Complex scalar field. O(2) model.

<u>3d</u> statistical mechanics O(N) model

$\phi_{a=1 \ldots N}$

$$E = \int d^3x \left[ \frac{1}{2} (\nabla \phi_a)^2 + \frac{1}{2} m^2 \phi_a^2 + \frac{1}{4!} \lambda (\phi_a^2)^2 \right]$$

↑ function of T etc.

<u>N=2</u>

$$E = \int d^3x \left[ (\nabla \phi^*)(\nabla \phi) + m^2 |\phi|^2 + \frac{\lambda}{3!} |\phi|^4 \right]$$

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2)$$

in <u>4d</u>

$$\underbrace{\boxed{S = \int d^4x \quad \partial_\mu \phi^* \partial_\mu \phi + m^2 |\phi|^2}}_{\text{free field}} + \frac{\lambda}{6} (|\phi|^2)^2$$

$$\partial_\mu \frac{\delta L}{\delta(\partial_\mu \phi^*)} - \frac{\delta L}{\delta \phi^*} = 0$$

$$\partial_\mu \partial^\mu \phi = -m^2 \phi - \frac{\lambda}{3} \phi |\phi|^2$$

$$\Pi = \frac{\delta L}{\delta \dot{\phi}^*} = \dot{\phi} \qquad \pi = \dot{\phi}^*$$

$$\mathcal{H} = \bar\pi \dot\phi^* + \pi \dot\phi - \mathcal{L}$$

$$= \cancel{2}|\dot\phi|^2 - \cancel{|\dot\phi|^2} + |D\phi|^2 + m^2|\phi|^2 + \frac{\lambda}{6}|\phi^2|^2$$

$$\mathcal{H} = \bar\pi\pi + |D\phi|^2 + m^2|\phi|^2 + \frac{\lambda}{6}|\phi^2|^2$$

$$H = \int d^3x \left(\bar\pi\pi + |D\phi|^2 + m^2|\phi|^2 + \frac{\lambda}{6}|\phi^2|^2\right)$$

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial^\nu \phi_a^*}\partial_\nu \phi_a^* + \frac{\partial \mathcal{L}}{\partial \partial^\nu \phi_a}\partial_\mu \phi_a - \eta_{\mu\nu}\mathcal{L}$$

$$= \partial_\nu \phi_a \partial_\mu \phi^* + \partial_\nu \phi^* \partial_\mu \phi - \eta_{\mu\nu}\mathcal{L}$$

$$T_{00} = \cancel{2}|\dot\phi|^2 \cancel{+ |\dot\phi|^2} + m^2|\phi|^2 + \frac{\lambda}{6}|\phi^2|^2 = \mathcal{H} \checkmark$$

$$T_{0i} = \partial_i \phi \, \dot\phi^* + \partial_i \phi^* \dot\phi = \bar\pi \partial_i \phi + \pi \partial_i \phi^* = T_{i0}$$

$$P_i = \int d^3x \left(\partial_i \phi \, \dot\phi^* + \partial_i \phi^* \dot\phi\right)$$

$$\phi \to e^{i\alpha}\phi \qquad \delta\phi = i\phi$$

$$j_\mu = \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi^*}\delta\phi^* + \frac{\delta \mathcal{L}}{\delta \partial^\mu \phi}\delta\phi = i\left(-\partial_\mu \phi \, \phi^* + \partial_\mu \phi^* \phi\right)$$

$$j_\mu = i(\phi\, \partial_\mu \phi^\dagger - \phi^\dagger \partial_\mu \phi)$$

$$\partial^\mu j_\mu = i\Big(\partial^\mu \phi\, \partial_\mu \phi^\dagger + \phi\, (m^2 \phi^\dagger + \frac{\lambda}{3} \phi^\dagger |\phi|^2) - \partial^\mu \phi^\dagger \partial_\mu \phi - m^2 \phi\, \phi^\dagger -$$

$$- \frac{\lambda}{3} |\phi^2|^2\Big) = 0 . \qquad \text{(using e.o.m.)}$$

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Quantization (free field $\lambda = 0$)

$$\partial_0^2 \phi - \nabla^2 \phi + m^2 \phi = 0$$

$$\phi = \xi_k(t)\, e^{i\vec{k}\vec{x}}$$

$$\ddot{\xi}_k(t) + \vec{k}^2 \xi_k + m^2 \xi_k = 0$$

$$\omega_k = \sqrt{m^2 + \vec{k}^2} \qquad \boxed{\ddot{\xi}_k + \omega_k^2 \xi_k = 0} \quad \text{harmonic oscillator.}$$

$$\xi_k = \xi_k(0)\, e^{\pm i \omega_k t}$$

Set of decoupled harmonic oscillators

change of variables.

$$\phi = \int \frac{d^3k}{(2\pi)^{3/2}} \, \xi_k(t) \, e^{i\vec{k}\cdot\vec{x}} \qquad \phi^* = \int \frac{d^3k}{(2\pi)^{3/2}} \, \xi_k^* \, e^{-i\vec{k}\cdot\vec{x}}$$

$$\mathcal{L} = \frac{1}{(2\pi)^3}\int d^3x \int d^3k \int d^3k' \left(e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\right)\left(\dot{\xi}_k \dot{\xi}_k^* - \omega_k^2 \xi_k \xi_k^*\right)$$

$$= \int d^3k \left(\dot{\xi}_k \dot{\xi}_k^* - \omega_k^2 \xi_k \xi_k^*\right)$$

$$\xi_k = \frac{1}{\sqrt{2}}\left(\eta_k + i\zeta_k\right)$$

$$\mathcal{L} = \int d^3k \left(\frac{1}{2}\dot{\eta}_k^2 - \frac{1}{2}\omega_k^2\eta_k^2\right) + \int d^3k \left(\frac{1}{2}\dot{\zeta}_k^2 - \frac{1}{2}\omega_k^2\zeta_k^2\right)$$

$$a_k = \sqrt{\frac{\omega_k}{2}}\left(\eta_k + \frac{i}{\omega_k}\dot{\eta}_k\right) \qquad a_k^\dagger = \sqrt{\frac{\omega_k}{2}}\left(\eta_k - \frac{i}{\omega_k}\dot{\eta}_k\right)$$

$$b_k = \sqrt{\frac{\omega_k}{2}}\left(\zeta_k + \frac{i}{\omega_k}\dot{\zeta}_k\right) \qquad b_k^\dagger = \sqrt{\frac{\omega_k}{2}}\left(\zeta_k - \frac{i}{\omega_k}\dot{\zeta}_k\right)$$

$$p_k = \dot{\eta}_k \qquad \tilde{p}_k = \dot{\zeta}_k \qquad \left[\left[\, a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right)\right]\right] \text{usual def.}$$

