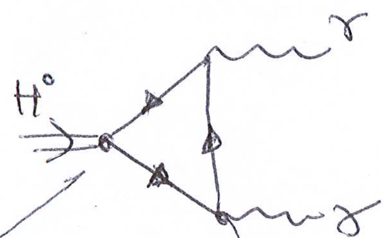
Perturbation theory

a) Compute probabilities of different physical processes.

- Mean life of particles
- cross sections.

Example $H^0 \longrightarrow \gamma + \gamma$

one diagram



$$\lambda \int d^4x \, \underbrace{H^0 \bar{\psi} \psi}_{\dim = 4} \qquad e \int d^4x \, \underbrace{A_\mu \bar{\psi} \gamma^\mu \psi}_{\dim = 4}$$

effective vertex $\qquad \lambda_{eff} \int d^4x \, \underbrace{H^0(x) F_{\mu\nu} F^{\mu\nu}}_{\dim 5}$

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a) From a theoretical perspective it is useful to compute Green functions:

$$\langle 0 | \hat{T} \{ \phi_{a_1}(x_1) \, - - \, \phi_{a_n}(x_n) \} | 0 \rangle$$

Schrödinger picture: $\mathcal{O}_S$; $|\psi(t)\rangle_S = e^{-iHt}|\psi(0)\rangle_S$

or $\partial_t|\psi(t)\rangle_S = -iH(t)|\psi(t)\rangle_S$

can be time-dep.

$|\psi(t)\rangle_S = U(t)|\psi(0)\rangle_S$; for H time indep. $U(t) = e^{-iHt}$

otherwise solve $\begin{cases} \partial_t U(t) = -iH(t)U(t) \\ U(0) = \mathbb{1} \end{cases}$

Heisenberg picture:

$$\langle\psi(t)|\mathcal{O}_S|\psi(t)\rangle = \langle\psi(0)|U^\dagger(t)\mathcal{O}_S U(t)|\psi(0)\rangle$$

$$\boxed{\mathcal{O}_H = U^\dagger(t)\mathcal{O}_S U(t)} \qquad \boxed{U^\dagger(t) = U^{-1}(t)}$$

Interaction picture

$$H(t) = H_0 + \lambda\hat{V}(t) \; ; \quad \boxed{\partial_t U(t) = -iH(t)U(t)}$$

λ: perturbation parameter.

$$U(t) = e^{-iH_0t}\underbrace{U_I(t)}_{=\mathbb{1} \text{ if } \lambda=0 \text{ (no perturbation)}}$$

$$U_I(t) = e^{iH_0t}U(t)$$

$$\partial_t U_I(t) = \cancel{e^{iH_0t}iH_0U(t)} + e^{iH_0t}(\cancel{-iH_0} - i\lambda\hat{V}(t))U(t)$$

$$= -i\lambda\underbrace{e^{iH_0t}\hat{V}(t)e^{-iH_0t}}_{\hat{V}_I(t)}U_I(t)$$

$$\partial_t U_I(t) = -i\lambda \hat{V}_I(t)\, U_I(t)$$

↘ if V_s is a function of fields $\phi_s(x)$ then V_I is the same function replacing $\phi_s \to \phi_I$

$$\phi_I(x) = e^{iH_0 t}\phi_s(\vec{x})e^{-iH_0 t}$$

(same as Heisenberg for H_0)

$$\phi_H = U^{-1}\phi_s U = U_I^{-1} e^{iH_0 t}\phi_s e^{-iH_0 t} U_I$$

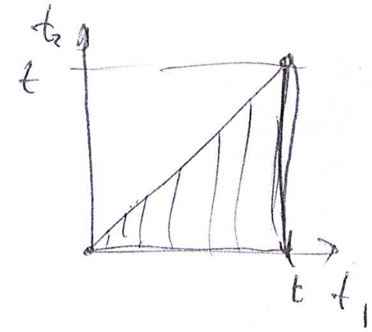
$$\boxed{= U_I^{-1}(t)\phi_I(t) U_I(t)}$$

