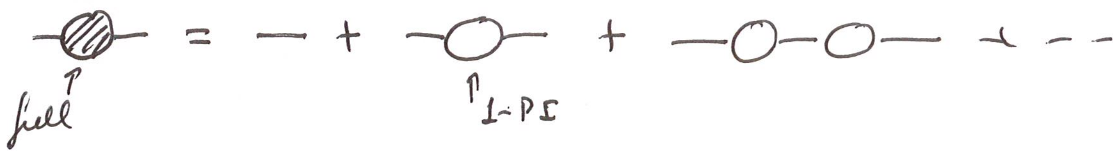
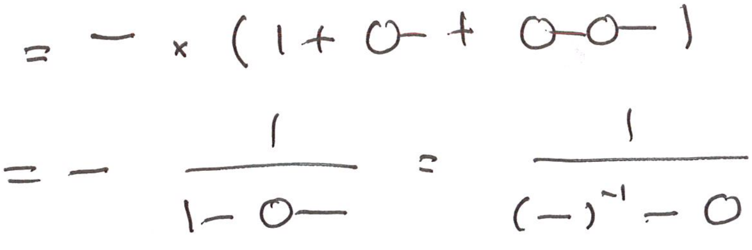
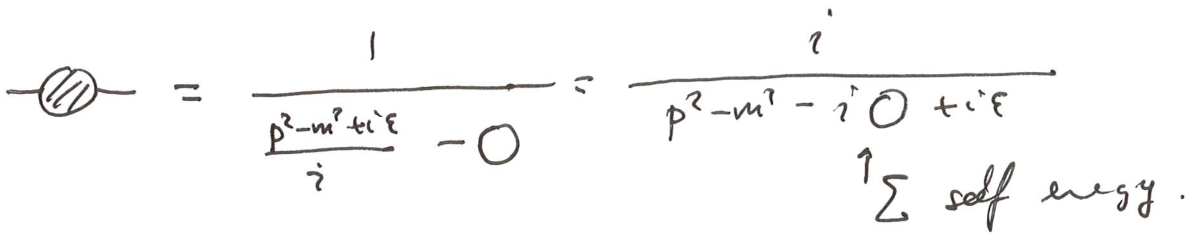
Renormalization of ϕ^4 theory

power counting gives only 2 & 4 point functions divergences.







 $$= \frac{i}{p^2 - m^2 - i\,\Sigma + i\epsilon}$$

$\Sigma =$  1-PI diagrams.

Counter terms

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi_R\,\partial^\mu\phi_R - \frac{1}{2}\,m_R^2\,\phi_R^2 - \frac{\lambda_R}{4!}\,\phi_R^4 + \frac{1}{2}\,\delta_Z\,\partial_\mu\phi_R\,\partial^\mu\phi_R - \frac{1}{2}\,\delta_m\,\phi_R^2 - \frac{\delta_\lambda}{4!}\,\phi_R^4$$

From the Lagrangian we see that this is equivalent to $m^2 \to m^2 + \delta_m \qquad p^2 \to p^2(1+\delta_Z)$

 $\rightarrow \dfrac{i}{p^2 - m^2 - i\Sigma + \delta_Z p^2 - \delta_m + i\varepsilon}$

equivalently

$$V \to V - \frac{1}{2}\delta_Z\, \partial_\mu \phi_R\, \partial^\mu \phi_R + \frac{1}{2}\delta_m \phi_R^2$$

perturbation

 vertex

$e^{-i\int V} \rightarrow$ 

2 ways to connect it

$$\frac{\cancel{2}}{\cancel{2}}(-i)(-\delta_Z p^2 + \delta_m) = i(\delta_Z p^2 - \delta_m)$$