$(m=1, \hbar=1)$

$$[a_k, a_{k'}^\dagger] = \delta(k-k') \qquad [b_k, b_{k'}^\dagger] = \delta(k-k')$$

$$a_k + a_k^\dagger = \sqrt{2\omega_k}\,\eta_k \qquad \eta_k = \frac{1}{\sqrt{2\omega_k}}(a_k + a_k^\dagger)$$

$$a_k - a_k^\dagger = \frac{2i}{\sqrt{2\omega_k}}\dot{\eta}_k = \sqrt{\frac{2}{\omega_k}}\, i\dot{\eta}_k \qquad \dot{\eta}_k = -i\sqrt{\frac{\omega_k}{2}}\,(a_k - a_k^\dagger)$$

Also we can introduce:

$$c_k = \frac{1}{\sqrt{2}}(a_k + i b_k) \qquad c_k^\dagger = \frac{1}{\sqrt{2}}(a_k^\dagger - i b_k^\dagger)$$

$$d_{-k} = \frac{1}{\sqrt{2}}(a_k - i b_k) \qquad d_{-k}^\dagger = \frac{1}{\sqrt{2}}(a_k^\dagger + i b_k^\dagger)$$

$$[c_k, c_{k'}^\dagger] = \frac{1}{2}\delta(k-k') + \frac{1}{2}\delta(k-k') = \delta(k-k') \checkmark$$

$$[c_k, d_{-k'}^\dagger] = \frac{1}{2}\delta(k-k') - \frac{1}{2}\delta(k-k') = 0 \checkmark$$

$$[d_{-k}, d_{-k'}^\dagger] = \delta(k-k')$$

$$\xi_k = \frac{1}{\sqrt{2}}(\eta_k + i\zeta_k) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2\omega_k}}(a_k + a_k^\dagger) + \frac{i}{\sqrt{2\omega_k}}(b_k + b_k^\dagger)\right)$$

$$= \frac{\sqrt{2}}{2\sqrt{\omega_k}}(c_k + d_{-k}^\dagger) = \frac{1}{\sqrt{2\omega_k}}(c_k + d_{-k}^\dagger)$$

$$\dot{\xi}_k = \frac{1}{\sqrt{2}}(\dot{\eta}_k + i\dot{\zeta}_k) = \frac{1}{\sqrt{2}}\left(-i\sqrt{\frac{\omega_k}{2}}(a_k - a_k^\dagger) + i\sqrt{\frac{\omega_k}{2}}(b_k - b_k^\dagger)\right)$$

$$= -\frac{i\sqrt{\omega_k}}{\sqrt{2}}(c_k - d_{-k}^\dagger)$$

$$\phi = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} (c_k + d^\dagger_{-k}) e^{i\tilde{k}\tilde{x}}$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} (c_k e^{i\tilde{k}\tilde{x}} + d^\dagger_k e^{-i\tilde{k}\tilde{x}})$$

$$\Pi = \dot{\phi} = -i \int \frac{d^3k}{(2\pi)^{3/2}} \sqrt{\frac{\omega_k}{2}} (c_k e^{i\tilde{k}\tilde{x}} - d^\dagger_k e^{-i\tilde{k}\tilde{x}})$$

$$\Pi = \dot{\phi}^* = i \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\omega_k}{\sqrt{2\omega_k}} (c^\dagger_k e^{-i\tilde{k}\tilde{x}} - d_k e^{i\tilde{k}\tilde{x}})$$

↑ notice no $\underline{t}$ dependence for Schroedinger operators.

Fields are operators acting on Hilbert space of occupation number $|n_k, \bar{n}_k\rangle$

$$\phi^* = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} (c^\dagger_k e^{-ikx} + d_k e^{ikx})$$

Fundamental commutation relation

$$[\Pi(\vec{x}), \phi(\vec{y})] = i\int \frac{d^3k\, d^3k'}{(2\pi)^3 2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\left(-e^{-ik'x+iky}\delta(k-k') + e^{i\vec{k}\vec{x}-i\vec{k}'\vec{y}}\delta(k-k')\right)$$

$$= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left(e^{ik(y-x)} + e^{ik(\vec{x}-\vec{y})}\right) = -i\,\delta(\vec{x}-\vec{y})$$

Hamiltonian

$$H = \int d^3x\,\left(\bar{\Pi}\Pi + |\nabla\phi|^2 + m^2|\phi|^2\right)$$

$$\int d^3x\,\Pi\bar{\Pi} = \int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^3}\frac{\sqrt{\omega_k\omega_{k'}}}{2}\Big(c_k^\dagger c_{k'}\, e^{-i\vec{k}\vec{x}+i\vec{k}'\vec{x}} - c_k^\dagger d_{k'}^\dagger\, e^{-i(k+k')x} - d_k c_{k'}\, e^{i(k+k')x} + d_k d_{k'}^\dagger\, e^{i(k-k')x}\Big)$$

$$= \int d^3k\,\frac{\omega_k}{2}\left(c_k^\dagger c_k - c_k^\dagger d_{-k}^\dagger + d_k c_{-k} + d_k d_k^\dagger\right)$$

$$\int d^3x\,\nabla\phi^*\nabla\phi = \int d^3x\int \frac{d^3k\, d^3k'}{(2\pi)^3}\frac{1}{2\sqrt{\omega_k\omega_{k'}}}\Big(c_k^\dagger c_{k'}(\vec{k}\cdot\vec{k}')\, e^{-ikx+ik'x} + c_k^\dagger d_{k'}^\dagger\, e^{-ikx-ik'x}(-k\cdot k') + d_k c_{k'}\, e^{ikx+ik'x}(-k k') + d_k d_{k'}^\dagger\, e^{ikx-ik'x}\, k\cdot k'\Big)$$