<u>Perturbative solution</u>

$$U_I(t) = \sum_{j=0}^{\infty} \lambda^j U_I^{(j)}(t) \quad ; \quad \boxed{U_I^{(0)}(t) = \mathbb{1}}$$

$$\partial_t U_I^{(j)}(t) = -i\hat{V}_I(t)\, U_I^{(j-1)}(t)$$

$$\boxed{U_I^{(j)}(t) = -i\int_0^t \hat{V}_I(t)\, U_I^{(j-1)}(t)}$$



$$U_I^{(1)}(t) = -i\int_0^t \hat{V}_I(t_1)\,dt_1$$

$$U_I^{(2)}(t) = (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2\, \hat{V}_I(t_1)\hat{V}_I(t_2)\, \cancel{\mathbb{1}}$$

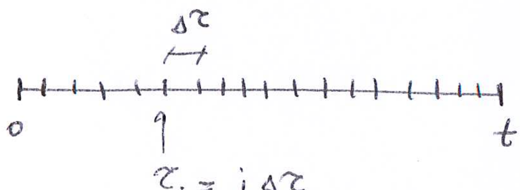
$$U_I^{(n)}(t) = (-i)^n \int_0^t \int_0^{t_1} \int_0^{t_2} \cdots \int_0^{t_{n-1}} dt_1 \ldots dt_n\, \hat{T}\{V(t_1) \ldots V(t_n)\}$$

$$= \frac{(-i)^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n\, \hat{T}\{V(t_1) \ldots V(t_n)\}$$

$$U_I^{(n)} = \frac{(-i)^n}{n!} \hat{T}\left\{\left(\int_0^t V(t')dt'\right)^n\right\}$$

$$U_I(t) = \sum_{j=0}^{\infty} \frac{(-i\lambda)^n}{n!} \hat{T}\left\{\left(\int_0^t V(t')dt'\right)^n\right\}$$

$$= \hat{T}\left\{e^{-i\int_0^t \lambda V(t')dt'}\right\}$$



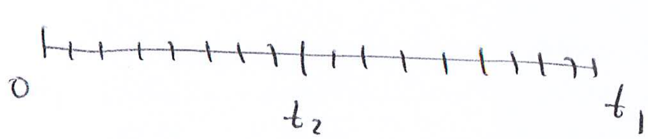
$$\hat{T}\left\{e^{-i\int_0^t \lambda V(t')dt'}\right\} = e^{-i\lambda V(t)\Delta\tau} \cdots\cdots e^{-i\lambda V(j\Delta\tau)\Delta\tau} \cdots\cdots e^{-i\lambda V(0)\Delta\tau}$$

$$U_I(t_1)\, U_I^{-1}(t_2) = e^{-i\lambda V(t_1)\Delta\tau} \cdots\cdots e^{-i\lambda V(0)\Delta\tau}\; e^{i\lambda V(0)\Delta\tau} \cdots$$

$t_2 < t_1$

$$\cdots e^{i\lambda V(t_2)\Delta\tau}$$

$$= e^{-i\lambda V(t_1)\Delta\tau} \cdots\cdots e^{-i\lambda V(t_2)\Delta\tau}$$



$$= \hat{T}\left\{e^{-i\int_{t_1}^{t_2} V(t')dt'}\right\}$$

$$U_I^{-1}(t_1) = U_t(t_2 \to 0)$$

Green's functions

ground state of full H.

$${}_H\langle 0| \hat{T}\{\phi_H(x_1) - - \phi_H(x_n)\} |0\rangle_H$$

Suppose $t_1 > t_2 > - - - > t_n$.

$$= \langle 0| \phi_H(x_1) - - \phi_H(x_n) |0\rangle =$$

$$= \langle 0| U_I^{-1}(t_1)\, \phi_I(t_1)\, U_I(t_1)\, U_I^{-1}(t_2)\, \phi_I(t_2)\, U_I(t_2) - - U_I^{-1}(t_n)\, \phi_I(t_n)\, U_I(t_n) |0\rangle$$

$$= \langle 0| U_I^{-1}(T)\, U_I(T)\, U_I^{-1}(t_1) - - - - U_I(t_n)\, U_I^{-1}(-T)\, U_I(-T) |0\rangle$$