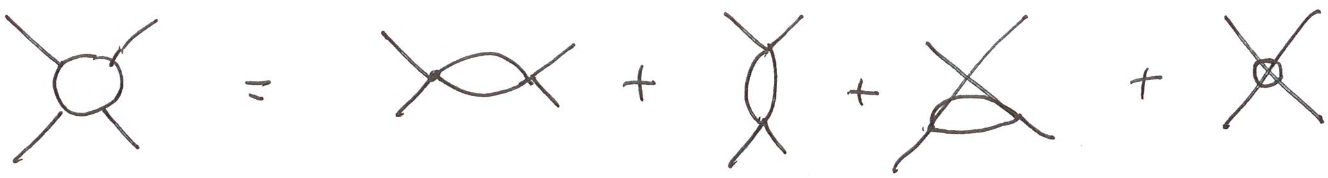
$$-i\Sigma \to -i\, i(\delta_Z p^2 - \delta_m) = \delta_Z p^2 - \delta_m \quad \checkmark \ \underline{\text{same}}$$

$$\Sigma \to \Sigma + i(\delta_Z p^2 - \delta_m)$$

 $i(\delta_Z p^2 - \delta_m)$

 $-i\delta_\lambda$ (same as  $-i\lambda$)

1-loop.



We only need the divergent part for renormalization.
We use dim-reg. with MS scheme.

 $=$  $= \dfrac{-i\lambda}{4!} \dfrac{4\times 3}{} \displaystyle\int \frac{d^dk}{(2\pi)^d} \frac{i}{(k^2-m^2+i\varepsilon)}$

$$= -\underbrace{\frac{i\lambda}{2} I_1(m^2)}_{\text{indep. of } p^2} \quad ; \quad I_1(m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2+i\varepsilon}$$

1-loop.

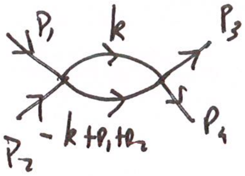
$$\Sigma = -\frac{i\lambda}{2} I_1(m^2) + i\left(\delta_Z^{(1)} p^2 - \delta_m^{(1)}\right)$$

divergent piece.

we take $\delta_Z^{(1)} = 0 \qquad \delta_m^{(1)} = -\dfrac{\lambda}{2} I_1^{div}(m^2)$

There is an ambiguity in subtracting a finite part.
e.g. we can subtract $\delta_m^{(1)} = -\frac{\lambda}{2} I_1(m^2)$. The end result is the same up to redefinitions of constants (here m^2).

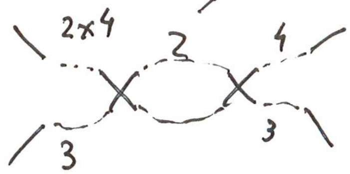
We need



$(2\pi)^d \delta^{(d)}(p_3+p_4-p_1-p_2)$: we do not include

We also do not include external propagators.

Symmetry factor $\frac{1}{2}$: $\frac{(-i\lambda)^2}{2} \cancel{\frac{1}{(4!)^2}} \cancel{2\times4\times3\times2\times4\times3} = -\frac{\lambda^2}{2}$



$$-\frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2+i\varepsilon}\frac{i}{(k-p_1-p_2)^2-m^2+i\varepsilon} =$$  $= B(p_1+p_2)$

However we know that the divergence is only a constant, indep of p_1, p_2

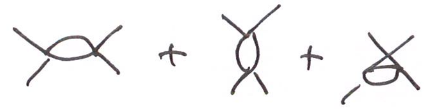
then we can take

$$B(0) = -\frac{\lambda^2}{2}\int\frac{d^dk}{(2\pi)^d}\left(\frac{i}{(k^2-m^2+i\varepsilon)}\right)^2$$

but

$$\frac{\partial I_1(m^2)}{\partial m^2} = \int\frac{d^dk}{(2\pi)^d}\frac{i}{(k^2-m^2+i\varepsilon)^2}$$

$$\Rightarrow B^{div}(p_1,p_2) = -\frac{i\lambda^2}{2}\left(\frac{\partial I_1}{\partial m^2}\right)^{div}$$

