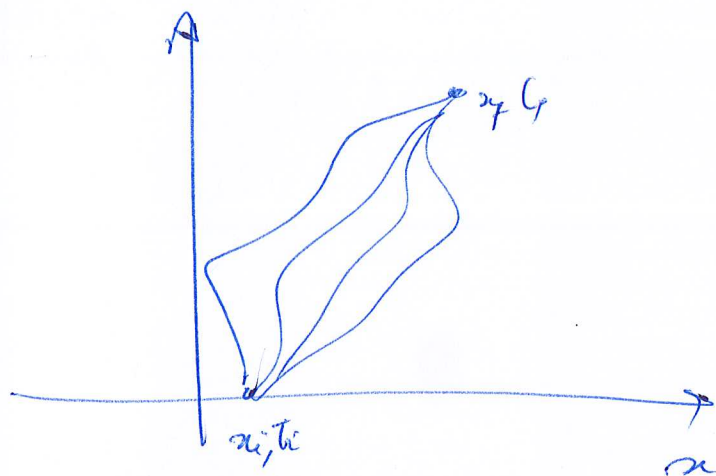


# Path integrals in QM.

①



$$\begin{aligned} \langle x_f, t_f | x_i, t_i \rangle &= \\ &= \langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle \\ &= K(x_f, t_f; x_i, t_i) \end{aligned}$$

$$K(x_f, t_f; x_i, t_i) = \int [Dx(t)] e^{iS[x(t)]}$$

$iS[x(t)]$

$$\begin{aligned} \psi(x_f, t_f) &= \int dx_i K(x_f, t_f; x_i, t_i) \psi(x_i, t_i) \\ K(x_f, t_f; x_i, t_i) &= \int dx K(x_f, t_f; x, t) K(x, t; x_i, t_i) \end{aligned}$$

$t_f > t_i$

example: free particle

$$K(x_f, t_f; x_i, t_i) = \int \frac{d^3p}{2\pi} \langle x_f | e^{-i\frac{p^2}{2m}(t_f - t_i)} | p \rangle \langle p | x_i \rangle$$

$$= \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}(t_f - t_i)} e^{ip(x_f - x_i)}$$

$$= \frac{1}{2\pi} \sqrt{\frac{\pi}{i\frac{1}{2m}(t_f - t_i)}} e^{-\frac{(x_f - x_i)^2}{4i\frac{1}{2m}(t_f - t_i)}} = \frac{\sqrt{m}}{\sqrt{2\pi}} \frac{e^{-\frac{i0}{4}}}{\sqrt{(t_f - t_i)}} e^{\frac{m^2(x_f - x_i)^2}{2(t_f - t_i)}}$$

$$K(x_f, t_f; x_i, t_i) = e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi(t_f - t_i)}} e^{\frac{i m (x_f - x_i)^2}{2(t_f - t_i)}}$$

$t_f - t_i \frac{e^{-i\pi/4}}{\sqrt{\pi}} e^{i\pi/4} \rightarrow \delta(x)$  by Fourier  $\int_{-\infty}^{\infty} dx e^{ix/\sigma} f(x) = \int_{-\infty}^{\infty} dk e^{ik\sigma} \tilde{f}(k)$   
 $= \sqrt{\frac{\sigma}{2\pi}} \int_{-\infty}^{\infty} dk \tilde{f}(k) e^{i\frac{k\sigma}{2}} = \sqrt{\frac{\sigma}{2\pi}} \tilde{f}(\frac{\sigma}{2})$

$$K = e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi T}} e^{\frac{im}{2} \frac{(x_f - x_i)^2}{T}} \quad \left[ \begin{array}{l} h=1 \\ \text{here} \end{array} \right] \quad (1)$$

$$\psi(x_i) = \frac{1}{(\pi a)^{1/4}} e^{-\frac{(x_i - x_0)^2}{2a}}$$

$$\int |\psi|^2 = \frac{1}{\sqrt{\pi a}} \int e^{-\frac{x^2}{a}} = \sqrt{\frac{\pi}{4a}} \frac{1}{\sqrt{\pi a}} = 1 \quad \checkmark$$

$$\psi(x_f) = \int_{-\infty}^{\infty} dx_i \frac{1}{(\pi a)^{1/4}} e^{-\frac{(x_i - x_0)^2}{2a}} e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi T}} e^{\frac{im}{2} \frac{(x_f - x_i)^2}{T}}$$

$$x_f - x_i = y \quad x_i = x_f - y$$

$$= \int_{-\infty}^{\infty} dy \frac{1}{(\pi a)^{1/4}} \sqrt{\frac{m}{2\pi T}} e^{-\frac{im}{2} y^2} e^{\frac{im}{2T} y^2 - \frac{1}{2a} (x_f - x_0 + y)^2}$$

$$= \frac{e^{-im/4}}{(\pi a)^{1/4}} \sqrt{\frac{m}{2\pi T}} \int_{-\infty}^{\infty} dy e^{\frac{im}{2T} y^2 - \frac{1}{2a} y^2 + \frac{1}{a} y \Delta x - \frac{\Delta x^2}{2a}}$$

$$= \frac{e^{-im/4}}{(\pi a)^{1/4}} \sqrt{\frac{m}{2\pi T}} \sqrt{\frac{\pi}{\frac{1}{2a} - \frac{im}{2T}}} e^{\frac{\Delta x^2}{a^2} \frac{1}{4(\frac{1}{2a} - \frac{im}{2T})} - \frac{\Delta x^2}{2a}}$$

$$- \frac{\Delta x^2}{2a} \left( 1 - \frac{1}{a} \frac{1}{(\frac{1}{a} - \frac{im}{T})} \right) = - \frac{\Delta x^2}{2a} \left( 1 - \frac{1}{1 - \frac{imo}{T}} \right) = \frac{\Delta x^2}{2a} \frac{(1 - im/T)}{1 - im/T}$$

$$= \frac{i m \Delta x^2}{2T} \frac{1 + \frac{i m a}{T}}{1 + \frac{m^2 a^2}{T^2}} \quad (2)$$

$$\psi(x_f) = \frac{e^{-i\pi/4}}{(\pi a)^{1/4}} \sqrt{\frac{m}{2T}} \frac{\sqrt{ka}}{\sqrt{1 - \frac{i m a}{T}}} e^{i m \Delta x^2 / 2T} \frac{1 + \frac{i m a}{T}}{1 + \frac{m^2 a^2}{T^2}}$$

$$= \frac{e^{-i\pi/4}}{(\pi a)^{1/4}} \sqrt{\frac{1}{\frac{T}{m a} - i}} e^{-\frac{m^2 a}{2T^2} \frac{\Delta x^2}{1 + \frac{m^2 a^2}{T^2}} + \frac{i m}{2T} \frac{\Delta x^2}{1 + \frac{m^2 a^2}{T^2}}}$$

$$|\psi(x_f)|^2 = \frac{1}{\sqrt{\pi a}} \frac{1}{\sqrt{1 + T^2/m^2 a^2}} e^{-\frac{m^2 a^2}{T^2} \frac{\Delta x^2}{1 + \frac{m^2 a^2}{T^2}} \frac{1}{a}}$$

$$= \frac{1}{\sqrt{\pi a (1 + T^2/m^2 a^2)}} e^{-\frac{\Delta x^2}{a (1 + T^2/m^2 a^2)}}$$

correctly normalized  $\int |\psi|^2 = 1$

$$a(T) = a (1 + T^2/m^2 a^2)$$

$$T=0 \quad \psi(x_f) = \psi(x_i) \Rightarrow K \rightarrow \delta(x_f - x_i)$$

$T \rightarrow 0$

$T \rightarrow \infty$   $a(T) \rightarrow \infty$  delocalized.

also  $\left[ \frac{\partial \psi_f}{\partial T} = \frac{i}{2m} \frac{\partial \psi_f}{\partial x^2} \right]$  time dep. Sch eqn.

