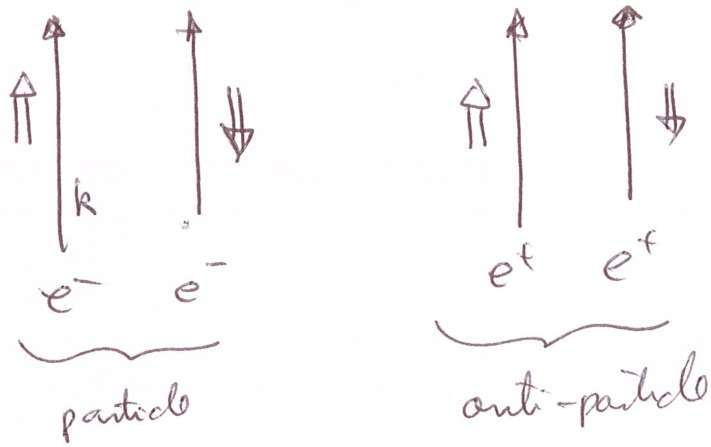
spin 1/2 Fermions in 3+1 dim

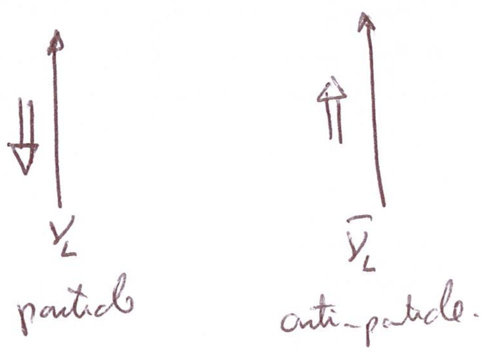


$|e^-, \vec{k}, \sigma\rangle$ $\pm\frac{1}{2}$

$|e^+, \vec{k}, \sigma\rangle$ $\pm\frac{1}{2}$

4 states

Weyl fermion



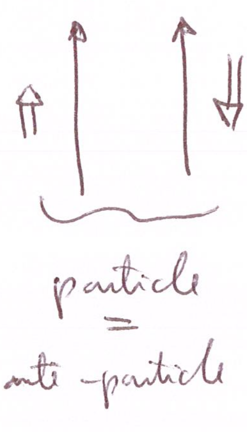
it has to be massless.

Otherwise by a boost we can change the helicity.

<u>Violates parity</u>.

Can have charge (ν_L does not)

Majorana fermion



·) Can have mass

·) <u>Cannot</u> have charge

ν ?

Neutrinoless double beta decay.

Spin 1/2 Fermions

$SO(3) \cong SU(2)$

$\begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix} = \mathbb{X}$ $\qquad \mathbb{X} = \mathbb{X}^\dagger \qquad \mathrm{Tr}\,\mathbb{X} = 0$ most general matrix satisfying this

$x_i \sigma_i$

$$\det \mathbb{X} = -x_1^2 - x_2^2 - x_3^2$$

$$\tilde{\mathbb{X}} = U \mathbb{X} U^\dagger \leftarrow \tilde{\mathbb{X}}^\dagger = U \mathbb{X}^\dagger U^\dagger = U \mathbb{X} U^\dagger = \tilde{\mathbb{X}}$$

$$\mathrm{Tr}\,\tilde{\mathbb{X}} = \mathrm{Tr}\,(U \mathbb{X} U^\dagger) = \mathrm{Tr}\,(U^\dagger U \mathbb{X}) \stackrel{?}{=} \mathrm{Tr}\,\mathbb{X}$$

We require $U^\dagger U = \mathbb{1}$ $\quad \boxed{U^\dagger = U^{-1}}$

$$\det \tilde{\mathbb{X}} = \det(U \mathbb{X} U^\dagger) = |\det U|^2 \det \mathbb{X}$$ already ok.

We require $|\det U| = 1.$ $\quad \boxed{U \in U(2)}$

$\tilde{\mathbb{X}}$ linear function of $\mathbb{X}$. $\Rightarrow$ rotation.

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \qquad U^\dagger = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \frac{1}{\alpha\delta - \beta\gamma}$$

$$= \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix}$$

$$U = e^{i\varphi}\, \mathbb{1}$$

$$\tilde{\mathbb{X}} = e^{i\varphi} \mathbb{X} e^{-i\varphi} = \mathbb{X}$$ nothing happens

take $\det U = 1$ $\quad SU(2) \quad U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \quad |\alpha|^2 + |\beta|^2 = 1$

S^3 3-sphere

Double cover

$U = -\mathbb{1}$ $\qquad \tilde{\mathbb{X}} \to \mathbb{X}$ identity.

Spinors

$\xi \to U\xi \qquad ; \tilde{\xi} = U\xi = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}\xi$

$SO(3,1) \cong SL(2,\mathbb{C}) \qquad (SO(3) \equiv SU(2))$

$$\mathbb{X} = \begin{pmatrix} x_0+x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} = x_0 \mathbb{1} + x_j \sigma_j$$

$\det \mathbb{X} = x_0^2 - x_3^2 - x_1^2 - x_2^2$: interval

$\mathbb{X} = \mathbb{X}^\dagger$ most general.