Since there are 3 diagrams 

we get

$$\left(\right.$$  $$\left. \right)_{div} = - \frac{3i\lambda^2}{2} \left(\frac{\partial I_1}{\partial m^2} \right)_{div}$$

 $$= -i\delta_\lambda^{(1)}$$

we take $\delta_\lambda^{(1)} = - \dfrac{3\lambda^2}{2} \left(\dfrac{\partial I_1^{div}}{\partial m^2} \right)$

Summary at 1-loop

$$\delta_Z^{(1)} = 0 \; ; \quad \delta_m^{(1)} = - \frac{\lambda}{2} I_1^{div} \; ; \quad \delta_\lambda^{(1)} = - \frac{3\lambda^2}{2} \frac{\partial I_1^{div}}{\partial m^2}$$

we have not yet decided how to regularize and defined divergent part.

Only now we use dim. reg. and <u>MS</u>.

For dim. reg. we need a scale.

$$S = \int d^d x \left(\frac{1}{2} \partial\phi\, \partial\phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 \right)$$

mass units $\frac{d-2}{2}$ units mass

but $[\lambda] = d - 4\,\frac{d-2}{2} = d - 2d + 4 = 4 - d$

$d = 4-\varepsilon$ $[\lambda] = \varepsilon$

We can get an adim λ. by writing $\lambda \to \mu^\varepsilon \lambda$ new scale

$$S = \int d^d x \left(\frac{1}{2} \partial_\mu\phi\, \partial^\mu\phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda \mu^\varepsilon}{4!} \phi^4 \right)$$

μ^ε keeps units OK.

Let's compute

$$I_1(m^2) = \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2 + i\varepsilon}$$

We use.

$$\frac{i}{k^2-m^2+i\varepsilon} = \int_0^\infty d\alpha \; e^{i(k^2-m^2+i\varepsilon)\alpha}$$

$$\int \frac{d^dk}{(2\pi)^d} \; e^{ik^2\alpha - 2ikp} = \frac{1}{(2\pi)^d}\left(\frac{\pi}{\alpha}\right)^{d/2} e^{-i\frac{\pi}{4}(d-2) - ip^2/\alpha}$$

then

$$I_1(m^2) = \int_0^\infty d\alpha \; \frac{1}{(4\pi\alpha)^{d/2}} \; e^{-i\frac{\pi}{4}(d-2)} \; e^{i(i\varepsilon - m^2)\alpha}$$

$$= \frac{e^{-i\frac{\pi}{4}(d-2)}}{(4\pi)^{d/2}} \int_0^\infty d\alpha \; \alpha^{1-d/2-1} \; e^{-(\varepsilon+im^2)\alpha}$$

$$= \frac{e^{-i\frac{\pi}{4}(d-2)}}{(4\pi)^{d/2}} \; \frac{\Gamma(1-d/2)}{(\varepsilon+im^2)^{1-d/2}} = \frac{e^{-i\frac{\pi}{4}(d-2)}\,\Gamma(1-d/2)}{(4\pi)^{d/2}\; e^{i\frac{\pi}{2}(1-d/2)}\,(m^2-i\varepsilon)^{1-d/2}}$$

$$= \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} \; \frac{e^{\cancel{-\frac{i\pi d}{4}} \cancel{+\frac{i\pi}{2}} \cancel{-\frac{i\pi}{2}} \cancel{+\frac{i\pi d}{4}}}}{(m^2-i\varepsilon)^{1-d/2}}$$

$$\boxed{I_1(m^2) = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} \; (m^2-i\varepsilon)^{d/2-1}}$$ in dim. reg.