$$= \langle 0| U_I^{-1}(T)\, \hat{T} e^{-i\int_{t_1}^{T} V(t)dt}\, \phi_I(t_1)\, \hat{T} e^{-i\int_{t_2}^{t_1} V(t)dt} - - \phi_I(t_n)\, \hat{T} e^{-i\int_{-T}^{t_n} V(t)dt}\, U_I(-T)|0\rangle$$

$$= {}_H\langle 0| U_I^{-1}(T)\, \hat{T}\left\{ e^{-i\int_{-T}^{T} V(t)dt}\, \phi_I(x_1) - - \phi_I(x_n) \right\} U_I(-T) |0\rangle_H$$

$$|\psi_S(t)\rangle = U(t)|\psi(0)\rangle = U(t)|\psi_H\rangle = e^{-iH_0 t}\, U_I(t)|\psi_H\rangle$$

"empty state"

$$U_I(-T)|\psi_H\rangle = e^{iH_0(-T)}\, |\psi_S(-T)\rangle$$

$-T \to -\infty$ we can adiabatically switch off the interactions $\to$ $|\psi_S(t)\rangle \to |0\rangle_{H_0}$ eigenstates of H_0 $t \to -\infty$

equivalently $|\psi_S(-T)\rangle = e^{iHT}|\psi(0)\rangle_H = \sum_n e^{iE_nT}|E_n\rangle\langle E_n|0\rangle_H$

$T + i\varepsilon T$ $\qquad e^{iE_nT}\, e^{-\varepsilon E_n T} \to 0$ $T \to \infty$ except $|0\rangle$ vacuum

One approach

adiabatically switch off interactions for $t \to \pm\infty$.

$$|\psi_s(-\infty)\rangle = |0_{H_0}\rangle \qquad \text{Define this (namely only phase).}$$

$$|\psi_s(+\infty)\rangle = U(+\infty,-\infty)|0_{H_0}\rangle \approx e^{i\alpha}|0_{H_0}\rangle$$

but with a phase.

$$\langle\psi_s(-\infty)|\psi_s(+\infty)\rangle = \langle 0_{H_0}|U(+\infty,-\infty)|0_{H_0}\rangle = e^{i\alpha}$$

$$= \langle 0_{H_0}| e^{-iH_0 t} U_I(+\infty,-\infty)|0_{H_0}\rangle$$

$2\times\infty$

$E_0 = 0$

$$\Rightarrow e^{i\alpha} = \langle\psi_s(-\infty)|\psi_s(+\infty)\rangle = \langle 0_{H_0}|\hat{T}\left\{e^{-i\int_{-\infty}^{\infty}\hat{V}_I(t)dt}\right\}|0_{H_0}\rangle$$

Going back to Green's function.

$$U_I(-T)|0\rangle_H \underset{T\to\infty}{\to} |\psi_s(-\infty)\rangle \qquad \left(e^{-iH_0 T}U(-T)|0\rangle\right)$$

0

$${}_H\langle 0|U_I^{-1}(T) = \left(U_I(T)|0\rangle_H\right)^{+} \underset{T\to\infty}{\Rightarrow} \langle\psi_s(+\infty)|$$

$$\langle\psi_s(+\infty)|\hat{T}\left\{\phi(x_1)\ldots\phi(x_n)\,e^{-i\int_{-\infty}^{\infty}V(t)dt}\right\}\underbrace{|\psi_s(-\infty)\rangle}_{|0_{H_0}\rangle}$$

$$e^{-i\alpha}\langle 0_{H_0}|\hat{T}\left\{\phi(x_1)-\phi(x_n)\,e^{-i\int_{-\infty}^{\infty}V(t)dt}\right\}|0_{H_0}\rangle$$

$$\Rightarrow \quad \langle 0_H | \hat{T}\{\phi(x_1)\cdots\phi(x_n)\} | 0\rangle = \frac{\langle 0_{H_0}| \hat T\{\phi(x_1)\cdots\phi(x_n)\, e^{-i\int_{-\infty}^{\infty}\hat V_I(t)\,dt}\}|0_{H_0}\rangle}{\langle 0_{H_0}|\hat T\{e^{-i\int_{-\infty}^{\infty} \hat V_I(t)\,dt}\}|0_{H_0}\rangle}$$

(*)

Alternative approach.