units

$T \sim \text{length}$

$m \sim \sqrt{\text{length}}$

$a \sim \text{length}^2$

h.o

2

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$x(t) = e^{-iHt} x e^{iHt}$$

$$p(t) = e^{-iHt} p e^{iHt}$$

$$[x(t), p(t)] = i \quad ; \quad H = \frac{1}{2m} p^2(t) + \frac{1}{2} m \omega^2 x^2(t)$$

$$\partial_t x(t) = -i H x(t) + x(t) i H = -i [H, x(t)] =$$

$$= -\frac{i}{m} p(-i) = -\frac{1}{m} p(t)$$

$$\partial_t p(t) = -i m \omega^2 i x(t) = m \omega^2 x(t)$$

$$\partial_t^2 x(t) = -\frac{1}{m} m \omega^2 x(t) = -\omega^2 x(t)$$

$$\begin{aligned} (\partial_t^2 + \omega^2) x(t) &= 0 & x(t) &= \cos \omega t \hat{A} + \sin \omega t \hat{B} \\ \dot{x}(t) &= -\omega \sin \omega t \hat{A} + \omega \cos \omega t \hat{B} & &= -\frac{1}{m} p(t) \end{aligned}$$

$$x_0(t) = \hat{x}_s \cos \omega t - \frac{1}{m\omega} \sin \omega t \hat{p}_s$$

$$X(t) \underbrace{e^{-iHt} |x_i\rangle}_{|\psi\rangle} = x_i \underbrace{e^{-iHt} |x_i\rangle}_{|\psi\rangle}$$

$$\langle x_f | \psi \rangle = \psi(x_f)$$

$$\cos \omega t x_f \psi + \frac{i}{m\omega} \sin \omega t \partial_{x_f} \psi = x_i \psi$$

$$\frac{\partial_{x_f} \psi}{\psi} = -i \frac{m\omega}{\sin \omega t} (x_i - x_f \cos \omega t)$$

$$\partial_{x_f} \ln \psi = -i \frac{m\omega}{\sin \omega t} (x_i - x_f \cos \omega t)$$

$$\psi = e^{-\frac{i m \omega}{\sin \omega t} (x_i x_f - \frac{1}{2} x_f^2 \cos \omega t)} \cdot A(t, x_i)$$

Property  $\langle x_f | e^{-iHt} |x_i\rangle^* = \langle x_i | e^{iHt} |x_f\rangle$

$$K(x_f, t; x_i, 0)^* = K(x_i, -t; x_f, 0)$$

$\psi^*(-t)$ : same as  $x_i \leftrightarrow x_f$

$$= e^{-\frac{i m \omega}{\sin \omega t} (x_i x_f - \frac{1}{2} x_f^2 \cos \omega t)}$$

$$\psi = A(t) e^{-\frac{i m \omega}{\sin \omega t} (x_i x_f - \frac{1}{2} (x_f^2 + x_i^2) \cos \omega t)}$$

$$\boxed{A^*(-t) = A(t)}$$

$$\partial_t \psi = -iH\psi = -i \left( -\frac{1}{2m} \partial_x^2 \psi + \frac{1}{2} m\omega^2 x^2 \psi \right)$$

(4)

$$\psi = A e^{\chi} \quad \partial_x \psi = A \partial_x \chi e^{\chi}$$

$$\dot{A} e^{\chi} + A \dot{\chi} e^{\chi} = \frac{i}{2m} A \chi'' e^{\chi} + \frac{i}{2m} A \chi'^2 e^{\chi} -$$

$$-\frac{i}{2} m\omega^2 x^2 A e^{\chi}$$

$$\frac{\dot{A}}{A} + \left( \frac{i m \omega^2 c}{s^2} x_i x_f + \frac{i m \omega^2}{2} (x_f^2 + x_i^2) \frac{1}{s^2 a t} \right) =$$

$$= \frac{i}{2m} \left( \frac{i m \omega}{s a t} c a t \right) + \frac{i}{2m} \left( -\frac{i m \omega}{s a t} (x_i - x_f) c a t \right)^2 -$$

$$-\frac{i}{2} m \omega^2 x_f^2$$

$$\frac{\dot{A}}{A} = -\frac{i \omega}{2 s a t} c a t - \frac{i m \omega^2 c}{s^2} x_i x_f + \frac{i m \omega^2}{2} (x_i^2 + x_f^2) \frac{1}{s^2 a t} +$$

$$+ \frac{i \omega^2 m}{2 s^2 a t} (x_i^2 + x_f^2) c a t - \left[ 2 x_i x_f c a t \right] - \frac{i}{2} m \omega^2 x_f^2$$

$$= \frac{i \omega c a t}{s a t} + \frac{i m \omega^2 x_f^2}{2 s^2 a t} (1 - c^2 a t) - \frac{i}{2} m \omega^2 x_f^2$$

$$\frac{\dot{A}}{A} = -\frac{m\omega c \alpha t}{2 \sin \omega t}$$

$$\partial_t \ln A = -\frac{1}{2} \ln \sin \omega t$$

$$A = \frac{A_0}{\sqrt{\sin \omega t}}$$

$$\psi = \frac{A_0}{\sqrt{\sin \omega t}} e^{-\frac{i m \omega}{\sin \omega t} (x_i x_f - \frac{1}{2} c \omega t (x_i^2 + x_f^2))}$$

$$t \rightarrow 0 \quad \psi \rightarrow \delta(x_i - x_f)$$

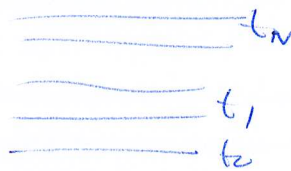
$$\psi \approx \frac{A_0}{\sqrt{\omega t}} e^{+\frac{i m \omega}{\omega t} \frac{1}{2} (x_i - x_f)^2}$$

$$\int \psi = \frac{A_0}{\sqrt{\omega t}} \sqrt{\frac{2\pi i t}{-i m}} = A_0 \sqrt{\frac{2\pi}{-i m \omega}} = 1$$

$$A_0 = \sqrt{\frac{-i m \omega}{2\pi}} = e^{-\frac{i\pi}{4}} \sqrt{\frac{m \omega}{2\pi}}$$

$$R(x_f t_f; x_i t_i) = e^{-\frac{i\pi}{4}} \sqrt{\frac{m \omega}{2\pi \sin(\omega(t_f - t_i))}} e^{-\frac{i m \omega}{\sin(\omega(t_f - t_i))} (x_i x_f - \frac{1}{2} c \omega (t_f - t_i) (x_i^2 + x_f^2))}$$

Free particle.



(6)

$$\int dx_1 \dots dx_{N-1} e^{+i \frac{(x_N - x_{N-1})^2}{2m \Delta t}}$$

$$\xi_1 = x_1 - x_0 \quad \xi_2 = x_2 - x_1 \quad \dots \quad \xi_N = x_N - x_{N-1}$$

$$\int d\xi_1 \dots d\xi_{N-1} d\xi_N e^{+i \frac{1}{2m \Delta t} (\xi_1^2 + \dots + \xi_{N-1}^2 + \xi_N^2)} \delta(\xi_1 + \dots + \xi_N - \Delta X)$$

$$\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \int d\xi_1 \dots d\xi_N e^{i\lambda \xi_1 + \dots + i\lambda \xi_N - \lambda \Delta X - i \frac{1}{2m \Delta t} \xi_1^2 - \dots - i \frac{1}{2m \Delta t} \xi_N^2}$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \left( \sqrt{\frac{\pi 2m \Delta t}{-i}} \right)^N e^{-\frac{\lambda^2 2m \Delta t N}{4(-i)} - i\lambda \Delta X}$$

$$= (+2\pi i m \Delta t)^N \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{-\frac{i}{2} m \lambda^2 (t_f - t_i) - i\lambda \Delta X}$$

$$= \frac{(+2\pi i m \Delta t)^N}{2\pi} \sqrt{\frac{\pi}{-i \frac{m}{2} (t_f - t_i)}} e^{+\frac{\Delta X^2}{2} \frac{4i m (t_f - t_i)}{2}}$$

$$= \frac{(+2\pi i m \Delta t)^N}{2\pi} \sqrt{\frac{2\pi i}{m (t_f - t_i)}} e^{-\frac{i \Delta X^2}{2m (t_f - t_i)}}$$

same up to  
N factors.