$d = 4-\varepsilon \qquad 1-d/2 = -1+\varepsilon/2$

$$I_1(m^2) = \frac{\Gamma(-1+\varepsilon/2)}{(4\pi)^{2-\varepsilon/2}} \, (m^2 - i\epsilon)^{1-\varepsilon/2}$$

different ε!

$$\Gamma(-1+\varepsilon/2) = \frac{\Gamma(\varepsilon/2)}{-1+\varepsilon/2} = \frac{\Gamma(1+\varepsilon/2)}{(-1+\varepsilon/2)(\varepsilon/2)} = \frac{1+\frac{\varepsilon}{2}\Gamma'(1)+\cdots}{(\varepsilon/2)(-1+\varepsilon/2)}$$

$\varepsilon \to 0$ div. piece $-\frac{2}{\varepsilon}$

$$I_1^{div}(m^2) = -\frac{2}{\varepsilon}\frac{1}{16\pi^2} m^2 = -\frac{m^2}{8\pi^2}\frac{1}{\varepsilon}$$

MS scheme only pole counts.

$\Rightarrow$

$$\delta_z^{(1)} = 0 \; ; \quad \delta_m^{(1)} = \frac{\lambda m^2}{16\pi^2}\frac{1}{\varepsilon} \; ; \quad \delta_\lambda^{(1)} = \frac{3}{16}\frac{\lambda^2}{\pi^2}\frac{1}{\varepsilon}$$

It is convenient to define $\boxed{g = \frac{\lambda}{8\pi^2}}$

$$\delta_z^{(1)} = 0 \; ; \quad \delta_m^{(1)} = \frac{g m^2}{2\varepsilon} \; ; \quad \delta_g^{(1)} = \frac{3}{2}\frac{g^2}{\varepsilon}$$

Notice. $\lambda + \delta_\lambda^{(1)} = \lambda + \frac{3}{16}\frac{\lambda^2}{\pi^2}\frac{1}{\varepsilon}$

$$g + \delta_g^{(1)} = \frac{\lambda}{8\pi^2} + \frac{3}{8\times 16}\frac{\lambda^2}{\pi^4}\frac{1}{\varepsilon} \Rightarrow \delta g^{(1)} = \frac{3}{2} g^2 \frac{1}{\varepsilon}$$

examples of finite 1-loop calc.

①

 $= \frac{(-i\lambda)^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{i}{(k^2-m^2+i\epsilon)}\frac{i}{(p_1+p_2-k)^2-m^2+i\epsilon}$

$$\downarrow$$

$$= -\frac{(\lambda\mu^\epsilon)^2}{2}\int\frac{d^dk}{(2\pi)^d}\int_0^\infty d\alpha_1\, e^{i(k^2-m^2+i\epsilon)\alpha_1}\int_0^\infty d\alpha_2\, e^{i[(p-k)^2-m^2+i\epsilon]\alpha_2}$$

$$= -\frac{(\lambda\mu^\epsilon)^2}{2}\int_0^\infty d\alpha_1 d\alpha_2\int\frac{d^dk}{(2\pi)^d}\, e^{i(\alpha_1+\alpha_2)k^2-2ip\alpha_2 k}\,.$$

$$\cdot\, e^{-i(\alpha_1+\alpha_2)m^2-(\alpha_1+\alpha_2)\epsilon+i\alpha_2 p^2}$$

$$= -\frac{(\lambda\mu^\epsilon)^2}{2}\frac{1}{(2\pi)^d}\int_0^\infty d\alpha_1 d\alpha_2\left(\frac{\pi}{\alpha_1+\alpha_2}\right)^{d/2} e^{-\frac{i\pi}{4}(d-2)}\,.$$

$$\cdot\, e^{-i(\alpha_1+\alpha_2)m^2-(\alpha_1+\alpha_2)\epsilon+i\alpha_2p^2-i\frac{\alpha_2^2p^2}{\alpha_1+\alpha_2}}$$

$$\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}$$

$$= -\frac{(\lambda\mu^\epsilon)^2}{2}\frac{1}{(2\pi)^d}\int_0^\infty d\xi\int_0^\infty d\alpha_1 d\alpha_2\,\delta(\xi-\alpha_1-\alpha_2)\left(\frac{\pi}{\xi}\right)^{d/2}\cdot e^{-\frac{i\pi}{4}(d-2)}$$