$$= \int d^3k\,\frac{\vec{k}^2}{2\omega_k}\left(c_k^\dagger c_k + c_k^\dagger d_{-k}^\dagger + d_k c_{-k} + d_k d_{-k}^\dagger\right)$$

$$\int d^3x\,\phi^*\phi = \int d^3k\,\frac{m^2}{2\omega_k}\left(c_k^\dagger c_k + c_k^\dagger d_{-k}^\dagger + d_k c_{-k} + d_k d_k^\dagger\right)$$

$$H = \int d^3k \, \frac{\omega_k}{2} \left( c_k^\dagger c_k - \cancel{c_k^\dagger d_{-k}^\dagger + d_k c_{-k}} + d_k d_k^\dagger \right)$$

$$+ \int d^3k \, \frac{\omega_k}{2} \left( c_k^\dagger c_k + \cancel{c_k^\dagger d_{-k}^\dagger + d_k c_{-k}} + d_k d_k^\dagger \right)$$

$$H = \int d^3k \, \omega_k \left( c_k^\dagger c_k + d_k d_k^\dagger \right)$$

$$d_k d_k^\dagger = d_k^\dagger d_k + \delta(k-k)$$

$$H = \int d^3k \, \omega_k \left( c_k^\dagger c_k + d_k^\dagger d_k \right) + \int d^3k \, \omega_k \, \delta^{(3)}(k=0)$$

$$\delta(k) = \int \frac{d^3x}{(2\pi)^3} e^{ikx} \qquad \delta(0) = \frac{1}{(2\pi)^3} \int d^3x = \frac{V}{(2\pi)^3}$$

$$V \int \frac{d^3k \, \omega_k}{(2\pi)^3} \qquad \frac{\mathcal{E}}{V} = \int \frac{d^3k}{(2\pi)^3} \omega_k = \frac{4\pi}{8\pi^3} \int_0^\infty k^2 dk \sqrt{k^2+m^2}$$

cut-off $|k| \leq \Lambda$ $\qquad \dfrac{\mathcal{E}}{V} \simeq \dfrac{1}{2\pi^2} \displaystyle\int^\Lambda k^3 dk = \dfrac{\Lambda^4}{8\pi^2}$

$$H = \int d^3k \, \omega_k \left( n_{c,k} + n_{d,k} \right) + \xcancel{\frac{\Lambda^4}{8\pi^2}}$$

vacuum energy.

$$j_0 = i(\phi \partial_0 \phi^* - \phi^* \partial_0 \phi)$$

$$= i\,(\phi \pi - \phi^* \bar{\pi})$$

$$Q = i \int d^3x \,(\phi \pi - \phi^* \bar{\pi}) \to \underbrace{i \int d^3x\,(\phi \pi - \bar{\pi} \phi^*)}_{\text{hermitian}}$$

$$[Q, \phi(\vec{y})]_{E.T.} = i \int d^3x\, \phi(x)\, [\pi(x), \phi(y)] =$$

$$= \int d^3x\, \phi(x)\, \delta(x-y) = \phi(y)$$

$$\delta\phi(\vec{y}) = i\alpha\phi = i\alpha\,[Q, \phi(\vec{y})]$$

Q generates symmetry.

$$Q = i \int d^3x \left( \int \frac{d^3k\, d^3k'}{(2\pi)^3} \right) \left( \frac{i}{2} \sqrt{\frac{\omega_{k'}}{\omega_k}} \right) \Big( c_k c_{k'}^\dagger e^{ikx - ik'x} - c_k d_{k'} e^{ikx + ik'x} + d_k^\dagger c_k^\dagger e^{-ikx - ik'x} - d_k^\dagger d_k e^{-ikx + ik'x} \Big)$$

$+$ h.c.

$$= -\frac{1}{2} \int d^3k \,(c_k c_k^\dagger - \cancel{c_k d_{-k}} + \cancel{d_k^\dagger c_{-k}^\dagger} - d_k^\dagger d_k)$$

$$-\frac{1}{2} \int d^3k \,(c_k c_k^\dagger - \cancel{d_{-k}^\dagger c_k} + \cancel{c_k d_k^\dagger} - d_k^\dagger d_k)$$

$$= -\int d^3k\, c_k c_k^\dagger + \int d^3k\, d_k^\dagger d_k = \int d^3k\,(d_k^\dagger d_k - c_k^\dagger c_k) \underbrace{- V \int d^3k}_{}$$

$$c_k c_k^\dagger = c_k^\dagger c_k + \delta(0)$$

remove for $\underline{Q|0\rangle = 0}$

$$Q = \int d^3k \, (d_k^\dagger d_k - c_k^\dagger c_k)$$

$$= \int d^3k \, (n_{d,k} - n_{c,k})$$

$d, c$ particles carry same mass but opposite charge.

particle $\leftrightarrow$ antiparticle.

Also

$$P^i = \int d^3k \, \vec{k}^i \, (n_{c,k} + n_{d,k})$$

$$[P^i, \phi(\vec{x})] = \int d^3k \, \vec{k}^i \left[ c_k^\dagger c_k + d_k^\dagger d_k \, , \int \frac{d^3k'}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{k'}}} (c_{k'} e^{i\vec{k}'\vec{x}} + d_{k'}^\dagger e^{-i\vec{k}'\vec{x}}) \right]$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \vec{k}^i \int dk' \left( -\delta(k-k') \, c_k \, e^{i\vec{k}\vec{x}} + \delta(k-k') \, d_k^\dagger \, e^{-i\vec{k}\vec{x}} \right)$$

$$= \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \left( -\vec{k}^i c_k e^{i\vec{k}\vec{x}} + \vec{k}^i d_k^\dagger e^{-i\vec{k}\vec{x}} \right)$$

$$= i \, \partial_{x_i} \phi \quad \leftarrow \text{generates translations}$$

## Heisenberg picture

$$\phi_H(\vec{x},t) = e^{iHt}\,\phi_s(x)\,e^{-iHt}$$

$$\langle \psi_1|\,\phi_H\,|\psi_2\rangle = \langle \psi_1|\,e^{iHt}\,\phi_S\,e^{-iHt}\,|\psi_2\rangle$$

$$= \langle \psi_1(t)|\,\phi_s\,|\psi_2(t)\rangle$$

$$i\frac{\partial \phi_H}{\partial t} = -[H,\phi_H]$$

$$\left.\begin{aligned} e^{iHt}\,c_k^{\dagger}\,e^{-iHt}\,|E\rangle &= e^{-iEt}\,e^{iHt}\,c_k^{\dagger}\,|E\rangle = \\ &= e^{-iEt}\,e^{iHt}\,|E+\omega_k\rangle = e^{i\omega_k t}\,c_k^{\dagger}\,|E\rangle \\ c_k^{\dagger}(t) = e^{i\omega_k t}\,c_k^{\dagger} \qquad & c_k(t) = e^{-i\omega_k t}\,c_k \end{aligned}\right\}$$

only for free theory!