$H|0_H\rangle = \tilde E_0|0_H\rangle$

$$U_I(-T)|0\rangle_H = e^{-iH_0T}\overbrace{U(-T)}^{e^{iHT}}|0\rangle_H = e^{i\tilde E_0 T}\sum_{(n)} e^{-iE_n T}|E_n\rangle\langle E_n|0_H\rangle$$

(n) eigenstates of H_0

$T \to \infty(1-i\varepsilon)$ only $E_0=0$ survives.

$H_0|0_H\rangle = 0$

$$U_I(-T)|0\rangle_H = e^{i\tilde E_0 T}|0_{H_0}\rangle\langle 0_{H_0}|0_H\rangle$$

$$\langle 0_H|U_I^{-1}(T) = \langle 0_H|\underbrace{U^{-1}(T)}_{e^{iHT}}\, e^{-iH_0T} \xrightarrow[T\to\infty(1-i\varepsilon)]{} e^{i\tilde E_0 T}\langle 0_H|0_{H_0}\rangle\langle 0_{H_0}|$$

$$\langle 0_H|\hat T\{\phi(x_1)\cdots\phi(x_n)\}|0_H\rangle = e^{2i\tilde E_0 T}|\langle 0_H|0_{H_0}\rangle|^2\langle 0_{H_0}|\hat T\{\phi(x_1)\cdots\phi(x_n)\,e^{-i\int V dt}\}|0\rangle$$

$$\langle 0_H|0_H\rangle = 1 = e^{2i\tilde E_0 T}|\langle 0_H|0_{H_0}\rangle|^2\langle 0_{H_0}|\hat T\{e^{-i\int_{-\infty}^{\infty} V dt}\}|0\rangle$$

Dividing we get same result above (*)

H_0: single particle eigenstates.

asymptotic states at $t \to \pm\infty$ are eigenstates of H_0.

$$|i\rangle = |p_1, p_2, \ldots p_n\rangle$$

$$|f\rangle = |p_1', p_2', \ldots p_m'\rangle$$

$$\langle p_1', p_2', \ldots p_m'| e^{-2iH_0T} U_I(+T, -T) |p_1 \ldots p_n\rangle$$

$$\underbrace{e^{-2iE_f T}}_{\text{phase}} \quad \langle 0| a_{p_1'} \ldots a_{p_m'} T\{e^{-i\int_{-\infty}^{\infty} V(t)dt}\} a^\dagger_{p_1} \ldots a^\dagger_{p_n}|0\rangle$$

$$V(t) = \int d^3x \, \mathcal{V}(\vec{x}, t)$$

$$T\{e^{-i\int_{-\infty}^{\infty} d^4x \, \mathcal{V}(\vec{x}, t)}\}$$

Lorentz inv. if $\mathcal{V}$ is a scalar

and commutes at space-like separation.

Example ϕ^4 theory.

$$V = \frac{\lambda}{4!}\phi^4$$

$$\langle 0 | T\{\phi_I(x)\phi_I(y)\, e^{-i\int_{-\infty}^{\infty} dt\, d^3x \frac{\lambda}{4!}\phi_I^4(x)}\} | 0\rangle$$

2-point function

$$= \langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle - i\frac{\lambda}{4!} \langle 0| \int d^4z \ \langle 0|T\{\phi_I(x)\phi_I(y)\phi_I^4(z)\}|0\rangle$$

$$+\left(\frac{i\lambda}{4!}\right)^2 \frac{1}{2} \int d^4z_1 d^4z_2 \ \langle 0|T\{\phi_I(x)\phi_I(y)\,\phi_I^4(z_1)\,\phi_I^4(z_2)\}|0\rangle \cdots$$

$$- - -$$

We need

$$\langle 0|T\{\phi_I(x_1) \cdots \phi_I(x_n)\}|0\rangle \text{ for free fields.}$$

Wick's theorem.

$$= \sum_{\substack{\text{all possible}\\ \text{contractions}\\ \text{or pairs}}} D_F(x_{i_1} - x_{i_1'})\, D_F(x_{i_2} - x_{i_2'}) \cdots D_F(x_{i_{n/2}} - x_{i_{n/2}'})$$

the main reason is that commuting two fields we get a number.