$$e^{-i\xi m^2-\xi\epsilon+\frac{i\alpha_1\alpha_2}{\xi}p^2}$$

$\alpha_1=\beta_1\xi \qquad \alpha_2=\beta_2\xi$

$$= -\frac{(\lambda\mu^\epsilon)^2}{2}\frac{1}{(2\pi)^d}\int_0^\infty d\xi\,\frac{\xi^2}{\cancel{\xi}}\int_0^1 d\beta_1 d\beta_2\,\delta(1-\beta_1-\beta_2)\cdot\frac{\pi^{d/2}}{\xi^{d/2}}$$

$$e^{-\frac{i\pi}{4}(d-2)}\; e^{-\xi(\epsilon+im^2-i\beta_1\beta_2p^2)}$$

(2)

$$= -\frac{(\lambda\mu^\varepsilon)^2}{2}\frac{e^{-\frac{i\pi}{4}(d-2)}}{(4\pi)^{d/2}}\int_0^1 d\beta_1\int_0^\infty d\xi\ \xi^{2-d/2-1}\cdot e^{-\xi(\varepsilon+im^2-i\beta_1(1-\beta_1)p^2)}$$

$$\Gamma(2-d/2)\ (\varepsilon+im^2-i\beta_1(1-\beta_1)p^2)^{d/2-2}$$

$$e^{i\frac{\pi}{2}\left(\frac{d}{2}-2\right)} = e^{\frac{i\pi d}{4}-i\pi}$$

$$= \frac{(\lambda\mu^\varepsilon)^2}{2}\frac{e^{i\frac{\pi}{2}}}{(4\pi)^{d/2}}\int_0^1 d\beta\ (m^2-\beta(1-\beta)p^2-i\varepsilon)^{d/2-2}\ \Gamma(2-d/2)$$

$d=4-\varepsilon$

$$= \frac{(\lambda\mu^\varepsilon)}{2}(\lambda\mu^\varepsilon)\frac{e^{i\pi/2}}{(4\pi)^{2-\varepsilon/2}}\Gamma(\varepsilon/2)\int_0^1 d\beta\ (m^2-\beta(1-\beta)p^2-i\varepsilon)^{-\varepsilon/2}$$

$$\Gamma(\varepsilon/2) = \frac{2}{\varepsilon}\Gamma(1+\varepsilon/2) = \frac{2}{\varepsilon}(1+\varepsilon/2\,\Gamma'(1)) = \frac{2}{\varepsilon}+\Gamma'(1)+\ldots$$

$$= i\frac{(\lambda\mu^\varepsilon)}{\cancel{2}}\lambda\,(1+\varepsilon\ln\mu)\,\frac{\cancel{2}}{\varepsilon}(1+\frac{\varepsilon}{2}\gamma)\frac{1}{16\pi^2}(1+\frac{\varepsilon}{2}\ln 4\pi)$$

$$\int_0^1 d\beta\,(1-\frac{\varepsilon}{2}\ln(m^2-\beta(1-\beta)p^2-i\varepsilon))$$

$$= i(\lambda\mu^\varepsilon)\frac{\lambda}{16\pi^2}\frac{1}{\varepsilon}\Big(1+\frac{\varepsilon}{2}\gamma+\frac{\varepsilon}{2}\ln 4\pi+\varepsilon\ln\mu\ \cancel{+}$$

$$-\frac{\varepsilon}{2}\int_0^1 d\beta\ \ln(m^2-\beta(1-\beta)p^2-i\varepsilon)\Big)$$

$$= i(\lambda\mu^\varepsilon)\frac{\lambda}{16\pi^2}\frac{1}{\varepsilon} + i(\lambda\mu^\varepsilon)\frac{\lambda}{16\pi^2}\Big(\frac{1}{2}\gamma+\frac{1}{2}\ln 4\pi\ \cancel{\text{tot}} -$$

$$-\frac{1}{2}\int_0^1 d\beta\ \ln\Big(\frac{m^2}{\mu^2}-\beta(1-\beta)\frac{p^2}{\mu^2}-i\varepsilon\Big)\Big)$$