$\tilde{\mathbb{X}} = U \mathbb{X} U^\dagger$ $\qquad$ but we do not need $U^\dagger = U^{-1}$ no.

we still need $|\det U| = 1$. $\to$ $\det U = 1$ since phase is irrelevant

$$\boxed{U \in SL(2,\mathbb{C})}$$

Spinor $\tilde{\xi} = U\xi$ $\qquad$ left

conjugate spinor $\tilde{\xi}^* = U^* \xi^*$ $\qquad$ right.

not equivalent.

Notice equivalency in $SU(2)$ case. $\sigma_a \sigma_b = \delta_{ab} + i \epsilon_{abc} \sigma_c$

$$\tilde{\xi} = U \xi \qquad \tilde{\xi}^* = U^* \xi^*$$

$$U = e^{i\theta_j \frac{\sigma_j}{2}} \quad ; \quad U^* = e^{-i\theta_j \frac{\sigma_j^*}{2}}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$-\sigma_1^* = -\sigma_1 \qquad -\sigma_2^* = \sigma_2 \qquad -\sigma_3^* = -\sigma_3$$

$$\sigma_2 \sigma_1 \sigma_2 = i \sigma_2 \sigma_3 = i i \sigma_1 = -\sigma_1$$

$$\sigma_2 \sigma_2 \sigma_2 = \sigma_2$$

$$\sigma_2 \sigma_3 \sigma_2 = i \sigma_1 \sigma_2 = -\sigma_3$$

$$U^* = \sigma_2 U \sigma_2$$

$$\tilde{\xi}^* = \sigma_2 U \sigma_2 \xi^*$$

$$\underbrace{\sigma_2 \tilde{\xi}^*} = U (\underbrace{\sigma_2 \xi^*})$$

change of basis

Same rep.

Works because of $\begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}^* \Longrightarrow \begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix}$

But in $SL(2,\mathbb{C})$ $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}$ no relation

Dirac spinor

(ξ_L, ξ_R) pair of 2 spinors. $e^-, \uparrow, \downarrow$ $e^+ \uparrow \downarrow$

Weyl spinor

$\xi_R = 0 \Rightarrow \xi_L \neq 0$ or $\xi_L = 0$ $\xi_R \neq 0$

no mass

$\nu_L, \bar{\nu}_R$ violates parity $k \Uparrow$, $k \Downarrow$

$\nu_L^+, \bar{\nu}_R^+$ massless

half the runs fast. components.

Majorana. differ in the action.

$\xi_R = \xi_L^*$ reality condition

can have mass

maybe $\bar{\nu}_R = \nu_L^+$

Charge $\xi \to e^{i\alpha}\xi$ phase.

Dirac $(\xi_L, \xi_R) \to (e^{i\alpha}\xi_L, e^{i\alpha}\xi_R)$

Weyl $\xi_L \to e^{i\alpha}\xi_L$ or $\xi_R \to e^{i\alpha}\xi_R$

Majorana

$e^{i\alpha}\xi_R \neq e^{-i\alpha}\xi_L^*$ not possible

they are <u>neutral</u>.

Lorentz algebra.

$$M = \begin{pmatrix} 0 & m_{01} & m_{02} & m_{03} \\ m_{01} & 0 & m_{12} & m_{13} \\ m_{02} & -m_{12} & 0 & m_{23} \\ m_{03} & -m_{13} & -m_{23} & 0 \end{pmatrix}$$

generators

$$M^{01} = i\begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad M^{02} = i\begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix} \quad M^{03} = i\begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix}$$

$$M^{12} = i\begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix} \quad M^{13} = i\begin{pmatrix} 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&0&0 \\ 0&1&0&0 \end{pmatrix} \quad M^{23} = i\begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix}$$

$$(M^{\alpha\beta})^{\mu}{}_{\nu} = i\left(\eta^{\alpha\mu}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\delta^{\alpha}{}_{\nu}\right)$$

$$\Lambda = e^{-i\,\theta_{\alpha\beta}M^{\alpha\beta}}$$

$$[M^{\alpha\beta}, M^{\mu\nu}]^{\rho}{}_{\sigma} = i\left(\eta^{\mu\beta}M^{\alpha\nu} - \eta^{\beta\nu}M^{\alpha\mu} - \eta^{\alpha\mu}M^{\beta\nu} + \eta^{\alpha\nu}M^{\beta\mu}\right)$$

Dirac method (generic to any dimension).

Find matrices γ^μ / $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$

$(\gamma^0)^2 = \mathbb{1}$; $(\gamma^i)^2 = -\mathbb{1}$.

example $\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$

but we can change basis $\gamma^\mu \to A\gamma^\mu A^{-1} = \tilde{\gamma}^\mu$.