Normal order:

creation to the left annihilation to the right.

$$\phi = \underset{a^{+}}{\phi^{(+)}} + \underset{a}{\phi^{(-)}}$$

$T\{\phi(x_1) \dots \phi(x_n)\}$ assume $t_1 > t_2 > \dots > t_n$

$$\langle 0| \phi(x_1) \dots \phi(x_n) |0\rangle = \langle 0| (\phi_1^{+} + \phi_1^{-}) \dots (\phi_n^{+} + \phi_n^{-}) |0\rangle$$

same # of ϕ^{+} and ϕ^{-} $\Rightarrow$ even #. of fields.

$$(\phi_{n-1}^{+} + \phi_{n-1}^{-})\phi_n^{+} = \phi_{n-1}^{+}\phi_n^{+} + \phi_{n-1}^{-}\phi_n^{+} =$$

$$= \phi_{n-1}^{+}\phi_n^{+} + [\phi_{n-1}^{-}, \phi_n^{+}] + \phi_n^{+}\phi_{n-1}^{-}$$

$|0\rangle$

$$(\phi_{n-2}^{+} + \phi_{n-2}^{-})\, \phi_{n-1}^{+}\, \phi_n^{+} |0\rangle + |0\rangle [\phi_{n-1}^{-}, \phi_n^{+}]$$

$$\phi_{n-2}^{+}\phi_{n-1}^{+}\phi_n^{+} + [\phi_{n-2}^{-}, \phi_{n-1}^{+}]\, \phi_n^{+}|0\rangle + \phi_{n-1}^{+}\phi_{n-2}^{-}\phi_n^{+}|0\rangle$$

$$+ \phi_{n-1}^{+}|0\rangle + |0\rangle [\phi_{n-2}^{-}, \phi_n^{+}] + |0\rangle [\phi_{n-1}^{-}, \phi_n^{+}]$$

$\langle 0 | \phi$

$$\langle 0 | \phi(x_1) \ldots \phi(x_n) | 0 \rangle = \sum_{\substack{\text{all possible} \\ \text{pairs}}} [\phi^-, \phi^+] \ldots [\phi^-, \phi^+]$$

$$[\phi_1^-, \phi_2^+] \overset{\text{because it's a number}}{=} \langle 0 | \phi_1^- \phi_2^+ - \phi_2^+ \phi_1^- | 0 \rangle = \langle 0 | \phi_1^- \phi_2^+ | 0 \rangle =$$

$$= \langle 0 | \phi_1(x_1) \phi_1(x_2) | 0 \rangle$$

$$\langle 0 | \hat{T} \{ \phi(x_1) \ldots \phi(x_n) \} | 0 \rangle = \sum_{\substack{\text{all} \\ \text{possible} \\ \text{pairs}}} \underbrace{\langle 0 | \hat{T} \{ \phi(x_{i_1}) \phi(x_{i_1}) \} | 0 \rangle}_{D_F} \ldots$$

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example

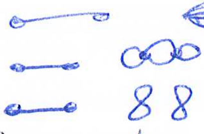
$$-i \frac{\lambda}{4!} \int d^4 z \quad \langle 0 | T \{ \phi(x) \phi(y) \phi(z) \phi(z) \phi(z) \phi(z) \} | 0 \rangle$$

3

4×3

$$-\frac{i\lambda}{4!} \int d^4 z \Big(3 D_F(x-y) D_F^2(0) +$$

$$+ 12 D_F(x-z) D_F(y-z) D_F(0) \Big)$$



$S = 2\times 4!$ $\quad$ $\cancel{4\times4\times3\times3\times2\times}\,4!$

$S = 16$ $\quad$ $\cancel{4\times4\times3\times3}$ $\quad 6\times6\times2$

$S = 2\times8\times8$ $\quad$ 8 8 $\quad 3\times3$

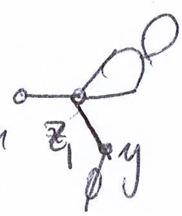
$$\left(-\frac{i\lambda}{4!}\right)^2 \frac{1}{2}\int d^4z_1 \int d^4z_2 \,\langle 0|T\{\phi(x)\phi(y)\,\phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)\,\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2)$$

$$\left(-\frac{i\lambda}{4!}\right)^2 \frac{1}{2}\int d^4z_1 d^4z_2 \quad D_F(x-z_1)\, D_F(y-z_1)\, D_F^2(z_1-z_2)\, D_F(0)$$