 $=$

$$= i(\lambda\mu^{\varepsilon})\,\frac{3\lambda}{16\pi^2}\,\frac{1}{\varepsilon} + B_f(s) + B_f(t) + B_f(u)$$

$$B_f(s) = i(\lambda\mu^{\varepsilon})\,\frac{\lambda}{16\pi^2}\left[\frac{1}{2}\gamma + \frac{1}{2}\ln 4\pi - \frac{1}{2}\int_0^1 d\beta\,\ln\left(\frac{m^2}{\mu^2} - \beta(1-\beta)\frac{\cancel{p^2}\,s}{\mu^2} - i\varepsilon\right)\right]$$

$$s = (p_1+p_2)^2 \qquad t = (p_1-p_4)^2 \qquad u = (p_1-p_3)^2$$

 $= -i\lambda\mu^{\varepsilon}\Big(1 - \frac{\lambda}{32\pi^2}\left[\gamma + \ln 4\pi - \int_0^1 d\beta\,\ln\left(\frac{m^2}{\mu^2} - \beta(1-\beta)\frac{s}{\mu^2} - i\varepsilon\right)\right]$

$\qquad - \frac{\lambda}{32\pi^2}\int \text{---} \quad t \;]$

$\qquad - \frac{\lambda}{32\pi^2}\int \text{—} \quad t \;]\Big)$

 $= -\frac{i\lambda}{2}\, I_1^f$

$$I_1(m^2) = \frac{\Gamma(-1+\varepsilon/2)}{(4\pi)^{2-\varepsilon/2}}\,(m^2 - i\varepsilon)^{1-\varepsilon/2}$$

$$\Gamma(-1+\varepsilon/2) = -\frac{2}{\varepsilon}\left(1 + \frac{\varepsilon}{2}\gamma + \frac{\varepsilon}{2}\right)$$

$$I_1(m^2) = -\frac{2}{\varepsilon}\,\frac{1}{16\pi^2}\,m^2\left(1 + \frac{\varepsilon}{2}\gamma + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\ln 4\pi - \frac{\varepsilon}{2}\ln m^2\right)$$

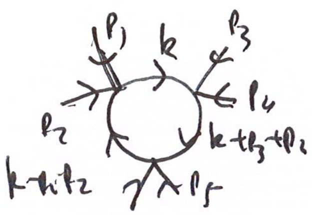
$$-\frac{i\lambda}{2}\left(-\frac{2}{\varepsilon 16\pi^2}\, m^2\left(1+\frac{\varepsilon}{2}\gamma+\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\ln 4\pi-\frac{\varepsilon}{2}\ln m^2\right)\right)$$

$$\frac{i\lambda}{\varepsilon 16\pi^2}m^2+\frac{i\lambda m^2}{16\pi^2}\left(\frac{\gamma}{2}+\frac{1}{2}+\frac{\ln 4\pi}{2}-\ln\frac{m}{\mu}\right)$$

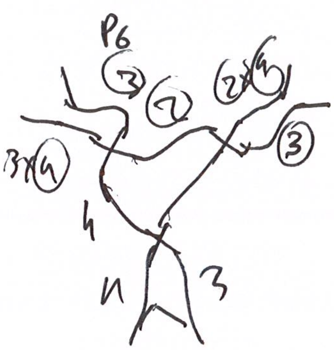
$$=\frac{i\lambda m^2}{16\pi^2}\left(\frac{\gamma}{2}+\frac{1}{2}+\frac{\ln 4\pi}{2}-\ln\frac{m}{\mu}\right)$$

$$=\frac{i}{p^2-m^2+\frac{\lambda m^2}{16\pi^2}\left(\frac{\gamma}{2}+\frac{1}{2}+\frac{\ln 4\pi}{2}-\ln\frac{m}{\mu}\right)+i\varepsilon}$$