H.O.

$$\int \mathcal{D}x e^{i \int dt \left( \frac{m \dot{x}^2}{2} + \frac{1}{2} m \omega^2 x^2 \right)}$$

$$X = X_{cl} + \delta X \quad \delta X(t) = \delta X(T) = 0 \quad - m \dot{x} \delta x + m \omega^2 x \delta x$$

$$\int \mathcal{D}x e^{i \int dt \left( \frac{m \dot{X}_{cl}^2}{2} + \underbrace{2m (\dot{x} \delta \dot{x})}_{\text{boundary terms}} + \frac{1}{2} m \delta \dot{x}^2 + \frac{1}{2} m \dot{X}_{cl}^2 + m \omega^2 x \delta x + \frac{1}{2} m \omega^2 \delta x^2 \right)}$$

$$\int \mathcal{D}x e^{i S_{cl}} e^{i \int dt \left( \frac{1}{2} m \dot{y}^2 - \frac{1}{2} m \omega^2 y^2 \right)}$$

$$y(t) = y(T) = 0.$$

Sol :  $X = A \cos \omega t + B \sin \omega t$

$$X = X_i \cos \omega t + B \sin \omega t$$

$$X_f = X_i \cos \omega T + B \sin \omega T$$

$$B = \frac{1}{\sin \omega T} (X_f - X_i \cos \omega T)$$

$$X = X_i \cos \omega t + \frac{\sin \omega t}{\sin \omega T} (X_f - X_i \cos \omega T)$$

$$\dot{X} = -X_i \sin \omega t \omega + \frac{\omega \cos \omega t}{\sin \omega T} (X_f - X_i \cos \omega T)$$

(8)

$$S_d = \int dt \left( \frac{m \dot{x}^2}{2} - \frac{1}{2} m \omega^2 x^2 \right) =$$

$$= \int dt \left( -\frac{m \ddot{x} x}{2} - \frac{1}{2} m \omega^2 x^2 + \partial_t \left( \frac{m x \dot{x}}{2} \right) \right)$$

$$\ddot{x} = -\omega^2 x$$

$$= \left. \frac{m x \dot{x}}{2} \right|_0^T = \frac{m x_f}{2} \dot{x}(T) - \frac{m x_i}{2} \dot{x}(0)$$

$$= \frac{m}{2} \left( x_f (-\omega x_i \sin \omega T) + \frac{\omega \cos \omega T}{\sin \omega T} (x_f - x_i \cos \omega T) - \right.$$

$$\left. - x_i \frac{\omega}{\sin \omega T} (x_f - x_i \cos \omega T) \right)$$

$$= \frac{m}{2} \left( -\omega \sin \omega T x_i x_f + \omega x_f^2 \frac{\cos \omega T}{\sin \omega T} - x_i x_f \omega \frac{\cos^2 \omega T}{\sin \omega T} - \right.$$

$$\left. - \frac{x_i x_f \omega}{\sin \omega T} + x_i^2 \omega \frac{\cos \omega T}{\sin \omega T} \right)$$

$$= \frac{m}{2} \left[ \frac{\omega x_i x_f}{\sin \omega T} (-\sin^2 \omega T - \cos^2 \omega T - 1) + \frac{\omega \cos \omega T}{\sin \omega T} (x_i^2 + x_f^2) \right]$$

(9)

$$S_{cl} = \frac{m\omega}{2} \cot \omega T (x_i^2 + x_f^2) - \frac{m\omega x_i x_f}{\sin \omega T}$$

$$K = e^{\frac{i m \omega}{2} \cot \omega T (x_i^2 + x_f^2) - \frac{i m \omega}{\sin \omega T} x_i x_f} \int \mathcal{D}y(t) e^{i S[y(t)]}$$

$$y(0) = y(T) = 0$$

$$\int \mathcal{D}y(t) e^{i \int \frac{m}{2} \dot{y}^2 - \frac{1}{2} m \omega^2 y^2} = \int \mathcal{D}y(t) e^{-\frac{i m}{2} \int y (\ddot{y} + \omega^2 y)}$$

$$= \int \mathcal{D}y(t) e^{-\frac{i m}{2} \int y (\partial_t^2 + \omega^2) y} = \det^{-1/2} \left( \frac{i m}{2} (\partial_t^2 + \omega^2) \right)$$

eigenvalues  $y = A \sin \left( \frac{n t \pi}{T} \right)$

$$= \prod_{n=1}^{\infty} \frac{1}{\sqrt{\frac{i m}{2} \left( \omega^2 - \frac{n^2 \pi^2}{T^2} \right)}} = \frac{1}{\left( \frac{i m \omega^2}{2} \right)^{N/2}} \prod_{n=1}^{\infty} \frac{1}{\left( 1 + \frac{n^2 \pi^2}{\omega^2 T^2} \right)^{1/2}}$$

$$\frac{\sin x}{x} = \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{k^2 \pi^2} \right)$$

$$\ln K = \sum_{n=1}^{\infty} \ln \left( \frac{\omega^2 T^2}{\omega^2 T^2 + n^2 \pi^2} \right)$$

Use them:

$$\frac{\det(\partial_t^2 + \omega^2)}{\det(\partial_t^2)} = \prod_{n=1}^{\infty} \frac{\omega^2 - \frac{n^2 \pi^2}{T^2}}{-\frac{n^2 \pi^2}{T^2}} = \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{n^2 \pi^2}\right)$$

$$= \frac{\sin(\omega T)}{\omega T}$$

$$\det^{-1/2}(\partial_t^2 + \omega^2) = \sqrt{\frac{\omega T}{\sin \omega T}} \underbrace{\det^{-1/2}(\partial_t^2)}_{\text{free particle}} K_{f.p.}(0, t_f, 0, t_i)$$

$$= e^{-\frac{i\theta}{\hbar}} \sqrt{\frac{m}{2\pi T}}$$

$$i \frac{m\omega}{2} \left[ \cotan(\omega T) (x_i^2 + x_f^2) - \frac{2x_i x_f}{\sin \omega T} \right]$$

$$K_{ho} = e^{i\theta/\hbar} \sqrt{\frac{m\omega}{2\pi \sin \omega T}} e$$

↑ path-integral result agrees.

Derivation

$$\langle x_f | e^{-iHt} | x_i \rangle = \int dx_1 \dots dx_{N-1} \langle x_f | e^{-iH\Delta t} | x_{N-1} \rangle \langle x_{N-1} | \dots e^{-iH\Delta t} | x_0 \rangle$$

$$= \int dx_1 \dots dx_{N-1} K(x_f, \Delta t; x_{N-1}, 0) \dots K(x_1, \Delta t; x_0, 0)$$

$$K(x_2, \Delta t; x_1, 0) = \langle x_2 | e^{-i\frac{p^2}{2m}\Delta t + iV(x)\Delta t} | x_1 \rangle =$$

$$= \langle x_2 | e^{-i\frac{p^2}{2m}\Delta t} e^{-iV(x)\Delta t} | x_1 \rangle =$$

↙  
to order Δt

$$= \int dp e^{-i\frac{p^2}{2m}\Delta t} e^{ip(x_2-x_1)} e^{-iV(x_2)\Delta t}$$

$$= \int = \sqrt{\frac{2m\pi}{i\Delta t}} e^{-\frac{(x_2-x_1)^2}{4i\frac{\Delta t}{2m}} - iV(x_2)\Delta t}$$

$$= \sqrt{\frac{2m\pi}{i\Delta t}} e^{\frac{im(x_2-x_1)^2}{2\Delta t} - iV(x_2)\Delta t} = N_{\Delta t} e^{iS(x_2, x_1, \Delta t)}$$

$$K(x_f, T; x_i, 0) = N \int dx_1 \dots dx_{N-1} e^{i \sum_j S(x_j, \Delta t; x_j, 0)}$$

$$\lim_{N \rightarrow \infty} = N \int dx_1 \dots dx_{N-1} e^{i \int L}$$

What if

$$H = \frac{p^2}{2m} f(x) + V(x)$$

$$\langle x_2 | e^{-i \frac{p^2}{2m} f(x) \Delta t} | p \rangle$$

$$\int dp e^{-\frac{i}{2m} f(x) p^2 \Delta t} e^{ip(x_2 - x_1)} = \sqrt{\frac{\pi}{\frac{i}{2m} f(x)}} e^{\frac{im (\Delta x)^2}{2 f \Delta t}}$$

$$\prod_i \frac{1}{i \sqrt{f(x_i)}} = e^{\frac{1}{2} \sum_i \ln f(x_i)} = e^{\frac{1}{2} \frac{1}{\Delta t} \sum_i \ln f(x_i) \Delta t}$$

$$= e^{\frac{1}{2} \delta(0) \int dt \ln f(x(t))}$$

↗  
erhalten.

$$S_{\text{eff}} = \int dt L(q, \dot{q}) - \frac{i}{2} \delta(0) \int dt \ln f(q)$$

$$\int \left[ \sqrt{f(q)} dq \right] e^{iS}$$

# Fermionic path integral.

Grassmann numbers

$$\theta \eta = -\eta \theta$$

$$\theta^2 = 0$$

$$f(\theta) = a + b\theta$$

$$\int d\theta (a + b\theta) = \int d\theta (a + b\theta + b\eta) \quad ; \quad \int d\theta (a + b\theta) = b$$

$$\int d\theta 1 = 0 \quad \int d\theta \theta = 1$$

Fermion zero mode very important.

$$\int d\theta_1 \dots d\theta_n e^{\theta_i A_{ij} \theta_j} = \int d\xi_1 \dots d\xi_n e^{\dots}$$