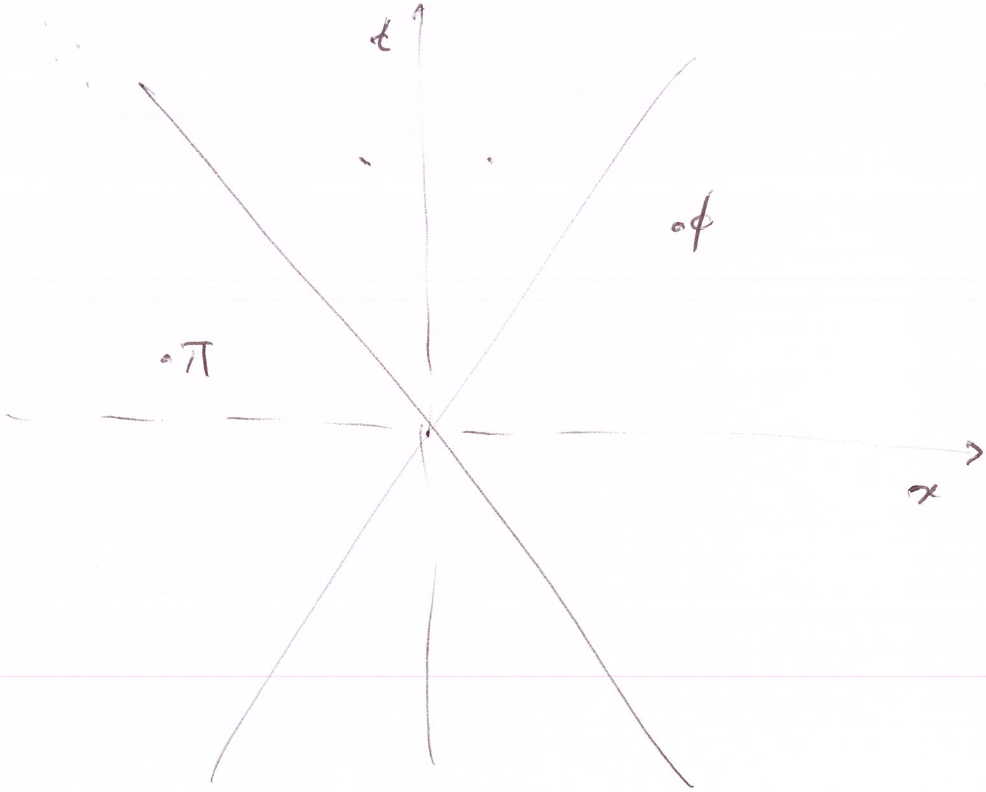
$c_k^{\dagger}|E\rangle = |E+\omega_k\rangle$

Same with $d_k$.

$$\phi_H(\vec{x},t) = \int \frac{d^3k}{(2\pi)^{3/2}}\frac{1}{\sqrt{2\omega_k}}\left(c_k\,e^{-i\omega_k t+ikx} + d_k^{\dagger}\,e^{i\omega_k t-i\vec{k}\vec{x}}\right)$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(c_k\,e^{-ikx} + d_k^{\dagger}\,e^{ikx}\right)$$

$$\Pi = i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_k}{2}}\left(c_k^{\dagger}\,e^{ikx} - d_k\,e^{-ikx}\right) = \dot{\phi}^* \checkmark$$




In the Heisenberg picture we have operator, $\phi, \pi$ at each point of space-time. More of a "field theory point of view" We can study space-time symmetries and causality, etc.

Operator act on a Hilbert space of particle occupation numbers. $|n_{k,c}, n_{k,d}\rangle$

$\uparrow$ "particle" point of view.

For curved space-time and conformal theories the "field theory" point of view is better.

## Commutator

$$[\phi(x,t),\ \phi^*(y,t')] = \frac{1}{(2\pi)^3}\int \frac{d^3k\, d^3k'}{\cancel{(2\pi)^3}} \frac{1}{\sqrt{4\omega_k \omega_{k'}}}\ .$$

$$\left\{ [c_k e^{-ikx},\ c^\dagger_{k'} e^{ik'y}] + [d^\dagger_k e^{ikx},\ d_{k'} e^{-ik'y}] \right\}$$

$$= \frac{1}{(2\pi)^3} \int \underbrace{\frac{d^3k}{2\omega_k}}_{\text{Lorentz inv.}} \left( e^{ik(y-x)} - e^{-ik(y-x)} \right) = \Delta_c(x-y).$$

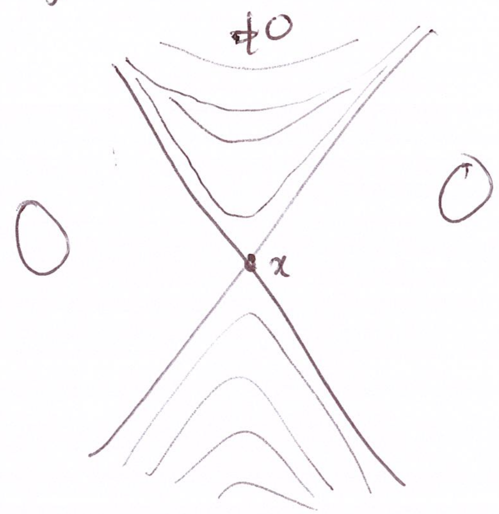
Lorentz inv.

$$\text{if} \quad t = t' \quad \Rightarrow \quad \Delta_c(x-y) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} \left( e^{-i\vec{k}(\vec{y}-\vec{y})} - e^{i\vec{k}(\vec{y}-\vec{x})} \right)$$

$k \to -\vec{k}$

$$= 0$$

$\Delta_c(x-y) = 0$ outside light-cone




We can measure the field at space-like separated points in agreement with causality

Depends on a cancellation between particles and anti-particles. Same mass and opposite charge.