Consider



$$\frac{(-i\lambda)^3}{3!\,(4!)^3}\int\frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(k+p_3+p_4)^2-m^2+i\varepsilon}\,\frac{i}{(k-p_1-p_2)^2-m^2+i\varepsilon}$$



$$\frac{\cancel{3\times4\times3\times4}}{\cancel{1\times4\times3\times4}}\quad \frac{1}{3!\times4}$$

$$+\, i\lambda^3\mu^{3\varepsilon}\int\frac{d^dk}{(2\pi)^d}\int d\alpha_1 d\alpha_2 d\alpha_3\; \times e^{i\alpha_1(k^2-m^2+i\varepsilon)+i\alpha_2((k+q)^2-m^2+i\varepsilon)}\, e^{i\alpha_3((k-p)^2-m^2+i\varepsilon)}$$

(5)

$$\frac{i\lambda^3\mu^{3\varepsilon}}{(2\pi)^d}\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \int d^dk\; e^{i(\alpha_1+\alpha_2+\alpha_3)k^2+2ik(\alpha_2 q-\alpha_3 p)}$$

$$e^{-i(\alpha_1+\alpha_2+\alpha_3)m^2}\; e^{-(\alpha_1+\alpha_2+\alpha_3)\varepsilon}\; e^{i\alpha_2 q^2+i\alpha_3 p^2}$$

$$\frac{i\lambda^3\mu^{3\varepsilon}}{(2\pi)^d}\int_0^\infty d\xi\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\;\delta(\xi-\alpha_1-\alpha_2-\alpha_3)\left(\frac{\pi}{\xi}\right)^{d/2} e^{-\frac{i\pi}{4}(d-2)}\;.$$

$$\cdot\; e^{-i\xi m^2-\xi\varepsilon+i\alpha_2 q^2+i\alpha_3 p^2-i\frac{(\alpha_2 q-\alpha_3 p)^2}{\xi}}$$

$$e^{-\frac{i\pi}{4}(d-2)}\frac{i\lambda^3\mu^{3\varepsilon}}{(2\pi)^d}\pi^{d/2}\int_0^1 d\beta_1 d\beta_2 d\beta_3\;\delta(1-\Sigma\beta_i)\int_0^\infty d\xi\;\xi^{3-1-d/2}\;.$$

$$\cdot\; e^{-\xi(\varepsilon+im^2-i\alpha_2 q^2\xi-i\alpha_3 p^2\xi+i\xi\alpha_2^2 q^2+i\alpha_3^2 p^2\xi-2i\alpha_2\alpha_3\xi qp}$$

$$e^{-\frac{i\pi}{4}(d-2)}\frac{i\lambda^3\mu^{3\varepsilon}}{(2\pi)^d}\pi^{d/2}\int_0^1 d\beta_1 d\beta_2 d\beta_3\;\delta(1-\Sigma\beta_i)\;.\;\Gamma(3-d/2)$$

$$(\varepsilon+im^2-i\beta_2(1-\beta_2)q^2-i\beta_3(1-\beta_3)p^2-2i\beta_2\beta_3 pq)^{d/2-3}$$

$\underline{d=4}$ $\Gamma(1)$ finite.

$$e^{-\frac{i\pi}{2}}\frac{i\lambda^3}{16\pi^4 2}\pi^2\int_0^1 d\beta_2\int_0^{1-\beta_2}d\beta_3\;(\varepsilon+im_a^2-i\beta_2(1-\beta_2)q^2-i\beta_3(1-\beta_3)p^2-2i\beta_3\beta_2 pq)^{-1}$$

finite.