↑ antisymmetric
↑ diagonalize

complex  $\theta = \frac{1}{\sqrt{2}} (\theta_1 + i\theta_2)$

$$\int d\theta^* d\theta (\theta\theta^*) = 1 \quad \int d\theta^* d\theta e^{-a\theta\theta^*} = \int d\theta^* d\theta (1 + a\theta\theta^*) = a$$

$$\int d\theta_i^* d\theta_j e^{-\theta_i^* A_{ij} \theta_j} = \int d\theta_i^* d\xi_i e^{-\xi_i^* a_i \xi_i} = \prod_i a_i = \det A$$

↑ diagonalize

$$\int d\theta_i^* d\theta_i \quad \theta_k \theta_l^* e^{-\theta_i^* B_{ij} \theta_j} =$$

$$\theta_k = U_{ke} \xi_e \quad \theta_k^* = \xi_e^* U_{ek}^+$$


$$\prod_k d\theta_k = \prod_k (U_{ke} d\xi_e) = \det U \prod_e d\xi_e = \prod_e d\xi_e$$

$$\int d\xi_e^* d\xi_e \quad U_{ki} \xi_i^* \xi_e^* U_{el}^+ e^{-\sum_p \xi_p^* \xi_p (\Lambda_B)_p}$$

$U^+ B U = \Lambda_B$   
↑  
diag.


$$= \int d\xi_e^* d\xi_e \quad U_{ki} \xi_i^* \xi_e^* U_{el}^+ \det B =$$

$$= \det B (U \Lambda_B^{-1} U^+)_{kl} = \det B B_{kl}^{-1}$$



$$\int d\theta(t) e^{i\int_0^T \theta \dot{\theta} dt} = \int d\theta_i e^{\sum_j i(\omega_j - \omega_i) \theta_j \Delta t}$$

$$= \det(i\partial_t - E) = \prod_n \left( \frac{i\omega_n}{T} - E \right)$$



$$\prod_n e^{\frac{2\omega_n T}{i\omega_n T} - E} \quad i\vec{\omega} - E\vec{0}$$



QM example.

\_\_\_\_\_  $\omega/2$ 

$$L = i\bar{\theta}\dot{\theta} - \omega\bar{\theta}\theta$$

\_\_\_\_\_  $-\omega/2$ 

$$S = \int dt \bar{\theta} (i\partial_t - \omega)\theta$$

$$\int_{0 \rightarrow T} d\bar{\theta} d\theta e^{iS} = \det(-\partial_t - i\omega) = \prod_{n=-\infty}^{\infty} \left( -\frac{i2\pi}{T}(n+1/2) - i\omega \right)$$

↑  
antiperiodic

$$e^{i(n+1/2)\frac{2\pi t}{T}} \rightarrow +\frac{2\pi}{T}(n+1/2)i \text{ eigenvalue of } \partial_t$$

$$= \prod_{n=-\infty}^{\infty} \left( -\frac{2\pi i}{T}(n+1/2) \right) \prod_{n=-\infty}^{\infty} \left( 1 + \frac{\omega T}{2\pi(n+1/2)} \right)$$

$$\underbrace{\qquad\qquad\qquad}_{\omega T \left( \frac{\omega T}{2} \right)}$$

$$\det(-\partial_t) = 2 \text{ regularized.}$$

$$\det(-\partial_t - i\omega) = 2 \omega T \left( \frac{\omega T}{2} \right) = e^{\frac{i\omega T}{2}} + e^{-\frac{i\omega T}{2}}$$

with periodic.  $\text{Tr}((-)F e^{-iHt})$

$$\int_{-\infty}^{\infty} d\omega e^{i\omega t} = \det(-\partial_t - i\omega) = \prod_{n=-\infty}^{\infty} \left( -i \frac{2\pi}{T} n - i\omega \right)$$

↑  
periodic

$$= \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( -\frac{2\pi i n}{T} \right) \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( 1 + \frac{\omega T}{2\pi n} \right) (-i\omega)$$

$$= -i\omega \prod_{n=-\infty}^{\infty} \left( -\frac{2\pi i n}{T} \right) \prod_{n=1}^{\infty} \left( 1 - \frac{\omega^2 T^2}{4n^2 \pi^2} \right)$$

$$= -i \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left( -\frac{2\pi i n}{T} \right) \frac{\sin\left(\frac{\omega t}{2}\right)}{\frac{\omega t}{2}}$$

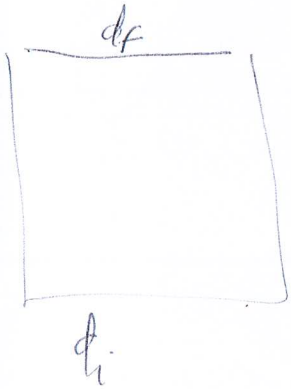
$$= -2i \prod_{n=1}^{\infty} \left( -\frac{2\pi i n}{T} \right)^2 \frac{\sin\left(\frac{\omega t}{2}\right)}{T} = -4\pi^2 i \text{sh}\left(\frac{\omega t}{2}\right) \quad (2A)$$

$$\prod_{n=1}^{\infty} \left( -\frac{4n^2 \pi^2}{T^2} \right) = 4n! \frac{T}{2}$$

$$\det(\partial_t^2)_{T/2} = \frac{4nT}{-i}$$

$$\left. \begin{aligned} & 2i \frac{e^{-i\omega t/2} - e^{i\omega t/2}}{2i} \\ & = -2 \text{sh}\left(\frac{\omega t}{2}\right) \end{aligned} \right\}$$

Path integral for scalar field



$$\int \mathcal{D}\phi e^{-\int_{-T}^T d^4x \mathcal{L}(\phi, \partial\phi)} = \langle \phi_f(x) | e^{-iHT} | \phi_i(x) \rangle$$

$$T \rightarrow \infty (1-i\epsilon) \quad e^{-2iH(\infty)(1-i\epsilon)} = e^{-2iH\omega - 2H\omega \cdot \epsilon} \quad | \phi_v \rangle$$

only the vacuum contributes.

$$\int \mathcal{D}\phi \phi(x_1) \phi(x_2) e^{-\int_{-T}^T d^4x \mathcal{L}(\phi, \partial\phi)} =$$

$$= \int \mathcal{D}\phi_1(x_1) \int \mathcal{D}\phi_2(x_2) \int \mathcal{D}\phi(x) e^{-\int_{-T}^T d^4x \mathcal{L}(\phi, \partial\phi)} \phi(x_1) \phi(x_2)$$

$\phi(x_1, \vec{x}) = \phi_1(\vec{x})$   
 $\phi(x_2, \vec{x}) = \phi_2(\vec{x})$

$$= \int \mathcal{D}\phi_1(x) \int \mathcal{D}\phi_2(x) \phi_1(\vec{x}) \phi_2(\vec{x}) e^{-iH(\phi_1, \phi_2)} e^{-iH(\phi_1, \phi_2)} | \phi_1 \rangle \langle \phi_2 | e^{-iH(T-x)}$$

$$= \langle \Omega | \hat{T} \{ \phi_H(x_2) \phi_H(x_1) \} | \Omega \rangle$$

$$\langle \Omega | \hat{T} \{ \phi_H(x_1) - - \phi_H(x_2) \} | \Omega \rangle = \int \mathcal{D}\phi \phi(x_1) - - \phi(x_2) e^{iS[\phi]} / \int \mathcal{D}\phi e^{iS[\phi]}$$

Generating function

(18)

$$Z[j(x)] = \int \mathcal{D}\phi e^{iS[\phi] + i \int d^4x \phi(x) j(x)} = \langle \Omega | T \{ e^{i \int d^4x \phi(x) j(x)} | \Omega \rangle$$

$$\frac{\delta^n Z[j(x)]}{\delta j(x_1) \dots \delta j(x_n)} = i^n \int \mathcal{D}\phi \phi(x_1) \dots \phi(x_n) e^{iS[\phi] + i \int d^4x \phi(x) j(x)}$$

$$\langle \Omega | T \{ \phi(x_1) \dots \phi(x_n) | \Omega \rangle = \frac{(-i)^n \delta^n Z[j(x)]}{Z[j]} \Big|_{j=0}$$

$-iW(j)$  ← connected diagrams

$$Z(j) = e$$

$$W(j) = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G_c(x_1, \dots, x_n) j(x_1) \dots j(x_n)$$