Define $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu} \Rightarrow S^{\mu\nu} = \frac{i}{2}\gamma^\mu\gamma^\nu + \frac{i}{2}\eta^{\mu\nu}$

$$[S^{\mu\nu}, \gamma^\alpha] = \frac{i}{2}(\gamma^\mu\gamma^\nu\gamma^\alpha - \gamma^\alpha\gamma^\mu\gamma^\nu)$$

$$= \frac{i}{2}(-\gamma^\mu\gamma^\alpha\gamma^\nu + 2\gamma^\mu\eta^{\nu\alpha} - \gamma^\alpha\gamma^\mu\gamma^\nu)$$

$$= \frac{i}{2}(\cancel{\gamma^\alpha\gamma^\mu\gamma^\nu} - 2\eta^{\mu\alpha}\gamma^\nu + 2\gamma^\mu\eta^{\nu\alpha} - \cancel{\gamma^\alpha\gamma^\mu\gamma^\nu})$$

$$= i(\eta^{\nu\alpha}\gamma^\mu - \eta^{\mu\alpha}\gamma^\nu)$$

$$[S^{\mu\nu}, S^{\alpha\beta}] = i(\eta^{\nu\beta}S^{\alpha\mu} - \eta^{\mu\beta}S^{\alpha\nu} + \eta^{\nu\alpha}S^{\mu\beta} - \eta^{\mu\alpha}S^{\nu\beta}) \checkmark$$

In example

$$S^{0i} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \; ; \; S^{ij} = \frac{i}{2}\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \epsilon^{ijk}\sigma^k & 0 \\ 0 & \epsilon^{ijk}\sigma^k \end{pmatrix} \checkmark$$

(6b)

$$S^{0i} = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \qquad S^{ij} = \frac{1}{2}\begin{pmatrix} \epsilon^{ijk}\sigma^k & 0 \\ 0 & \epsilon^{ijk}\sigma^k \end{pmatrix}$$

$$\Lambda_{1/2} = e^{\frac{1}{2}\beta_i\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} + \frac{i}{2}\theta_{ij}\begin{pmatrix} \epsilon^{ijk}\sigma^k & 0 \\ 0 & \epsilon^{ijk}\sigma^k \end{pmatrix}}$$

$$\sigma_2\sigma_k\sigma_2 = -\sigma_k^*$$

$$= e^{\frac{1}{2}\beta_i\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} + \frac{i}{2}\theta_k\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

$$= \begin{pmatrix} e^{-\frac{1}{2}\beta_i\sigma^i - \frac{i}{2}\theta_k\sigma_k} & 0 \\ 0 & e^{\frac{1}{2}\beta_i\sigma^i - \frac{i}{2}\theta_k\sigma^k} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & \tilde{U} \end{pmatrix}$$

$$U^* = e^{+\frac{1}{2}\beta_i\,\sigma_2\sigma^i\sigma_2 + \frac{i}{2}\theta_k\sigma_2\sigma_k\sigma_2} = \sigma_2\tilde{U}\sigma_2$$

$$\tilde{U} = \sigma_2 U^*\sigma_2$$

$$\Lambda_{1/2} = \begin{pmatrix} U & 0 \\ 0 & \sigma_2 U^*\sigma_2 \end{pmatrix} \quad \checkmark$$

Boost

$$\Lambda_{1/2} = \begin{pmatrix} e^{-\frac{1}{2}\beta\hat{\beta}_i\sigma_i} & 0 \\ 0 & e^{\frac{1}{2}\beta\hat{\beta}_i\sigma_i} \end{pmatrix} = \begin{pmatrix} \mathrm{ch}\frac{\beta}{2} - \mathrm{sh}\frac{\beta}{2}\hat{\beta}_i\sigma_i & 0 \\ 0 & \mathrm{ch}\beta/2 + \mathrm{sh}\frac{\beta}{2}\hat{\beta}_i\sigma_i \end{pmatrix}$$

$$e^{\frac{1}{2}\beta\hat{\beta}_i\sigma^i} = \sum_{n=0}^{\infty}\frac{\beta^n}{2^n}\frac{(\hat{\beta}_i\sigma^i)^n}{n!} = \sum_{n\ \mathrm{even}}\frac{\beta^n}{2^n n!} + \sum_{n\ \mathrm{odd}}\frac{\beta^n}{2^n n!}\hat{\beta}_i\sigma_i = \mathrm{ch}\frac{\beta}{2} + \mathrm{sh}\frac{\beta}{2}(\sigma_i\hat{\beta}_i)$$

$$m(1,0,0,0) \xrightarrow[\text{boost}]{} (\omega, \vec{k})$$

$\omega = m\,\mathrm{ch}\beta$ $\qquad 2\,\mathrm{sh}\beta/2\,\mathrm{ch}\beta/2 = \mathrm{sh}\beta \quad ; \quad \mathrm{ch}^2\beta/2 - \mathrm{sh}^2\beta/2 = 1$

$k = m\,\mathrm{sh}\beta$ $\qquad \mathrm{ch}\beta = \mathrm{ch}^2\beta/2 + \mathrm{sh}^2\beta/2 = 1 + 2\,\mathrm{sh}^2\beta/2$

$\hat{\beta}_i = \hat{k}_i$ $\qquad = 2\,\mathrm{ch}^2\beta/2 - 1$

$$\mathrm{sh}\beta/2 = \sqrt{\frac{\mathrm{ch}\beta - 1}{2}} \qquad \mathrm{ch}\beta/2 = \sqrt{\frac{\mathrm{ch}\beta + 1}{2}}$$

$$\mathrm{sh}\beta/2 = \sqrt{\frac{\omega/m - 1}{2}} = \frac{\sqrt{\omega - m}}{\sqrt{2m}} \qquad \mathrm{ch}\beta/2 = \frac{\sqrt{\omega + m}}{\sqrt{2m}}$$

$$A_{1/2}(\vec{k}) = \frac{1}{\sqrt{2m}} \begin{pmatrix} \sqrt{\omega+m} - \sqrt{\omega-m}\,\hat{k}_i \sigma_i & 0 \\ 0 & \sqrt{\omega+m} + \sqrt{\omega-m}\,\hat{k}_i \sigma_i \end{pmatrix}$$