## Wightman functions

$$\langle 0 | \phi^*(x)\, \phi(y) | 0 \rangle = \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{\sqrt{4\omega_k \omega_{k'}}}$$

$$\cdot \langle 0 | (c_k^\dagger e^{ikx} + d_k e^{-ikx})(c_{k'} e^{-ik'y} + d_{k'}^\dagger e^{ik'y}) | 0 \rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-ik(x-y)} \quad \leftarrow \text{Lorentz inv.}$$

$$= \int \frac{d^4k}{(2\pi)^3} \frac{1}{2k_0} \delta(k_0 - \omega_k) e^{-ik(x-y)}$$

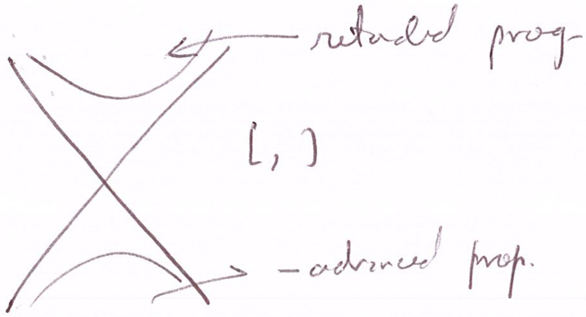
$$\delta(k_0^2 - \omega_k^2) = \frac{\delta(k_0 - \omega_k)}{2\omega_k} + \frac{\delta(k_0 + \omega_k)}{2\omega_k}$$

$$= \int \frac{d^4k}{(2\pi)^3} \delta(k_0^2 - \vec{k}^2 - m^2)\, \Theta(k_0)\, e^{-ik(x-y)}$$

$$\Delta_W(x-y) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-ik(x-y)} = \int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \underbrace{\Theta(k_0)}_{\text{Lorentz inv.}} e^{-ik(x-y)}$$

$$\Theta(k_0) = \begin{cases} 1 & k_0 > 0 \\ 0 & \text{otherwise.} \end{cases}$$

sign of $k_0$ is Lorentz inv.




$$D_R(x-y) = \Theta(x^0-y^0)\,\langle 0|\,[\phi(x), \phi^\dagger(y)]\,|0\rangle$$

$$D_A(x-y) = -\Theta(y^0-x^0)\,\langle 0|\,[\phi(x), \phi^\dagger(y)]\,|0\rangle$$

$$\Delta_F(x-y) = \langle 0|\,\underbrace{\hat{T}}_{\text{time-order.}}\{\phi(x)\,\phi^\dagger(y)\}\,|0\rangle = \begin{cases} \langle 0|\,\phi(x)\,\phi^\dagger(y)\,|0\rangle & \text{if } x^0 > y^0 \\ \langle 0|\,\phi^\dagger(y)\,\phi(x)\,|0\rangle & \text{if } x^0 < y^0 \end{cases}$$

It is Lorentz invariant because operators commute when space-like separated (i.e. when time order is not Lorentz inv.).

$$H = H_0 + \int d^3x\, j(x,t)\,\phi(\vec{x}) + \int d^3x\, \bar{j}(x,t)\,\phi^\dagger(x)$$

time dep.

recall ~~from~~ Q.M.

$$|\psi\rangle = |\psi_0\rangle + |\psi_1\rangle + \cdots$$

$$\partial_t|\psi\rangle = \partial_t|\psi_0\rangle + \partial_t|\psi_1\rangle = -iH|\psi_0\rangle - iH|\psi_1\rangle \cdots$$

$$\partial_t|\psi_0\rangle = -iH_0|\psi_0\rangle$$

$$\partial_t|\psi_1\rangle = -iV|\psi_0\rangle - iH_0|\psi_1\rangle$$

Suppose $H_0|\psi_0\rangle = 0$

$$|\psi_1\rangle = \sum_n c_n e^{-iE_n t}|n\rangle$$

$$\partial_t|\psi_1\rangle = \sum_n (-iE_n)c_n e^{-iE_n t}|n\rangle + \sum_n \partial_t c_n e^{-iE_n t}|n\rangle$$

$$= -iH_0|\psi_1\rangle - iV|\psi_0\rangle$$

$$\partial_t c_n e^{-iE_n t} = -i\langle n|V|\psi_0\rangle = -\phi$$

$$c_n = -i\int_0^t e^{iE_n t'}\langle n|V(t')|\psi_0\rangle$$

$$|\psi_1) = -i\sum_n \int_0^t e^{iE_n t' - iE_n t} \quad |n)\langle n|V(t')|\psi_0)$$

$$= -i \int_0^t e^{iH_0(t'-t)} \quad V(t')\,|\psi_0\rangle$$

$$\langle\psi|\mathcal{O}|\psi) = \langle\psi_0|\mathcal{O}|\psi_1) + \langle\psi_1|\mathcal{O}|\psi_0\rangle \qquad (\langle\psi_0|\mathcal{O}|\psi_0\rangle = 0)$$

assume.