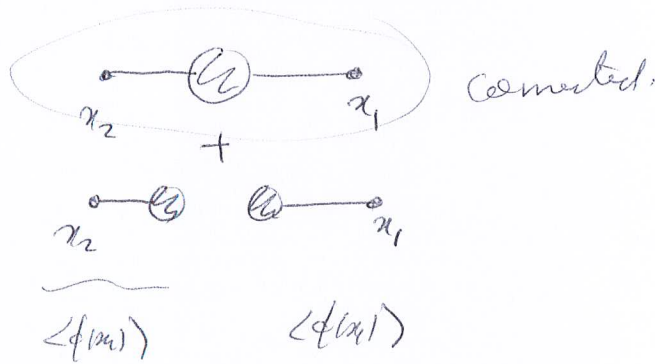
$$W = i \ln Z$$

$$\frac{\delta W}{\delta j} = i \frac{1}{Z} \frac{\delta Z}{\delta j} = - \langle \phi(x) \rangle_j$$

$$\frac{\delta^2 W}{\delta j(x_1) \delta j(x_2)} = i \frac{1}{Z} \frac{\delta^2 Z}{\delta j(x_1) \delta j(x_2)} - \frac{i}{Z^2} \frac{\delta Z}{\delta j(x_2)} \frac{\delta Z}{\delta j(x_1)}$$

$$\frac{\delta^2 W}{\delta j(x_2) \delta j(x_1)} = -i \langle \eta | \{ \phi(x_2) \phi(x_1) \} | \Omega \rangle + i \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle$$

$$= -i (\langle \phi(x_2) \phi(x_1) \rangle - \langle \phi(x_1) \rangle \langle \phi(x_2) \rangle)$$

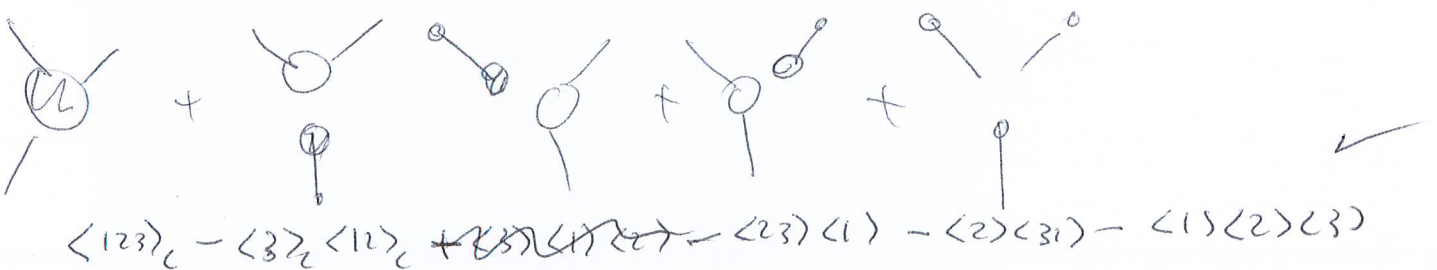


$$\frac{\delta^3 W}{\delta j(x_3) \delta j(x_2) \delta j(x_1)} = i \frac{1}{Z} \frac{\delta^3 Z}{\delta j(x_3) \delta j(x_2) \delta j(x_1)} - \frac{i}{Z^2} \frac{\delta Z}{\delta j(x_3)} \frac{\delta^2 Z}{\delta j(x_2) \delta j(x_1)}$$

$$+ \frac{2i}{Z^3} \frac{\delta Z}{\delta j(x_3)} \frac{\delta Z}{\delta j(x_2)} \frac{\delta Z}{\delta j(x_1)} - \frac{i}{Z^2} \frac{\delta^2 Z}{\delta j_3 \delta j_2} \frac{\delta Z}{\delta j_1} - \frac{i}{Z^2} \frac{\delta Z}{\delta j_2} \frac{\delta^2 Z}{\delta j_3 \delta j_1}$$

$$= \langle \phi(x_3) \phi(x_2) \phi(x_1) \rangle = \langle \phi(x_3) \rangle \langle \phi_1 \phi_2 \rangle - \langle \phi_2 \phi_3 \rangle \langle \phi_1 \rangle$$

$$- \langle \phi_2 \rangle \langle \phi_3 \phi_1 \rangle + 2 \langle \phi_1 \rangle \langle \phi_2 \rangle \langle \phi_3 \rangle$$



$$\frac{\delta^n W(j)}{\delta j(x_1) \dots \delta j(x_n)} = i^{n+1} \langle \phi(x_1) \dots \phi(x_n) \rangle_c$$

Effective action

$$\langle \Omega | \phi(x) | \Omega \rangle_j = - \frac{\delta W}{\delta j} = \phi_{cl}(x)$$

$$\phi_{cl}(x) \equiv j[\phi_{cl}(x)]$$

$$\Gamma[\phi_{cl}(x)] \equiv -W(j) - \int d^4x j(x) \phi_{cl}(x)$$

$\uparrow$   
effective action.

$$\frac{\delta \Gamma[\phi_{cl}(x)]}{\delta \phi_{cl}(x)} = \phi_{cl} \rightarrow = - \frac{\delta W}{\delta j_y} \frac{\delta j_y}{\delta \phi_x} - j(x) - \int d^4y \frac{\delta j_y}{\delta \phi_x}$$

$$= \int d^4y \phi_{cl}(y) \frac{\delta j_y}{\delta \phi_x} - j(x) - \int d^4y \frac{\delta j_y}{\delta \phi_x} = -j(x)$$

$$\frac{\delta}{\delta \phi_{cl}} \left[ \Gamma[\phi_{cl}] + \int_x j(x) \phi_{cl}(x) \right] = 0$$

$$\boxed{\frac{\delta \Gamma}{\delta \phi} = 0}$$

$$j=0 \quad \frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}} = 0$$

$$\frac{\delta \Gamma}{\delta \phi_{cl}} = -j$$

$$\frac{\delta}{\delta j(x)} \frac{\delta \Gamma}{\delta \phi_{cl}(y)} = -\delta^{(4)}(x-y)$$

$$\int_z \frac{\delta \phi_{cl}(z)}{\delta j(x)} \frac{\delta^2 \Gamma}{\delta \phi_{cl}(z) \delta \phi_{cl}(y)} = -\delta^{(4)}(x-y)$$

$$\int_z \frac{\delta^2 W}{\delta j(x) \delta j(z)} \frac{\delta^2 \Gamma}{\delta \phi_{cl}(y) \delta \phi_{cl}(z)} = \delta^{(4)}(x-y)$$

$$-i \langle \phi(x) \phi(z) \rangle_c \quad \Gamma_{yz}^{(2)} = \delta^{(4)}$$

$$\Gamma_{(y,z)}^{(2)} = i D^{-1}(y,z)$$

$$\begin{aligned}
 \textcircled{a} &= \textcircled{a} - + - 0 - + - 0 - 0 - + - \\
 &= - ( 1 + 0 - + 0 - 0 - + - ) \\
 &= \frac{1}{1 - 0 -} = \frac{1}{(-)^{-1} - 0}
 \end{aligned}$$

$$\begin{aligned}
 (\textcircled{a})^{-1} &= (-)^{-1} - 0 \\
 &= -i (p^2 - m^2 - \Sigma(p^2))
 \end{aligned}$$

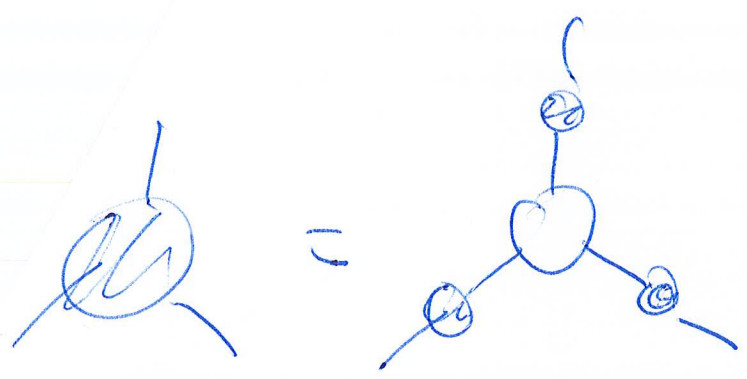
$$\frac{\delta^2 W}{\delta J \delta J} = \left( \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \right)^{-1}$$

$$\delta M^{-1} = -M^{-1} \delta M M^{-1}$$

$$\frac{\delta}{\delta J(x)} = \int d^4 y \frac{\delta \phi}{\delta J(y)} \frac{\delta}{\delta \phi} = i \int d^4 x D(x, y) \frac{\delta}{\delta \phi(y)}$$

$$\begin{aligned}
 \frac{\delta^3 W}{\delta J \delta J \delta J} &= -i \left( \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \right)^{-1} \int D \frac{\delta^3 \Gamma}{\delta \phi \delta \phi \delta \phi} \cdot \left( \frac{\delta^2 \Gamma}{\delta \phi \delta \phi} \right)^{-1} \\
 &= -i D D \frac{\delta^3 \Gamma}{\delta \phi \delta \phi \delta \phi} D
 \end{aligned}$$