Properties

$$\gamma_0 \gamma^\mu \gamma_0 = \begin{cases} \gamma_0 \; ; \; \mu = 0 \\ -\gamma^i \gamma_0 \gamma_0 = -\gamma_i \; ; \; \mu = i \end{cases}$$

$$(\gamma^\mu)^\dagger = \begin{cases} \gamma^0 & \mu = 0 \\ -\gamma^i & \mu = i \end{cases} \qquad (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0$$

Dirac spinor $\quad \psi \xrightarrow{\Lambda} \psi' = \Lambda_{1/2} \psi = e^{-i\theta_{\mu\nu} S^{\mu\nu}} \psi .$

4-component column vector.

$$\psi^\dagger \to \psi^\dagger \Lambda_{1/2}^\dagger = \psi^\dagger e^{i\theta_{\mu\nu} (S^{\mu\nu})^\dagger} = \psi^\dagger e^{i\theta_{\mu\nu} \gamma_0 S^{\mu\nu} \gamma_0}$$

$$= \psi^\dagger \gamma_0 e^{i\theta_{\mu\nu} S^{\mu\nu}} \gamma_0$$

$$\psi^\dagger \gamma_0 \Rightarrow \psi^\dagger \Lambda_{1/2}^\dagger \gamma_0 = \psi^\dagger \gamma_0 \Lambda_{1/2}^{-1} \gamma_0 \gamma_0 = (\psi^\dagger \gamma_0) \Lambda_{1/2}^{-1}$$

$$\bar\psi = \psi^\dagger \gamma_0$$

$\bar\psi \to \bar\psi \Lambda_{1/2}^{-1} \qquad \bar\psi \psi$ <u>scalar</u>.

Also $\quad \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu$

$$\tilde\gamma^\mu(t) = e^{it\theta_{\alpha\beta} S^{\alpha\beta}} \gamma^\mu e^{-it\theta_{\alpha\beta} S^{\alpha\beta}}$$

$$\tilde\gamma^\mu(t) = \left(e^{-i\theta_{\alpha\beta} M^{\alpha\beta} t}\right)^\mu{}_\nu \gamma^\nu$$

$$\tilde{\gamma}^\mu(0) = \gamma^\mu(0)$$

$$\partial_t \gamma^\mu(t) = i \Lambda_{1/2}^{-1} \left[\theta_{\alpha\beta} S^{\alpha\beta}, \gamma^\mu\right] \Lambda_{1/2}$$

$$= i\theta_{\alpha\beta} \Lambda_{1/2}^{-1} \left(i\eta^{\beta\mu}\gamma^\alpha - i\eta^{\alpha\mu}\gamma^\beta\right) \Lambda_{1/2}$$

$$= -\theta_{\alpha\mu}\gamma^\alpha(t) + \theta_{\mu\beta}\gamma^\beta(t)$$

$$= 2\theta_{\mu\alpha}\gamma^\alpha(t)$$

$$\partial_t \tilde{\gamma}^\mu(t) = -i\theta_{\alpha\beta}\left(M^{\alpha\beta}\right)^\mu{}_\nu \tilde{\gamma}^\nu(t) = -i\theta_{\alpha\beta}\left(i\eta^{\alpha\mu}\delta^\beta_\nu - i\eta^{\beta\mu}\delta^\alpha_\nu\right)\tilde{\gamma}^\nu(t)$$

$$= \theta_{\mu\nu}\tilde{\gamma}^\nu(t) - \theta_{\alpha\mu}\tilde{\gamma}^\alpha(t)$$

$$= 2\theta_{\mu\nu}\tilde{\gamma}^\nu(t)$$

Same diff. eq. (1st order) and same initial cond.

$$\Rightarrow \gamma^\mu(t) = \tilde{\gamma}^\mu(t) \Rightarrow \gamma^\mu(1) = \tilde{\gamma}^\mu(1)$$

$$\Rightarrow \boxed{\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu}$$

$\Rightarrow \quad V^\mu = \bar\psi \gamma^\mu \psi$ transforms as a vector.

γ^μ equivalent to Clebsch-Gordan coeff.

$$\bar\psi \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} \psi = \Lambda^\mu{}_\nu \bar\psi \gamma^\nu \psi$$

Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$

in chiral rep. $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \{\gamma^5, \gamma^\mu\} = 0 \quad (\gamma^5)^2 = 1$

$(i\gamma_5)^2 = -1$

(spatial matrix of extra dimension.) $\qquad \Lambda_{1/2}^{-1} \gamma^5 \Lambda_{1/2} = \gamma^5$

parity $\quad 0 \to 0$, $1 \to -1$, $2 \to -2$, $3 \to -3$

$\bar\psi \gamma_5 \psi$ pseudo-scalar.