$$D_F(x-z_1)\, D_F(y-z_2)\, D_F^2(z_1-z_2)\, D_F(0)$$

$\frac{1}{8}$

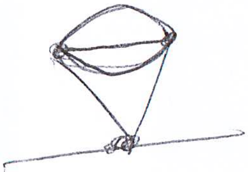
$\frac{1}{2}$



$\frac{1}{4}$

1/6

1/4



$\frac{1}{2}\times\frac{1}{6} = \frac{1}{12}$

3 loops

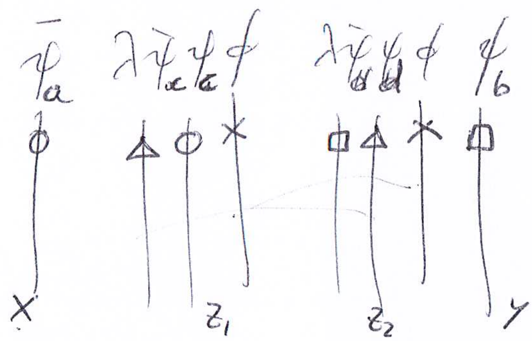
$\phi(x)\phi(y) \quad \phi\phi\phi\phi \quad \phi\phi\phi\phi \quad \phi\phi\phi\phi$

$$\frac{1}{3!}\left(\frac{1}{4!}\right)^3 \, 3\times4\times3\times2\times4\times4\times6 = \frac{1}{123}\,\frac{4!\,4!\,\times 12^2}{4!\,4!\,4!} = \frac{1}{12} \checkmark$$

vertex

$$\frac{1}{12}(-i\lambda)^3 \int d^4z_1 d^4z_2\, d^4z_3 \quad \Delta(x-z_3)\,\Delta(y-z_3)\,\Delta(z_3-z_1)\,\Delta(z_3-z_2)\,\big(\Delta(z_1-z_2)\big)^3$$

$\lambda \bar{\psi} \psi \phi$



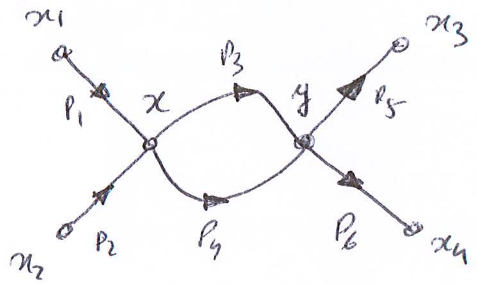
$D_F(x-y)\ S_{ac}$

$$D_F(z_1-z_2)\quad S_{ca}(x-z_1)\ S_{dc}(z_1-z_2)\ S_{bd}(z_2-y)$$

$$D_F(z_1-z_2)\quad S_{bd}\, S_{dc}\, S_{ca}$$

$$\int d^4z_1\, d^4z_2\ D_F(z_1-z_2)\ S_{bd}(z_2-y)\ S_{dc}(z_1-z_2)\ S_{ca}(x-z_1)$$

Fourier transform to momentum space.



$$= \int \frac{d^4p}{(2\pi)^4}\, i\, \frac{e^{ip(y-x)}}{p^2-m^2+i\epsilon}$$

$$= \int \frac{d^4p}{(2\pi)^4}\, \Delta(p)\, e^{ip(y-x)}$$

$$(-i\lambda)^2 \int d^4x\, d^4y\; D_F(x-x_1)\, D_F(x-x_2)\, \big(D_F(y-x)\big)^2\, D_F(x_3-y)\, D_F(x_4-y)$$

$$= \int d^4x\, d^4y \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_6}{(2\pi)^4}\; e^{ip_1(x-x_1)+ip_2(x-x_2)+ip_3(y-x)+ip_4(y-x)+ip_5(x_3-y)+ip_6(x_4-y)}$$

$$\cdot\, \Delta(p_1) \cdots \Delta(p_6)$$

$$= \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_6}{(2\pi)^4}\; \Delta(p_1) \cdots \Delta(p_6)\; e^{-ip_1x_1-ip_2x_2+ip_5x_3+ip_6x_4}$$

$$\cdot\, (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)\, (2\pi)^4 \delta^{(4)}(p_3+p_4-p_5-p_6)$$