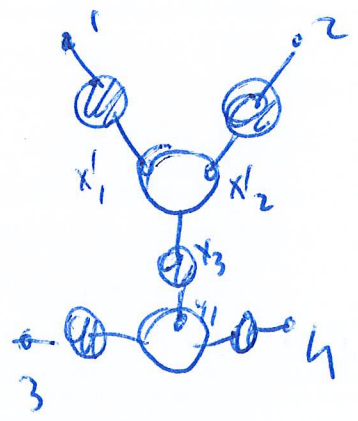
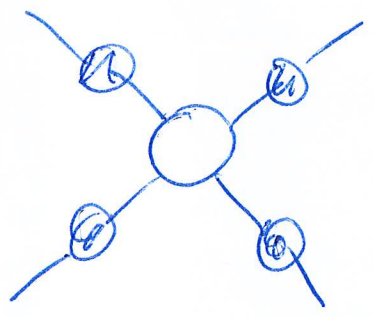




$$\frac{\delta^4 W}{\delta J_{x_4} \delta J_{x_3} \delta J_{x_2} \delta J_{x_1}} = -i \frac{\delta}{\delta J_{x_4}} \left( \frac{\delta^3 \Gamma}{\delta J_{x_3} \delta J_{x_2} \delta J_{x_1}} \right)$$

$$= -i \left( \frac{\delta^4 \Gamma}{\delta J_{x_4} \delta J_{x_3} \delta J_{x_2} \delta J_{x_1}} \right) \frac{\delta^3 \Gamma}{\delta J_{x_3} \delta J_{x_2} \delta J_{x_1}} + \dots$$

$$\left( \frac{\delta^4 \Gamma}{\delta J_{x_4} \delta J_{x_3} \delta J_{x_2} \delta J_{x_1}} \right)$$



+ (-) + (-)

Feynman rules from path integral.

①

$$Z[j(x)] = \int \mathcal{D}\phi \ e^{iS[\phi] + i \int j(x)\phi(x)}$$

generating function.

example  $\phi^4$

$$Z[j(x)] = \int \mathcal{D}\phi \ e^{i \int d^4x \left( \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right) + i \int j(x)\phi(x)}$$

$$= \int \mathcal{D}\phi \ e^{-\frac{i\lambda}{4!} \int d^4x \phi^4} \ e^{-\frac{i}{2} \int d^4x \phi(x) (\partial^2 + m^2) \phi(x) + i \int j(x)\phi(x)}$$

$$= \int \mathcal{D}\phi \ e^{-\frac{i\lambda}{4!} \int d^4x \frac{\delta^4}{\delta j^4(x)}} \ e^{-\frac{i}{2} \int d^4x \dots}$$

$$= e^{-\frac{i\lambda}{4!} \int d^4x \frac{\delta^4}{\delta j^4(x)}} \ \propto \ e^{-\frac{i}{4^{3/2}} \int j(x) (\partial^2 + m^2)^{-1} j(y)}$$

$$= e^{-\frac{i\lambda}{4!} \int d^4x \frac{\delta^4}{\delta j^4(x)}} \ \propto \ e^{\frac{i}{2} \int j(x) \Delta(x,y) j(y)}$$

$$(\partial_x^2 + m^2) \Delta(x,y) = \delta^{(4)}(x-y)$$

$$(-p^2 + m^2) \Delta(p) = 1 \quad \Delta(p) = -\frac{1}{p^2 - m^2}$$

$$e^{-\frac{i\lambda}{\hbar} \int d^4x \frac{\delta}{\delta j(x)}} \quad e^{-\frac{i}{2} \int j(x) \Delta_F(x-y) j(y)}$$

$$\Delta_F \rightarrow \frac{i}{p^2 - m^2}$$

$$(-i) \frac{\delta}{\delta j(x_1)} \dots (-i) \frac{\delta}{\delta j(x_n)} e^{-\frac{i}{2} \int j(x) \Delta_F(x-y) j(y)}$$

$$(-i) \frac{\delta}{\delta j(x_2)} \left[ \frac{i}{2} \left( +i \int j(x) \Delta_F(x-x_1) \right) e^{-\frac{i}{2} \int j(x) \Delta_F(x-y) j(y)} \right]$$

$$+ \Delta_F(x_2-x_1) + i \int j(x) \Delta_F(x-x_1) i \int j(y) \Delta_F(y-x_2)$$

Gives Wick's theorem.

Effective action  $Z(j) = e^{iW} = \int \mathcal{D}\phi e^{iS[\phi] + j\phi}$

$j$  chosen such that  $\langle \phi \rangle = \phi_{cl}(x)$

take  $\phi = \phi_{cl} + \eta$   $Z(j) = \ln \Gamma(\phi_{cl}) = e^{iW - iJ\phi} = \int \mathcal{D}\eta e$

$$= e^{iS(\phi_{cl})} \int \mathcal{D}\eta e^{i \frac{\delta S}{\delta \phi} \eta + \frac{\delta^2 S}{\delta \phi^2} \eta \eta + \dots + \left( \frac{\delta S}{\delta \phi} \eta + i\eta \right) + \dots + \delta j \phi + \dots} = \dots$$

$j = -\delta S / \delta \phi$

# Path integrals for gauge theories

$$S = \frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}$$

$$= \frac{1}{2} \int d^4x \partial_\mu A_\nu (\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= \frac{1}{2} \int d^4x A_\nu (\partial^2 A - \partial_\mu \partial^\mu A)$$

$$= \frac{1}{2} \int d^4x A_\nu (\partial^2 \eta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu) A_\nu$$

$$(-\eta^{\mu\nu} p^2 + p^\mu p^\nu) A_{\nu\alpha} = \delta^\mu_\alpha$$

$$0 = -p^\mu p^2 + p^\mu p^2 = p_\nu$$

zero ~~sub~~ eigenvalues  $\rightarrow$  no inverse

$A_\mu \rightarrow A_\mu + \partial_\mu \lambda$  ; extra degrees of freedom.

gauge fixing

(4)

$$\int \mathcal{D}A_\mu e^{iS[A_\mu]}$$

path integral here

space of  $A_\mu$  configurations

could be other  $\leftarrow$  should give the same.



gauge orbits

Gauge fixing  $\mathcal{F}(A_\mu) = 0$   $A_\mu^\lambda = A_\mu + \partial_\mu \lambda$

$$\Delta_f^{-1}[A_\mu] = \int \underbrace{\mathcal{D}\lambda(x)}_{\text{gauge transf.}} \delta(\mathcal{F}(A_\mu^\lambda))$$

$$\Delta_f^{-1}[A_\mu^{\tilde{\lambda}}] = \int \mathcal{D}\lambda(x) \delta(\mathcal{F}(A_\mu^{\lambda+\tilde{\lambda}}))$$

$$= \int \mathcal{D}\lambda(x) \delta(\mathcal{F}(A_\mu^\lambda)) = \Delta_f^{-1}[A_\mu]$$

↑  
gauge invariant

functional

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$$\int \mathcal{D}A_\mu \int \mathcal{D}\lambda \delta(\mathcal{F}(A_\mu)) \Delta_f(A_\mu) e^{iS[A_\mu]} =$$

gauge transf  $\bar{\lambda}$

$$= \left( \int \mathcal{D}\lambda \right) \int \mathcal{D}A_\mu \delta(\mathcal{F}(A_\mu)) \Delta_{FP}(A_\mu) e^{iS[A_\mu]}$$

Assuming measure & action are gauge inv.

Volume of gauge group  $\rightarrow \infty$  but indep. of  $A_\mu$ , etc.  
b.c.

$$\int \mathcal{D}A_\mu \delta(\mathcal{F}(A_\mu)) \int \mathcal{D}\bar{c} \mathcal{D}c e^{i\int \bar{c} A_{FP} c} e^{iS[A_\mu]}$$

$$\delta(\mathcal{F}(A_\mu) - B_\mu)$$

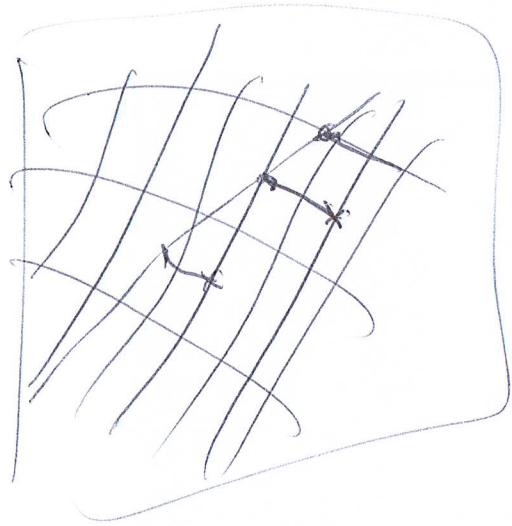
$$\int e^{-\frac{i}{\int x} \int B^2(x)} \mathcal{D}B(x)$$

$$i \int_{x\mu} \bar{c} A_{FP} c + \frac{i}{\int} \int \mathcal{F}^2 + iS[A_\mu]$$

$$\int \mathcal{D}A_\mu \int \mathcal{D}\bar{c} \mathcal{D}c e^{i \int_{x\mu} \bar{c} A_{FP} c + \frac{i}{\int} \int \mathcal{F}^2 + iS[A_\mu]}$$

$$\int \mathcal{D}\lambda \delta(\mathcal{F}(A_\mu^\lambda)) = \int \mathcal{D}\lambda \delta(\mathcal{F}(A_\mu^{\delta_0 + \lambda}))$$