$\bar\psi \gamma_5 \gamma^\mu \psi$ " vector $\qquad \gamma^5 \to -\gamma^5$

$\bar\psi \sigma^{\mu\nu} \psi$ antisymmetric tensor. $\qquad \bar\psi [\gamma^\mu, \gamma^\nu] \psi$

"

$$4 \times 4 \to 16 = \underset{\text{scalar}}{1} \oplus \underset{\text{vector}}{4} \oplus \underbrace{3 \oplus 3}_{\substack{\text{antisymmetric} \\ \text{tensor w/} \\ \text{2 indices.}}} \oplus \underset{\text{pseudovector}}{4} + \underset{\text{pseudoscalar}}{1}$$

Weyl fermion

$$\gamma_5 \psi = \psi \quad \text{or} \quad \gamma_5 \psi = -\psi$$

Majorana

reality condition : relation between ψ^* and ψ.

$$\psi \to \Lambda_{1/2} \psi \quad ; \quad \psi^* \to \Lambda_{1/2}^* \psi^*$$

$$\Lambda_{1/2} = e^{-i\theta_{\alpha\beta} S^{\alpha\beta}} = e^{-i\theta_{\alpha\beta} \frac{i}{4}[\gamma^\alpha, \gamma^\beta]} = e^{\frac{1}{4}\theta_{\alpha\beta}[\gamma^\alpha,\gamma^\beta]}$$

$$\Lambda_{1/2}^* = e^{\frac{1}{4}\theta_{\alpha\beta}[(\gamma^\alpha)^*, (\gamma^\beta)^*]}$$

$$(\gamma^0)^* = \gamma^0 \qquad (\gamma^1)^* = \gamma^1 \qquad (\gamma^2)^* = -\gamma^2 \qquad (\gamma^3)^* = \gamma^3$$

$$\gamma^2\gamma^\mu\gamma^2 = \begin{cases} \gamma^\mu & \mu \neq 2 \\ -\gamma^\mu & \mu = 2 \end{cases} \Rightarrow \boxed{(\gamma^\mu)^* = \gamma^2\gamma^\mu\gamma^2}$$

$$\Lambda_{1/2}^* = e^{\frac{1}{4}\theta_{\alpha\beta}[\gamma^2\gamma^\alpha\gamma^2, \gamma^2\gamma^\beta\gamma^2]} = e^{-\frac{1}{4}\theta_{\alpha\beta}\gamma^2[\gamma^\alpha,\gamma^\beta]\gamma^2}$$

$$e^{\gamma_2 A \gamma_2} = 1 + \gamma_2 A \gamma_2 + \frac{1}{2}\gamma_2 A \underbrace{\gamma_2\gamma_2}_{-1} A\gamma_2 + \frac{1}{3!}\gamma_2 A \underbrace{\gamma_2\gamma_2}_{-1} A \underbrace{\gamma_2\gamma_2}_{-1} A\gamma_2 + \cdots$$

$$= \gamma_2\left(-1 + A - \frac{1}{2}A^2 + \frac{1}{3!}A^3 \quad\right)\gamma_2$$

$$= -\gamma_2 e^{-A}\gamma_2$$

$$\Lambda_{1/2}^{*} = -\gamma^2 e^{\frac{1}{4}\theta_{\alpha\beta}[\gamma^\alpha,\gamma^\beta]} \gamma^2$$

$$= -\gamma^2 \Lambda_{1/2} \gamma^2$$

$$\psi^* = -\gamma^2 \Lambda_{1/2} \gamma^2 \psi^*$$

$$\underbrace{\gamma^2\psi^*}_{\text{transforms as } \psi.} = \Lambda_{1/2} \gamma^2 \psi^*$$

$$\boxed{\psi = \gamma^2 \psi^*}$$

Majorana condition in chiral rep.

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi \qquad \text{Dirac.}$$

$$= \psi^\dagger \gamma_0 (i\gamma^\mu \partial_\mu - m)\psi$$

<u>Weyl.</u>

e.g. $\gamma_5 \psi = \psi \qquad \psi^\dagger = \psi^\dagger \gamma_5 \qquad \bar{\psi} = \psi^\dagger \gamma_5 \gamma_0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = -\bar{\psi}\gamma_5$

$$\bar{\psi}\psi = +\bar{\psi}\gamma_5\gamma_5\psi = -\bar{\psi}\psi \Rightarrow \boxed{\bar{\psi}\psi = 0}$$

$$\bar{\psi}\gamma^\mu\psi = -\bar{\psi}\gamma_5\gamma^\mu\gamma_5\psi = \bar{\psi}\gamma^\mu\psi \quad \text{can be} \neq 0$$

$$\mathcal{L}_W = \bar{\psi}\, i\partial\!\!\!/\, \psi = \frac{1}{2}\bar{\psi}\, i\partial\!\!\!/\, \underbrace{(1+\gamma_5)}_{\text{for emphasis.}}\psi$$

Majorana

$$\bar{\psi} = \psi^\dagger \gamma_0 = (\psi^c)^t \gamma_0 = -\psi^t (\gamma^2)^t \gamma_0 = \psi^t \gamma^2 \gamma_0$$

$$\psi^c = -\gamma^2 \psi$$

$$\mathcal{L}_M = \psi^t \gamma^2 \gamma_0 (i\partial\!\!\!/ - m)\psi$$

$$\gamma^2\gamma_0 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}$$

$\psi_L^t \sigma^2 \psi_L$ seems like zero $\Rightarrow$ ψ anticommuting numbers.