$$= -i \int_0^t \langle\psi_0|\mathcal{O}e^{iH_0(t'-t)} \, V(t')\,|\psi_0) +$$

$$+ i \int_0^t \langle\psi_0|\,V(t')\,e^{-iH_0(t'-t)}\,\mathcal{O}\,|\psi_0\rangle$$

$$= -i \int_0^t \langle\psi_0|\,e^{iH_0 t}\,\mathcal{O}\,e^{-iH_0 t}\,e^{+iH_0 t'}\,V(t')\,e^{-iH_0 t'}\,|\psi_0)$$

$$+ i \int_0^t \langle\psi_0|\,e^{iH_0 t'}\,V(t')\,e^{-iH_0 t'}\,e^{iH_0 t}\,\mathcal{O}e^{-iH_0 t}\,|\psi_0)$$

$$= -i \int_0^t \langle\psi_0|\,\mathcal{O}_{H_0}(t)\,V_{H_0}(t') - V_{H_0}(t')\,\mathcal{O}_{H_0}(t)\,|\psi_0\rangle$$

$$= -i \int_0^t \langle\psi_0|\,[\mathcal{O}_{H_0}(t), V_{H_0}(t')]\,|\psi_0\rangle\,dt'$$

$$V_{H_0}(t') = \int d^3x \; j(x, t') \, \phi_{H_0}(\vec{x}, t') +$$

$$\mathcal{O} = \phi$$

$$\langle \psi_t | \phi | \psi_t \rangle = -i \int_0^t \langle 0 | [\phi_{H_0}(\vec{x}, t), \int d^3y \, J(y, t') \phi_{H_0}(t')] | 0 \rangle$$

$$= -i \int_0^t dt' d^3y \; \langle 0 | [\phi_{H_0}(\vec{x}, t), \phi_{H_0}(y, t')] | 0 \rangle J(y, t)$$

$$= -i \int_0^t dt' d^3y \quad \Delta_c(\vec{x}, t; y, t') \, J(\vec{y}, t)$$

$$= -i \int d^4y \;\; \Theta(x^0 - y^0) \;\; \Delta_c(x-y) \; J(y)$$

$$\boxed{\langle 0 | \phi_H(\vec{x}) | 0 \rangle = -i \int d^4y \; \underbrace{\Theta(x^0 - y^0) \;\; \Delta_c(x-y)}_{D_R(x-y)} J(y)}$$

$$D_R(x-y) = \Theta(x^0-y^0)\,[\phi(x), \phi^\dagger(y)]$$

$$-i\int \frac{d^4k}{(2\pi)^4}\,\frac{e^{ikx}}{k^2+m^2}$$


$$\int \frac{dk_0}{2\pi}\,\frac{e^{ik_0t}}{k_0^2-\omega_k^2}$$

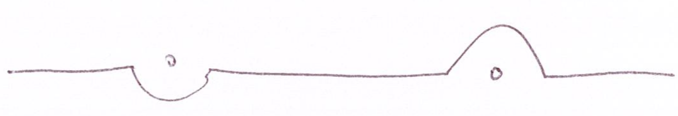
$k_0 + i\eta$

$$\int \frac{dk_0}{2\pi}\,\frac{e^{ik_0t-\eta t}}{(k_0-\omega_k)(k_0+\omega_k)}$$

$$\frac{\cancel{2\pi} i}{\cancel{2\pi}}\left(\frac{e^{i\omega_k t}}{2\omega_k}+\frac{e^{-i\omega_k t}}{-2\omega_k}\right) \qquad t>0.$$

$$D_R(x-y) = \Theta(x^0-y^0)\int \frac{d^3k}{(2\pi)^3}\left(\frac{e^{ik(x-y)}}{2\omega_k}-\frac{e^{-ik(x-y)}}{2\omega_k}\right)$$

Advanced



Feynmann.

Feynman propagator

$$i\int \frac{d^4p}{(2\pi)^4}\,\frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon} = \langle 0|\hat{T}\{\phi(x)\,\phi^\dagger(y)\}|0\rangle$$

i) $x^0>y^0$ $\quad \langle 0|\hat{T}\{\phi(x)\phi^\dagger(y)\}|0\rangle = \langle 0|\phi(x)\,\phi^\dagger(y)|0\rangle$

$$= \int\frac{d^3k\,d^3k'}{(2\pi)^3}\,\frac{1}{\sqrt{4\omega_k\omega_{k'}}}\,\langle 0|\left(c_k e^{-ikx} + \cancel{d_k^\dagger e^{ikx}}\right)\left(c_{k'}^\dagger e^{ik'y} + \cancel{d_k e^{-iky}}\right)|0\rangle$$

$$= \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\,e^{-ik(x-y)}$$

ii) $x^0<y^0$ $\quad \langle 0|\hat{T}\{\phi(x)\phi^\dagger(y)\}|0\rangle = \langle 0|\phi^\dagger(y)\,\phi(x)|0\rangle$

$$= \int\frac{d^3k\,d^3k'}{(2\pi)^3}\,\frac{1}{\sqrt{4\omega_k\omega_{k'}}}\,\langle 0|\left(\cancel{c_k^\dagger e^{iky}} + d_k e^{-iky}\right)\left(\cancel{c_{k'} e^{-ik'x}} + d_{k'}^\dagger e^{ik'x}\right)|0\rangle$$

$$= \int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\,e^{ik(x-y)}$$

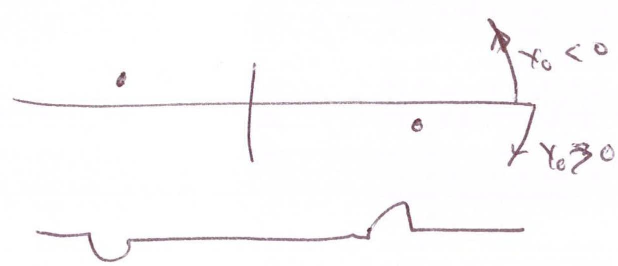
---

$$i\int\frac{dk_0}{2\pi}\,\frac{e^{-ik_0x_0}}{k_0^2-\omega_k^2+i\epsilon} = i\int\frac{dk_0}{2\pi}\,\frac{e^{-ik_0x_0+\eta x_0}}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k-i\epsilon)} = \begin{cases} x_0>0 & \cancel{0} + \dfrac{e^{-i\omega_k x_0}}{2\omega_k} \\ x_0<0 & +\dfrac{e^{i\omega_k x_0}}{+2\omega_k} \end{cases}$$

$$k_0^2 = \omega_k^2 - i\epsilon \qquad (k_0+i\eta)$$

$$k_0 = \pm\sqrt{\omega_k^2-i\epsilon}$$

$$= \pm\,\omega_k\left(1-\frac{i\epsilon}{2\omega_k^2}\right) = \pm\,\omega_k \mp i\epsilon$$