②



$$= \int \mathcal{D}\lambda \delta(\mathcal{F}(A_\mu^{\text{cur } \lambda}))$$

$$= \int \mathcal{D}\lambda \delta\left(\int \frac{\delta \mathcal{F}(x)}{\delta \lambda(y)} \lambda(y) + \dots\right)$$

$$= \det^{-1}\left(\frac{\delta \mathcal{F}(x)}{\delta \lambda(y)}\right)$$

$$\Delta_{FP} = \det\left(\frac{\delta \mathcal{F}(A_\mu(x))}{\delta \lambda(y)}\right) \Bigg|_{\mathcal{F}(A) = 0}$$

$$= \int \mathcal{D}\bar{c}(y) \mathcal{D}c \ e^{\int \bar{c}(x) \frac{\delta \mathcal{F}(x)}{\delta \lambda(y)} c(y)}$$

$\mathcal{F}(A) = 0$

$$\zeta(x) = \int \frac{\delta \mathcal{F}(x)}{\delta \lambda(y)} \lambda(y)$$

$$\frac{\delta \zeta(x)}{\delta \lambda(y)} = \frac{\delta \mathcal{F}(x)}{\delta \lambda(y)}$$

$$\int \mathcal{D}\zeta \ \det^{-1}\left(\frac{\delta \zeta(x)}{\delta \lambda(y)}\right) \delta(\zeta(x)) = \det\left(\frac{\delta \mathcal{F}(x)}{\delta \lambda(y)}\right)$$

examples

$$\partial_\mu A^\mu = 0$$

$$\Delta_{FP}^{-1}(A_\mu) = \int \mathcal{D}\lambda(x) \delta(\partial_\mu A^\mu + \partial_\alpha^2 \lambda) = \partial^{-2}$$

$$\Delta_{FP}(A_\mu) = \partial^2 \quad \text{indep. of } A_\mu$$

$$A_\mu A^\mu = 0 \quad \bar{c} \partial^2 c$$

$$\Delta_{FP}^{-1}(A_\mu) = \int \mathcal{D}\lambda(x) \delta((A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda))$$

$$A_\mu \rightarrow A_\mu + \partial_\mu \lambda \quad / \quad (A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda) = 0$$

$$A_\mu A^\mu + 2A^\mu \partial_\mu \lambda + \partial_\mu \lambda \partial^\mu \lambda = 0$$

$$\left. \frac{\delta \mathcal{L}(A_\mu)}{\delta \lambda(x)} \right|_{F=0} = \frac{\delta((A_\mu + \partial_\mu \lambda)(A^\mu + \partial^\mu \lambda))}{\delta \lambda(y)} = 2(A^\mu + \partial^\mu \lambda) \partial^\mu \delta(x-y)$$

$$= 2A^\mu_{,\lambda} \partial_\mu \delta(x-y)$$



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$$2 \bar{c} \partial_\mu c A^\mu$$

$$\int \mathcal{D}A_\mu e^{2 \int \bar{c} \partial_\mu c A^\mu} \delta(A_\mu A^\mu) e^{i \int \mathcal{L} d^4x}$$

$$\int \mathcal{D}A_\mu e^{2 \int \bar{c} \partial_\mu c A^\mu + \frac{1}{\xi} \int (A_\mu A^\mu)(A_\nu A^\nu) + i \int \mathcal{L} d^4x}$$

$$S = \int \bar{c} \delta c - \frac{i}{2\xi} \int (\partial A)^2 - \frac{1}{4} \int F_\mu F^\mu$$

$$+ \frac{1}{2\xi} \int A_\mu \partial_\nu A_\nu - \frac{i}{2} \int \partial_\mu A_\nu (\partial_\nu A_\mu - \partial_\mu A_\nu)$$

$$\frac{i}{2\xi} \int A_\mu \partial_\mu A_\nu + \frac{1}{2} \int A_\nu \partial_\nu^2 A_\mu - A_\nu \partial_\mu A_\nu$$

$$\frac{1}{2} \int A_\nu (\partial_\nu^2 A_\mu - (1 - \frac{1}{\xi}) \partial_\nu \partial_\mu A_\nu)$$

$$\xi = 1 \quad \frac{i}{2} \int A_\nu \partial_\nu^2 A_\mu \quad - \frac{i}{2} \int \phi \partial^2 \phi$$

Scale field

$$\langle A_\mu h \rangle = \frac{-i \eta_\mu}{p^2 + i\epsilon}$$

$$(-p_\mu^2 + (1 - \frac{1}{\xi}) p_\mu p_\nu) A_{\nu\rho} = \frac{\eta_\rho}{p\rho}$$

$a \eta^\mu + b p^\mu$

$$(-a p^2 \delta_\mu^\rho - b p^2 p_\mu p^\rho + (1 - \frac{1}{\xi}) a p_\mu p^\rho + (1 - \frac{1}{\xi}) p^2 b p_\mu p^\rho = \delta_\mu^\rho$$

$$a = -\frac{1}{p^2}$$

$$-b p^2 + a(1 - \frac{1}{\xi}) + (1 - \frac{1}{\xi}) p^2 b = 0$$

$$b (1 - \frac{1}{\zeta}) p^2 = \frac{1}{p^2} (1 - \frac{1}{\zeta})$$

(10)

$$b = \frac{1}{p^4} \int \left( \frac{1}{\zeta} - 1 \right) = \frac{1 - \zeta}{p^4}$$

$$\langle A_\mu A_\nu \rangle = \frac{-i}{p^2 + i\epsilon} \left( \eta_{\mu\nu} - \frac{(1 - \zeta)}{p^2} p_\mu p_\nu \right)$$

$\zeta = 0$  Lorenz gauge

---

To quantize we fixed the gauge.

However then there is no gauge symmetry to restrict the counterterms?!

BRST symmetry (abelian case)

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu})^2 + \bar{\psi} (i\not{D} - m) \psi + \frac{\xi}{2} B^2 + B \mathcal{O}e(A) + \bar{c} \mathcal{M}c$$

$$\delta \mathcal{O}e = \mathcal{M} \delta \lambda = \frac{\delta \mathcal{O}e}{\delta A(x)} \partial_{\mu} \lambda$$

example  $\mathcal{O}e = \partial_{\mu} A^{\mu}$   $\delta \mathcal{O}e = \partial^2 \lambda = \partial^{\mu} \partial_{\mu} \lambda$   
 $\mathcal{M} = \partial^2$

$$\left. \begin{aligned} \delta A_{\mu} &= \zeta \partial_{\mu} c \\ \delta \psi &= i \zeta c \psi \end{aligned} \right\} \text{gauge transf. w/ parameter } \zeta c$$

↑ fermionic (constant).

$-\frac{1}{4} F^2 + \bar{\psi} (i\not{D} - m) \psi$  is gauge inv.

$$\delta(B \mathcal{O}e(A)) = B \mathcal{M} \zeta c = \zeta B \mathcal{M}c$$

take  $\delta B = 0$

$$\delta(\mathcal{M}c) = \delta\left(\frac{\delta \mathcal{O}e}{\delta A(x)} \partial_{\mu} c\right) = \frac{\delta^2 \mathcal{O}e}{\delta A(x) \delta A(x)} \underbrace{\zeta \partial_{\mu} c \partial_{\mu} c}_{\text{antisym}} + \frac{\delta \mathcal{O}e}{\delta A(x)} \partial_{\mu} \underbrace{\delta c}_0 = 0$$

$$\delta(\bar{c} \mathcal{M}c) = \delta \bar{c} \mathcal{M}c \Rightarrow \boxed{\delta \bar{c} = -\zeta B}$$