antisymmetric

Dirac Field

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \psi^\dagger \gamma_0 (i\gamma^\mu \partial_\mu - m)\psi$$

$\psi = \begin{pmatrix} - \\ - \\ - \\ - \end{pmatrix}$ complex.

$$= i\psi^\dagger \dot{\psi} + i\psi^\dagger \gamma_0 \gamma^i \partial_i \psi - m\psi^\dagger \gamma_0 \psi$$

$$\Pi_\psi = \frac{\delta L}{\delta \dot{\psi}} = i\psi^\dagger \gamma_0 \gamma^0 = i\psi^\dagger$$

$$H = \Pi_\psi \dot{\psi} - \mathcal{L} = -i\psi^\dagger \gamma_0 \gamma^i \partial_i \psi + m\psi^\dagger \gamma_0 \psi$$

$\psi^\dagger = -i\Pi_\psi$ ← we can use $\psi^\dagger$ instead of Π_ψ

$$\boxed{\{\psi^\dagger_a(\vec{x}), \psi_b(\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}}$$

e.o.m.

For $\bar{\psi}$ (or $\psi^\dagger$): $\quad (i\not{\partial} - m)\psi = 0 \quad$ Dirac eqn.

Notice

$$(i\not{\partial} + m)\underbrace{(i\not{\partial} - m)\psi}_{0} = (-\not{\partial}\not{\partial} - m^2)\psi = 0$$

$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = \partial^2 \qquad (-\partial^2 - m^2)\psi = 0$$

Each component satisfies the Klein Gordon eqn.

$$(\partial^2+m^2)\psi_a=0$$

$$\psi_a=u_a(k)\,e^{-ikx} \qquad \psi_a=v_a(k)\,e^{ikx}$$

$$(\not{k}-m)\,u=0 \qquad (\not{k}+m)\,v=0$$

Take $\underline{\vec{k}=0}$ $\quad k^\mu=m(1,0,0,0)$

$$(\gamma_0-1)\,u=0 \qquad \gamma_0 u=u \qquad \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}=\begin{pmatrix}\xi_2\\\xi_1\end{pmatrix}=\begin{pmatrix}\xi_1\\\xi_2\end{pmatrix}$$

$$u=\begin{pmatrix}\xi\\\xi\end{pmatrix} \text{ same.} \qquad u^{(1)}=\sqrt{m}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \qquad u^{(2)}=\sqrt{m}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

Using a boost we get the solutions for arbitrary k^μ.

$$u^{(r)}(k)=\frac{1}{\sqrt{2}}\begin{pmatrix}\left[\sqrt{\omega+m}-\sqrt{\omega-m}\,(\hat{k}.\sigma)\right] & 0\\ 0 & \left[\sqrt{\omega+m}+\sqrt{\omega-m}\,(\hat{k}.\sigma)\right]\end{pmatrix}\begin{pmatrix}\xi^{(r)}\\\xi^{(r)}\end{pmatrix}$$

$$\xi^{(1)}=\begin{pmatrix}1\\0\end{pmatrix} \qquad \xi^{(2)}=\begin{pmatrix}0\\1\end{pmatrix}$$

$$\bar{u}^{(r)}_{\vec{k}}\,u^{(s)}_{\vec{k}}=2m\,\delta^{rs} \qquad \text{Lorentz invariant normalization}$$

Consider $\bar{u}^{(r)}_{\vec{k}} \gamma^\mu u^{(s)}_{\vec{k}} = \Lambda^\mu{}_\nu \, \bar{u}^{(r)}_{\vec{k}=0} \gamma^\nu u^{(s)}_{\vec{k}=0}$

$$\bar{u}^{(r)}_{\vec{k}=0} \gamma^\mu u^{(s)}_{\vec{k}=0} = \left(u^{(r)}_{\vec{k}=0}\right)^\dagger \gamma^0 \gamma^\mu u^{(s)}_{\vec{k}=0} =$$

$$= \begin{cases} (\mu=0) \quad \left(u^{(r)}_{\vec{k}=0}\right)^\dagger u^{(s)}_{\vec{k}=0} = m \left((\xi^{(r)})^\dagger (\xi^{(r)})^\dagger\right) \begin{pmatrix} \xi^{(s)} \\ \xi^{(s)} \end{pmatrix} = 2m \, \delta^{rs} \\ \mu = j \quad \left(u^{(r)}_{\vec{k}=0}\right)^\dagger \gamma^0 \gamma^j u^{(s)}_{\vec{k}=0} = m \left((\xi^{(r)})^\dagger (\xi^{(r)})^\dagger\right) \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \begin{pmatrix} \xi^{(s)} \\ \xi^{(s)} \end{pmatrix} \end{cases}$$

$$= m \left(- (\xi^{(r)})^\dagger \sigma_j \xi^{(s)} + (\xi^{(r)})^\dagger \sigma_j \xi^{(s)} \right) = 0 .$$

$$\bar{u}^{(r)}_{\vec{k}=0} \gamma^\mu u^{(s)}_{\vec{k}=0} = \begin{cases} 2m & \mu=0 \\ 0 & \mu=j \end{cases} \Bigg\} \; \underset{\text{for } k^\mu = (m,0,0,0)}{2k^\mu}$$

$$\Rightarrow \quad \bar{u}^{(r)}_{\vec{k}} \gamma^\mu u^{(s)}_{\vec{k}} = 2 k^\mu \delta^{rs}$$