$$i\int \frac{dk_0}{2\pi} \frac{e^{-ik_0 x_0}}{k_0^2-\omega_k^2+i\epsilon} = \frac{e^{-i\omega_k |x_0|}}{2\omega_k}$$

$$i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon} = \int \frac{d^3p}{(2\pi)^3} \frac{e^{-i\omega_k |x_0-y_0| + i\vec{p}(\vec{x}-\vec{y})}}{2\omega_k} =$$

$$= \begin{cases} x^0 > y^0 & \displaystyle\int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\omega_k(x_0-y_0)+i\vec{k}(\vec{x}-\vec{y})}}{2\omega_k} = \int \frac{d^3k}{(2\pi)^3} \frac{e^{-ik(x-y)}}{2\omega_k} \checkmark \\ x^0 < y^0 & \displaystyle\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\omega_k(x_0-y_0)+i\vec{k}(\vec{x}-\vec{y})}}{2\omega_k} \underset{k\to -\vec{k}}{=} \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik(x-y)}}{2\omega_k} \checkmark \end{cases}$$

---

Propagators

$$\Delta(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon \otimes}$$

some prescription.

$$(+\partial^2+m^2)\,\Delta(x-y) = i\int \frac{d^4p}{(2\pi)^4} \frac{(-p^2+m^2)\, e^{-ip(x-y)}}{p^2-m^2+i\epsilon \otimes} =$$

$$= -i\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} = -i\,\delta^{(4)}(x-y)$$

$$\boxed{(-\partial^2-m^2)\,\Delta(x-y) = i\,\delta^{(4)}(x-y) \qquad \text{Green function}}$$

$\epsilon$-prescription $\to$ boundary conditions.

Another derivation of $\Delta_F$ satisfies eq. of motion (Klein-Gordon eqn.) with a source.

$$(-\partial^2 - m^2)_x \langle 0 | \hat{T}\{\phi(x)\,\phi^\dagger(y)\} | 0 \rangle = ?$$

$$(-\partial_0^2 + \nabla^2 - m^2)_x \langle 0 | \hat{T}\{\phi(x)\,\phi^\dagger(y)\} | 0 \rangle =$$

$$= (-\partial_0^2 + \nabla^2 - m^2)\Big( \Theta(x^0-y^0) \langle 0| \phi(x)\,\phi^\dagger(y) |0\rangle +$$

$$+ \Theta(y^0-x^0) \langle 0| \phi^\dagger(y)\,\phi(x) |0\rangle \Big) =$$

$$\langle 0| [\phi(x), \phi^\dagger(y)] |0\rangle_{E.T.} = 0$$

$$= -\partial_0 \Big( \delta(x^0-y^0) \langle 0| \phi(x)\,\phi^\dagger(y) |0\rangle - \delta(x^0-y^0) \langle 0| \phi^\dagger(y)\,\phi(x) |0\rangle +$$

$$+ \Theta(x^0-y^0) \langle 0| \partial_0 \phi(x)\,\phi^\dagger(y) |0\rangle + \Theta(y^0-x^0) \langle 0| \phi^\dagger(y)\,\partial_0 \phi(x) |0\rangle \Big)$$

$$+ \Theta(x^0-y^0) \langle 0| (\nabla^2-m^2)\,\phi(x)\,\phi^\dagger(y) |0\rangle + \Theta(y^0-x^0) \langle 0| \phi^\dagger(y) (\nabla^2-m^2) \phi(x) |0\rangle$$

$$= -\delta(x^0-y^0) \langle 0| [\partial_0 \phi(x), \phi^\dagger(y)] |0\rangle + \Theta(x^0-y^0) \langle 0| (-\partial^2-m^2)\phi(x)\,\phi^\dagger(y) |0\rangle$$

$$+ \Theta(y^0-x^0) \langle 0| \phi^\dagger(y) (-\partial^2-m^2) \phi(x) |0\rangle$$

$$= -\delta(x^0-y^0) \langle 0| [\underbrace{\partial_0 \phi(x)}_{\pi(x)}, \phi^\dagger(y)]_{E.T.} |0\rangle + \langle 0| \hat{T}\{\underbrace{(-\partial^2-m^2)\phi(x)}_{\substack{0 \\ \text{from e.o.m.}}}\,\phi^\dagger(y)\} |0\rangle$$

$$= i\,\delta^{(4)}(x-y)$$

# Schwinger term



$$j^\mu = i\left(\phi^* \partial^\mu \phi - \phi\, \partial^\mu \phi^*\right) \qquad [\pi(\vec{x}), \phi(\vec{y})] = -i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$j^0 = i\left(\phi^* \bar{\pi} - \phi\,\pi\right) \qquad \pi = \partial^0 \phi^*$$

$$j^l = i\left(\phi^* \partial^l \phi - \phi\, \partial^l \phi^*\right)$$

$$[j^0(\vec{x}), j^0(\vec{y})] = -\left[\phi^*(x)\bar{\pi}(x) - \phi(x)\pi(x),\ \phi^*(y)\bar{\pi}(y) - \phi(y)\pi(y)\right]$$

$$= -\phi^*(x)(-i)\,\delta(x-y)\,\bar{\pi}(y) - \phi^*(y)\, i\delta(x-y)\,\bar{\pi}(x)$$

$$-\phi(x)(-i)\,\delta(x-y)\,\pi(y) - \phi(y)\, i\delta(x-y)\,\pi(x)$$

$$= i\,\delta(x-y)\left(\cancel{\phi^*(x)\bar{\pi}(x)} - \cancel{\phi^*(x)\bar{\pi}(x)} + \cancel{\phi(x)\pi(x)} - \cancel{\phi(x)\pi(x)}\right)$$

$$= 0$$

$$[j^0(\vec{x}), j^l(\vec{y})] = -\left[\phi^*(x)\bar{\pi}(x) - \phi(x)\pi(x),\ \phi^*(y)\partial^l\phi(y) - \phi(y)\partial^l\phi^*(y)\right]$$

$$= -\phi^*(x)(-i)\,\delta(x-y)\,\partial_y^l\phi(y) + \phi^*(x)\phi(y)(-i)\,\partial_y^l\delta(x-y)$$

$$+\phi(x)\phi^*(y)(-i)\,\partial_y^l\delta(x-y) - \phi(x)(-i)\,\delta(x-y)\,\partial^l\phi^*(y)$$