Thus:

$$\delta A_\mu = \zeta \partial_\mu c$$

$$\delta \psi = i \zeta c \psi$$

$$\delta c = 0$$

$$\delta B = 0$$

$$\delta \bar{c} = -\zeta B$$

$$Q A_\mu = \partial_\mu c$$

$$Q \psi = i c \psi$$

$$Q \bar{c} = -B$$

$$Q c = 0$$

$$Q B = 0$$

$$Q^2 A_\mu = 0$$

$$Q^2 \psi = \underbrace{ic ic}_{\text{minus}} \psi = 0$$

$$Q^2 \bar{c} = 0$$

$$Q^2 c = 0$$

$$Q^2 B = 0$$

$$\Rightarrow Q^2 = 0$$

Fermionic symmetry.

cohomology  $|\psi\rangle$  states

$Q|\psi\rangle$  are such that  $Q(Q|\psi\rangle) = 0$

Look for solutions of  $Q|\psi\rangle = 0$  modulo trivial  $Q|\psi\rangle$  solutions.

assume all possible pol.

$$A_\mu |0\rangle = \int e^{ikx} \sum_{\sigma \in \{1\}} \epsilon_\mu^\sigma |1_{k,\sigma}\rangle ; \quad \partial_\mu c |0\rangle = \int e^{ikx} k_\mu |1_k^c\rangle$$

$$Q A_\mu |0\rangle = \int e^{ikx} \sum_\sigma \epsilon_\mu^\sigma Q |1_{k,\sigma}\rangle = \int e^{ikx} k_\mu |1_k^c\rangle$$

$$\left( \sum_\sigma \epsilon_\mu^\sigma Q |1_{k,\sigma}\rangle = k_\mu |1_k^c\rangle \right) \epsilon_\mu^{\sigma'} \Rightarrow Q |1_{k,\sigma}\rangle = (\epsilon_\mu^{\sigma'} k^\mu) |1_k^c\rangle$$

$$\text{if } (k \cdot \epsilon) = 0 \quad Q |1_{k,\sigma}\rangle = 0$$

WT, ST, BRST identities

$$S = -\frac{1}{4} F^2 + \bar{\psi} (i \not{D} - m) \psi + \frac{1}{2} B^2 + \text{BR} + \bar{c} \mathcal{K} c \quad (3)$$

$$\int \mathcal{D}[A B c \bar{c}] e^{i \int d^4 x S + i \int d^4 x (j_\mu A^\mu + \bar{\eta} c + \bar{c} \eta + \bar{K} c \psi + \bar{\psi} c K + j_B B + \bar{J}_4 \psi + \bar{\psi} J_4)}$$

change of variables: BRST.

$$" i \int d^4 x (j_\mu \delta c - \delta B \eta + i \bar{J}_4 \delta c \psi - i \bar{\psi} c \delta J_4) \mathcal{G}_c = 0 "$$

$$i \int d^4 x \left( j_\mu \frac{\delta}{\delta \bar{\eta}} - \delta B \frac{\delta}{\delta j_B} + i \bar{J}_4 \frac{\delta}{\delta \bar{K}} - i \frac{\delta}{\delta K} \bar{J}_4 \right) \mathcal{G}_c = 0$$

$\mathcal{G}_c \rightarrow \Gamma$

$$i \int d^4 x \left( j_\mu \frac{\delta \Gamma}{\delta A_\mu} c - \delta B \frac{\delta \Gamma}{\delta \bar{c}} + i \frac{\delta \Gamma}{\delta \bar{\psi}} \frac{\delta \Gamma}{\delta \bar{K}} - i \frac{\delta \Gamma}{\delta K} \frac{\delta \Gamma}{\delta \bar{\psi}} \right) = 0$$

$$i \int d^4 x \left( \frac{\delta \Gamma}{\delta A_\mu(x)} c(x) - \delta B(x) \frac{\delta \Gamma}{\delta \bar{c}(x)} + i \frac{\delta \Gamma}{\delta \bar{\psi}(x)} \frac{\delta \Gamma}{\delta \bar{K}(x)} - i \frac{\delta \Gamma}{\delta K(x)} \frac{\delta \Gamma}{\delta \bar{\psi}(x)} \right) = 0$$

$$i S \rightarrow i \bar{c} \partial^2 c \quad \text{gauge } \partial_\mu A^\mu$$

$$i \partial^2 \frac{\delta \Gamma}{\delta \bar{\eta}} + \eta \mathcal{G} = 0$$

$$i \partial^2 c + \frac{\delta \Gamma}{\delta \bar{c}} = 0$$

$$\Gamma = i \bar{c} \partial^2 c$$

$$B \rightarrow B + dB$$

$$\int (\xi B + \partial_\mu A^\mu) G_c = 0$$

$$B = -\frac{1}{\xi} \partial_\mu A^\mu$$

$$e^{-\frac{F^2}{4\xi}} \quad e^{-\frac{i}{2\xi} (\partial A)^2}$$

$$\mathcal{P} = -\frac{1}{2\xi} (\partial A)^2$$

(4)

$$i \int d^4x \left( \delta_c^g \Gamma + \frac{\delta \Gamma}{\delta c} \frac{1}{\xi} \partial_\mu A^\mu \right) = 0$$

↓  
 $\delta_c^g$

$$\delta_c^g \Gamma + \int \frac{1}{\xi} \partial_\mu A^\mu \delta^2 c = 0$$

$$\delta \left( \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right) = \frac{1}{\xi} \partial_\mu A^\mu \delta^2 c = 0$$

$$\delta_c^g \left( \Gamma + \frac{1}{2\xi} (\partial A)^2 \right) = 0.$$

$\Gamma$  is gauge invariant except for the gauge fixing term!

$\Gamma$  gauge inv.

$$\Gamma \sim \bar{\psi}\psi \quad \bar{\psi}\psi A \quad \bar{\psi}\psi AA \quad \dots$$

$$(\bar{\psi}\psi)(\bar{\psi}\psi) \quad - \quad 1 \quad -$$

$$\delta\psi = i\alpha\psi \quad \delta\bar{\psi} = -i\alpha\bar{\psi} \quad \delta A = \partial_\mu \alpha.$$

$\uparrow$   $\psi, \bar{\psi}$  preserved

$\uparrow$   
 $\# A$  derors by 1.

So we can look at

$$\Gamma = \int d^4x d^4y \bar{\psi}(x) S^{-1}(x-y) \psi(y) + \int d^4x d^4y d^4z$$

$$\bar{\psi}(x) \Gamma^\mu(x-z, y-z) \psi(y) A_\mu(z) + \dots$$

$$\delta\Gamma = \int d^4x d^4y (-i\alpha(x) \bar{\psi}(x) S^{-1}(x-y) \psi(y) + i \bar{\psi}(x) S^{-1}(x-y) \alpha(y) \psi(y))$$

$$+ \int d^4x d^4y d^4z \bar{\psi}(x) \partial_\mu \alpha(z) \Gamma^\mu(x-z, y-z) \psi(y) + \dots$$

$$= 0$$

$$-\partial_\mu^z \Gamma^\mu(x-z, y-z) = i \delta(z-x) S^{-1}(x-y) - i \delta(z-y) S^{-1}(x-y)$$



⑥

$$\Gamma^M(x, y) = \int d^4p d^4q e^{-ipx + iqy} \Gamma^M(p, q)$$

$$\int d^4p d^4q (-\partial_\mu^z) e^{-ip(x-z) + i(q-z)y} \Gamma^M(p, q) =$$

$$e^{i(p-q)z}$$

$$= \int d^4p d^4q (-i) (p-q)_\mu \Gamma^M(p, q) e^{-ip(x-z) + iq(y-z)}$$

$$S^{-1}(x-y) = i \int d^4p e^{-ip(x-y)} \tilde{S}^{-1}(p)$$

$$\delta(z-x) = \int d^4q e^{iq(z-x)}$$

$$= i \int d^4p d^4q \left( e^{iq(z-x) + ip(x-y)} \tilde{S}^{-1}(p) - e^{iq(z-y) + ip(x-y)} \tilde{S}^{-1}(p) \right)$$

$$x \rightarrow x+z \quad y \rightarrow y+z$$

$$-i \int d^4p d^4q (p-q)_\mu \Gamma^M(p, q) e^{-ip(x+z) + iqy} = -i(q+p)_\mu + ipx$$

$$= i \int d^4p d^4q e^{-iqx + ip(x-y)} \tilde{S}^{-1}(q) - i \int d^4p d^4q e^{-iqy + ip(x-y)} \tilde{S}^{-1}(q)$$

$$e^{-i(q+p)x + ipy} \tilde{S}^{-1}(q)$$

$$-i (p-q)_\mu \Gamma^M(p, q) = i (i \tilde{S}^{-1}(q) - i \tilde{S}^{-1}(q+p))$$

$$-i k_\mu \Gamma^M(p+k, p) = i (i \tilde{S}^{-1}(q) - i \tilde{S}^{-1}(q+p+k)) = \tilde{S}^{-1}(p+k) - \tilde{S}^{-1}(p)$$