In particular

$$\boxed{\left(u^{(r)}_{\vec{k}}\right)^\dagger \left(u^{(s)}_{\vec{k}}\right) = 2\omega}$$

Summary of identities

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

$$\Lambda_{1/2}(\vec{k}) = \begin{pmatrix} e^{-\frac{\beta}{2}\hat{k}\sigma} & 0 \\ 0 & e^{\frac{\beta}{2}\hat{k}\sigma} \end{pmatrix} \qquad \begin{array}{l} ch\beta = \omega/m \\ sh\beta = \vec{k}/m \end{array}$$

$$u^{(r)}_{\vec{k}} = \Lambda_{1/2}\, u^{(r)}_0 \quad ; \quad u^{(r)}_0 = \sqrt{m}\begin{pmatrix} \xi^r \\ \xi^r \end{pmatrix} \qquad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$v^{(r)}_{\vec{k}} = \Lambda_{1/2}\, v^{(r)}_0 \quad ; \quad v^{(r)}_0 = \sqrt{m}\begin{pmatrix} \xi^r \\ -\xi^r \end{pmatrix} \qquad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$(\not{k} - m)\, u^{(r)}_{\vec{k}} = 0$	$(\not{k} + m)\, v^r_{\vec{k}} = 0$
$\bar{u}^{(r)}_{\vec{k}}\, u^s_{\vec{k}} = 2m\delta^{rs}$	$\bar{v}^{(r)}_{\vec{k}}\, v^s_{\vec{k}} = -2m\delta^{rs}$
$(u^r_{\vec{k}})^\dagger u^s_{\vec{k}} = 2\omega\delta^{rs}$	$(v^r_{\vec{k}})^\dagger v^s_{\vec{k}} = 2\omega\delta^{rs}$
$\sum_r u^r_{\vec{k}}\, \bar{u}^r_{\vec{k}} = \not{k} + m$	$\sum_r v^r_{\vec{k}}\, \bar{v}^r_{\vec{k}} = \not{k} - m$
$(u^r_{\vec{k}})^\dagger v^s_{-\vec{k}} = 0$	$(v^r_{\vec{k}})^\dagger u^s_{-\vec{k}} = 0$
$\bar{u}^r_{\vec{k}}\, v^s_{\vec{k}} = 0$	$\bar{v}^{(r)}_{\vec{k}}\, u^s_{\vec{k}} = 0$

Quantum operators

$$\psi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \sum_{r=1,2} \left(u_{\vec{k}}^{(r)} e^{-ikx} c_{\vec{k},r} + v_{\vec{k}}^{r} d^{\dagger}_{k,r} e^{ikx} \right)$$

$$\bar{\psi} = \psi^{\dagger}\gamma_0 = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} \sum_{r=1,2} \left(\bar{u}_{\vec{k}}^{(r)} e^{ikx} c^{\dagger}_{\vec{k},r} + \bar{v}_{\vec{k}}^{r} d_{k,r} e^{-ikx} \right)$$

$$H = \int d^3x \left(-i\bar{\psi}\gamma^i \partial_i \psi + m\bar{\psi}\psi \right) = \int d^3x \left(\underbrace{-i\bar{\psi}\,\partial\!\!\!/\,\psi + m\bar{\psi}\psi}_{0 \text{ by Dirac eqn.}} + i\bar{\psi}\gamma^0\partial_0\psi \right)$$

$$= i\int d^3x \; \psi^{\dagger}\dot{\psi}$$

$$= i\int d^3x \int \frac{d^3k\, d^3k'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k 2\omega_{k'}}} \sum_{\substack{r=1,2\\ s=1,2}} \left((u_{\vec{k}}^{(r)})^{\dagger} e^{ikx} c^{\dagger}_{\vec{k},r} + (v_{\vec{k}}^{(r)})^{\dagger} d_{k,r} e^{-ikx} \right)$$

$$\left(-i\omega' u^{(s)}_{k'} e^{-ik'x} c_{k',s} + i\omega' v^{s}_{k'} d^{\dagger}_{k',s} e^{ik'x} \right)$$

$$= \int \frac{d^3k}{\cancel{(2\pi)^3}} \frac{\cancel{(2\pi)^3}\,\omega}{2\omega_k} \sum_{r,s=1,2} \Big\{ \underbrace{(u_{\vec{k}}^{(r)})^{\dagger}\cdot(u_{\vec{k}}^{(s)})}_{2\omega\delta^{rs}} c^{\dagger}_{k,r} c_{k,s} -$$

$$- e^{2i\omega t} \underbrace{(u_{\vec{k}}^{(r)})^{\dagger}(v_{-k}^{(s)})}_{0} c^{\dagger}_{\vec{k},r} d^{\dagger}_{-k,s} + \underbrace{(v_{\vec{k}}^{(r)})^{\dagger}(u_{-\vec{k}}^{(s)})}_{0} e^{-2i\omega t} d_{k,r} c_{-k,s}$$

$$- \underbrace{(v_{\vec{k}}^{(r)})^{\dagger}(v_{\vec{k}}^{(s)})}_{2\omega\delta^{rs}} d_{k,r} d^{\dagger}_{k,s} \Big\} = \int d^3k\, \omega_k \left(c^{\dagger}_{k,r} c_{k,r} - d_{k,r} d^{\dagger}_{k,r} \right)$$