$$= i\,\delta(x-y)\left(\phi^*(x)\,\partial_y^l\phi(y) + \phi(x)\,\partial_y^l\phi(y)\right)$$

$$-i\,\partial_y^l\delta(x-y)\left(\phi^*(x)\phi(y) + \phi(x)\phi^*(y)\right)$$

$\Rightarrow$ To understand better we integrate against test function $g(x,y)$

$$\int d^3x\, d^3y\; g(x,y)\,[j^0(x), j^l(\vec y)] =$$

$$= i\int d^3x\;\; g(x,x)\;(\phi^*(x)\,\partial_x^l \phi(x) + \phi(x)\,\partial_x^l \phi^*(x))$$

$$+ i\int d^3x\;\; \partial_y^l\,\big(\,g(x,y)\,(\phi^*(x)\,\phi(y) + \phi(x)\,\phi^*(y))\big)\Big|_{x=y}$$

$$= i\int d^3x\,\Big[\,g(x,x)\,\partial_x^l\,|\phi(x)|^2 + g(x,x)\,\partial_x^l\,|\phi(x)|^2$$

$$+\ \partial_2^l\, g(x,x)\;\; 2|\phi(x)|^2\,\Big]$$

derivative w/ respect to $2^{nd}$ argument.

$$= 2i\int d^3x\;\; g(x,x)\,\partial_x^l\,|\phi(x)|^2 + \partial_2^l\, g(x,x)\;\;|\phi(x)|^2$$

$$= 2i\int d^3x\;\; \underbrace{\partial_x^l\,(g(x,x)\;|\phi(x)|^2)}_{0 \text{ by parts}} - 2i\int d^3x\;\partial_1^l\, g(x,x)\,|\phi(x)|^2$$

$$= -2i\int d^3x\;\;\partial_1^l\; g(x,x)\;|\phi(x)|^2 = 2i\int d^3x\,d^3y\;\;|\phi(y)|^2\, g(x,y)\,\partial_x^l\delta(x-y)$$

(we used $\partial_x^l\, g(x,x) = \partial_1^l\, g(x,x) + \partial_2^l\, g(x,x)$ )

$$\Rightarrow\quad \boxed{[j^0(\vec x), j^l(\vec y)] = 2i\;|\phi(y)|^2\;\partial_x^l\;\delta(x-y)}$$

It should be a total derivative w/respect to $\vec{x}$

because

$$[\int d^3x\, j^0(\vec{x}),\, j^l(y)] = [Q,\, j^l(x)] = 0$$

invariant under

$\phi \to e^{i\alpha}\phi,\ \phi^\dagger \to e^{-i\alpha}\phi$

It cannot be zero. Consider hermitian op. $\mathcal{O}$.

and assume $H|0\rangle = 0$

$$\dot{\mathcal{O}} = i[H, \mathcal{O}]$$

$$\langle 0|[\mathcal{O}, \dot{\mathcal{O}}]|0\rangle = i\langle 0|[\mathcal{O}, [H, \mathcal{O}]]|0\rangle =$$

$$= i\langle 0|\mathcal{O}[H, \mathcal{O}]|0\rangle - i\langle 0|[H, \mathcal{O}]\mathcal{O}|0\rangle =$$

$$= i\langle 0|\mathcal{O}H\mathcal{O}|0\rangle + i\langle 0|\mathcal{O}H\mathcal{O}|0\rangle = 2i\langle 0|\mathcal{O}H\mathcal{O}|0\rangle$$

But defining $|\psi\rangle = \mathcal{O}|0\rangle$ we have

$$\langle 0|[\mathcal{O}, \dot{\mathcal{O}}]|0\rangle = 2i\langle\psi|H|\psi\rangle$$

$\langle\psi|H|\psi\rangle > 0$ unless $|\psi\rangle = \alpha|0\rangle$

Assume $\mathcal{O}|0\rangle \neq \alpha|0\rangle \Rightarrow \langle 0|[\mathcal{O}, \dot{\mathcal{O}}]|0\rangle \neq 0$

Take $\mathcal{O} = \int d^3x\, g(\vec{x})\, j^0(\vec{x})$

test function

$$[\mathcal{O}, \dot{\mathcal{O}}] = \int d^3x\, d^3y\; g(\vec{x})\, g(\vec{y})\, [j^0(x), \partial_0 j^0(\vec{y})]$$

$$\partial_0 j^0 + \partial_\ell j^\ell = 0$$

$$[\mathcal{O}, \dot{\mathcal{O}}] = -\int d^3x\, d^3y\; g(\vec{x})\, g(\vec{y})\, [j^0(x), \partial_\ell j^\ell(y)]$$

$$= \int d^3x\, d^3y\; g(\vec{x})\, \partial_\ell g(\vec{y})\, [j^0(x), j^\ell(y)]$$

But $\langle 0| [\mathcal{O}, \dot{\mathcal{O}}] |0\rangle = 2i\, \langle \psi | H | \psi \rangle \neq 0$

and then $[j^0(\vec{x}), j^\ell(\vec{y})] \neq 0$ in general

cannot be zero but has to be total $\vec{x}$ derivative.

Here:

$$[\mathcal{O}, \dot{\mathcal{O}}] = \int d^3x\, d^3y\; g(x)\, \partial_\ell g(y)\; 2i\, |\phi(y)|^2\, \partial_x^\ell\, \delta(x-y)$$

$$= -2i \int d^3x\, d^3y\; \partial^\ell g(x)\, \partial_\ell g(y)\, |\phi(y)|^2\, \delta(x-y)$$

$(\partial_\ell g = -\partial^\ell g)$

$$= 2i \int d^3x \sum_\ell (\partial_\ell g(x))^2\, |\phi(x)|^2$$

$$\langle 0| [\mathcal{O}, \dot{\mathcal{O}}] |0\rangle = 2i \int d^3x\, \underbrace{\sum_\ell \underbrace{(\partial_\ell g)^2}_{>0}\, \underbrace{\langle 0| |\phi(x)|^2 |0\rangle}_{>0}}_{>0}$$

$>0$ as it should be

$= \langle \psi | H | \psi \rangle$