int_0^\Lambda \frac{dk}{k^2+m^2} = \frac{\Lambda}{2\pi} - \frac{m}{2\pi^2} \underbrace{\int_0^{\Lambda/m} \frac{dk}{1+k^2}}_{\substack{\text{atan } \Lambda/m \\ \to \pi/2}}$$

$$= \frac{\Lambda}{2\pi^2} - \frac{|m|}{4\pi}$$

$$\frac{1}{g^2N} = \frac{\Lambda}{2\pi^2} - \frac{|m|}{4\pi}$$

$$|m| = \frac{2\Lambda}{\pi} - \frac{4\pi}{g^2N} > 0$$

$$\frac{1}{g^2N} < \frac{\Lambda}{2\pi^2} \qquad g^2N > \frac{2\pi^2}{\Lambda}$$

$$\lambda_c = \frac{2\pi^2}{\Lambda}$$

$\lambda > \lambda_c \rightarrow$ mass gap.
symmetry restore

$\lambda < \lambda_c \rightarrow \underline{m^2 < 0}$

symmetry broken.

$$\frac{1}{\lambda} = \frac{1}{\lambda_c} - \frac{|m|}{4\pi}$$

$$\frac{m}{4\pi} = \frac{1}{\lambda_c} - \frac{1}{\lambda} = \frac{\lambda - \lambda_c}{\lambda\lambda_c}$$

$\lambda \to \lambda_c$

$$m \sim (\lambda - \lambda_c)^1$$

critical exponent $\underline{1}$

units.

$$\frac{1}{g^2}\int (\partial n)^2 d^3x$$

$$\ell^{-2} \quad \ell^{3} \qquad\qquad M^{2-d}$$

$$\ell \qquad g^2 \sim \ell.$$

$$\frac{1}{g^2}\int (\partial n)^2 \, d^d x$$

$$\ell^d \quad M^{-d+2}$$

$$g^2 \sim M^{2-d} \qquad\qquad \lambda \sim M^{2-d}$$

$$\lambda^{\frac{1}{2-d}} \sim M$$

$$m \sim \lambda^{\frac{1}{2-d}}$$

$$c_1 \Lambda^{d-2} - c_2 m^{d-2} = \frac{1}{g^2 N}$$

$$\frac{1}{g^2_c} - \frac{1}{\lambda} = m^{d-2} \qquad\qquad m \sim (\lambda - \lambda_c)^{\frac{1}{d-2}}$$

$$m \sim (\lambda - \lambda_c)^{1/2}$$

$$c_1 \Lambda^{d-2} - c_2 \Lambda^{d-4} m^2 + \dots = \frac{1}{g^2 N}$$

$$\frac{1}{g_c} \qquad \frac{1}{\lambda_c} - \frac{1}{\lambda} = \Lambda^{d-4} m^2$$

units.

$$\frac{1}{g^2}\int (\partial n)^2 d^3x$$

ℓ^{-2} ℓ^3 $\qquad M^{2-d}$

ℓ $\qquad g^2 \sim \ell.$

$$\frac{1}{g^2}\int (\partial n)^2 d^dx$$

ℓ^d M^{-d+2}

$g^2 \sim M^{2-d}$ $\qquad \lambda \sim M^{2-d}$

$\lambda^{\frac{1}{2-d}} \sim M$

$m \sim \lambda^{\frac{1}{2-d}}$

$$C_1\Lambda^{d-2} - C_2 m^{d-2} = \frac{1}{g^2 N}$$

$\frac{1}{\cancel{g^2}\lambda_c} - \frac{1}{\lambda} = m^{d-2}$ $\qquad m \sim (\lambda-\lambda_c)^{\frac{1}{d-2}}$

$m \sim (\lambda-\lambda_c)^{1/2}$

$$C_1\Lambda^{d-2} - C_2\Lambda^{d-4} m^2 \;\; = \frac{1}{g^2 N}$$

$\frac{1}{g_c}$ $\qquad \frac{1}{\lambda_c} - \frac{1}{\lambda} = \Lambda^{d-4} m^2$

$$\int \frac{d^5k}{(2\pi)^6} \frac{1}{k^2+m^2}$$

$$\int_0^\Lambda \frac{k^4\,dk}{k^2+m^2} \qquad \frac{(k^2+m^2)\,k^2}{k^2+m^2} = \frac{m^2k^2}{k^2+m^2}$$

$$k^2\,dk \sim \Lambda^3 - m^2\Lambda^2 + -$$

$$\frac{1}{\lambda} = \Lambda^3 - m^2\Lambda^2$$

3d again

$$\frac{1}{\lambda_0} - \frac{\Lambda}{2\pi^2} = -\frac{|m|}{4\pi} \quad ; \text{ Define } \frac{1}{\lambda_0} - \frac{\Lambda}{2\pi^2} = \frac{1}{\lambda_R} - \frac{\mu}{2\pi^2}$$

then $\displaystyle \frac{1}{\lambda_R} - \frac{\mu}{2\pi^2} = -\frac{|m|}{4\pi} \qquad \frac{1}{\lambda_R^c} = \frac{\mu}{2\pi^2} \qquad \lambda_R^c = \frac{2\pi^2}{\mu}.$

$$-\frac{1}{\lambda_R^2}\frac{\partial\lambda_R}{\partial\mu} - \frac{1}{2\pi^2} = 0 \qquad \frac{\partial\lambda_R}{\partial\mu} = -\frac{\lambda_R^2}{2\pi^2}$$

λ_R has units of $M^{-d+2} = M^{-1}$ $\rightarrow$ define $\tilde{\lambda}_R = \mu\lambda_R$

$$\frac{\partial\tilde{\lambda}_R}{\partial\mu} = \lambda_R + \mu\frac{\partial\lambda_R}{\partial\mu} = \lambda_R - \frac{\mu\lambda_R^2}{2\pi^2} \;;\; \mu\frac{\partial\tilde{\lambda}_R}{\partial\mu} = \mu\lambda_R - \frac{\mu^2\lambda_R^2}{2\pi^2} = \tilde{\lambda}_R - \frac{\tilde{\lambda}_R^2}{2\pi^2}$$

$$\beta(\tilde{\lambda}_R) = \underset{\varepsilon\checkmark}{\tilde{\lambda}_R} - \frac{\tilde{\lambda}_R^2}{2\pi^2}$$

fixed point $\lambda_R = 0$ $\boxed{\tilde{\lambda}_R = 2\pi^2}$ $\checkmark$ $\lambda_R^c\mu$