If they were bosons $d_{k,r} d^{\dagger}_{k,r} = d^{\dagger}_{k,r} d_{k,r}$ + constant,

$$\Rightarrow \quad H = \int d^3k \; \omega_k \, (n_c - n_d)$$

negative energy! (unbounded).

we need

$$\{ d_{k,r} \, , \, d_{k',s} \} = \delta_{rs} \, \delta^{(3)}(k-k')$$

$$H = \int d^3k \; \omega_k \, (c^{\dagger}_{k,r} c_{k,r} + d^{\dagger}_{k,r} d_{k,r}) - \int d^3k \, \omega_k \, \overbrace{2}^{\sum_r} \cdot \delta(0)$$

$$\delta(k) = \int \frac{d^3x}{(2\pi)^3} e^{ikx} \qquad \delta(0) = V/(2\pi)^3$$

$$H = \int d^3k \; \omega_k \sum_r (\hat{n}_{k,c} + \hat{n}_{k,d}) - \frac{2V}{(2\pi)^3} \int d^3k \, \omega_k$$

< 0 & equivalent to 4 bosons

because we have

four particles $e^- \uparrow \downarrow$
$e^+ \uparrow \downarrow$

We could cancel zero point energy between bosons and fermions.

Consider now

$$\{\psi(x), \bar{\psi}(y)\}_{E.T.} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \sum_r \Big(u_k^r \bar{u}_k^r \, e^{ik(y-x)} + v_k^r \bar{v}_k^r \, e^{-ik(y-x)} \Big)$$

equal time

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \Big((\not{k}+m) \, e^{ik(y-x)} + (\not{k}-m) \, e^{-ik(y-x)} \Big)$$

$e^{-i\vec{k}(\vec{y}-\vec{x})}$ $\qquad$ $e^{i\vec{k}(\vec{y}-\vec{x})}$

$t_y = t_x$, $\qquad$ $\vec{k} \to -\vec{k}$,

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{-i\vec{k}(\vec{y}-\vec{x})} \Big(k_0\gamma^0 + \cancel{k_j\gamma^j} + \cancel{m} + k_0\gamma^0 - \cancel{k_j\gamma^j} - \cancel{m} \Big)$$

$2\omega\gamma^0$ $\qquad$ $k \to -k$

$$= \gamma^0 \, \delta^{(3)}(\vec{y}-\vec{x})$$

multiplying by γ^0 $\qquad$ $(\gamma^0)^2 = \mathbb{1}$.

$$\{\psi_a(x), \psi_b^\dagger(y)\}_{E.T.} = \delta(\vec{y}-\vec{x}) \, \delta_{ab}$$

Feynman propagator

$$\langle 0|\hat{T}\{\psi_a(x)\bar{\psi}_b(y)\}|0\rangle = \begin{cases} \langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle & x^0>y^0 \\ \text{Notice} - \text{sign} \\ -\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle & x^0<y^0 \end{cases}$$

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\sum_{r=1,2}\left(u_{\vec{k}}^{(r)}\right)_a\left(\bar{u}_{\vec{k}}^{(r)}\right)_b e^{ik(y-x)}$$

$$= \int\frac{d^3k}{(2\pi)^3}(\not{k}+m)_{ab}\,\cancel{\int\frac{d^3k}{(2\pi)^3}}\frac{1}{2\omega_k}e^{-ik(x-y)}$$

$$-\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = -\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\underbrace{\left(\bar{v}_{\vec{k}}^{(r)}\right)_b\left(v_{\vec{k}}^{(r)}\right)_a}_{(\not{k}-m)_{ab}} e^{-iky+ikx}$$

$$= \int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}(-\not{k}+m)_{ab}\,e^{ik(x-y)}$$

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = (i\not{\partial}_x+m)_{ab}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik(x-y)}$$

$$-\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = (i\not{\partial}_x+m)_{ab}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{ik(x-y)}$$

$\Delta_F(x-y)$ scalar.

$$\langle 0|\hat{T}\{\psi_a(x)\bar{\psi}_b(y)\}|0\rangle = (i\not{\partial}_x+m)_{ab}\,\Delta_F(x-y)$$

$$G_{F\,ab} = (i\not{\partial}_x + m)\int \frac{d^4k}{(2\pi)^4}\,\frac{i\,e^{-ik(x-y)}}{k^2-m^2+i\varepsilon}$$

$$= i\int \frac{d^4k}{(2\pi)^3}\,\frac{(+\not{k}+m)}{k^2-m^2+i\varepsilon}\,e^{-ik(x-y)}$$

$$G_F(k) = \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}$$

$$G_F(k) = \frac{i}{\not{k}-m}$$ understanding

Because $(\not{k}-m)(\not{k}+m) = k^2-m^2$