$p_4 = p_1+p_2-p_3$ $\qquad$ $p_1+p_2-p_5-p_6$

$$= \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_5}{(2\pi)^4} \int \frac{d^4p_6}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p_1+p_2-p_5-p_6)\; \Delta(p_1)\Delta(p_2)\Delta(p_3)\,\cdot$$

$$\Delta(p_1+p_2-p_3)\, \Delta(p_5)\, \Delta(p_6)$$

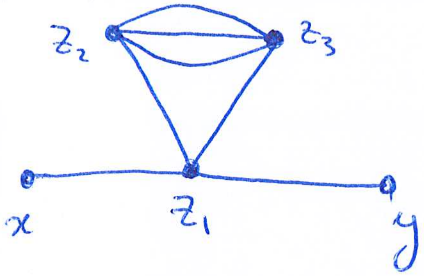
$$\cdot\, e^{-ip_1x_1-ip_2x_2+ip_5x_3+ip_6x_4}$$

e^{-ipx}

$(-i\lambda)$

$$\frac{i}{p^2-m^2+i\epsilon}$$

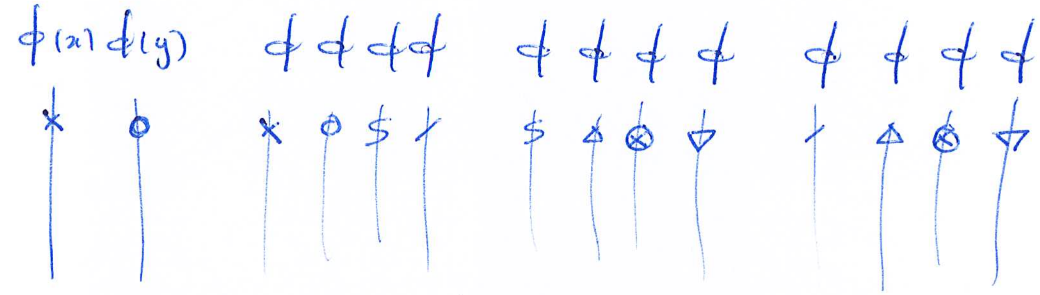
Divide by symmetry factor.



$$\frac{(-i\lambda)^3}{12}\int d^4z_1\, d^4z_2\, d^4z_3\ \Delta_F(x-z_1)\,\Delta_F(y-z_1)\ \times$$

$$\times\ \Delta_F(z_2-z_1)\,\Delta_F(z_3-z_1)\left[\Delta_F(z_2-z_3)\right]^3$$

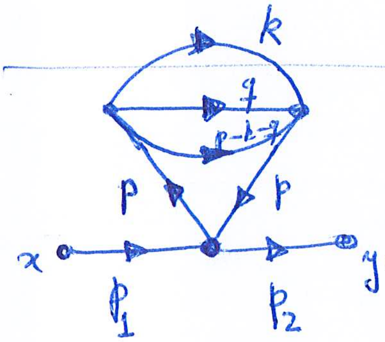
$S = 2\times 3! = 12$



$$\overset{V^3}{\frac{1}{3!}}\ \frac{(-i\lambda)^3}{(4!)^3}$$

$$3\times 4\times \underbrace{3\times 4}_{4!}\times \underbrace{4\times 6}_{4!}\times 2$$

$$\frac{3\times 4\times 4!\times 4!}{6\times 4!\times 4!\times 4!} = \frac{\cancel{3}\times\cancel{4}}{12\times 3\times 1\times 2\times\cancel{3}\times\cancel{4}} = \frac{1}{12}\ \checkmark$$



$$\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\ e^{-ip_1x+ip_2y}\ (2\pi)^4\,\delta^{(4)}(p_2-p_1)$$

$$\cdot\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}\left(\frac{i}{p^2-m^2+i\epsilon}\right)^2\frac{i}{k^2-m^2+i\epsilon}\ \frac{i}{q^2-m^2+i\epsilon}\ \frac{i}{(p-k-q)^2-m^2+i\epsilon}$$

3 loops

N_E : number of external vertices.

N_V : " " internal "

N_P : " " propagators

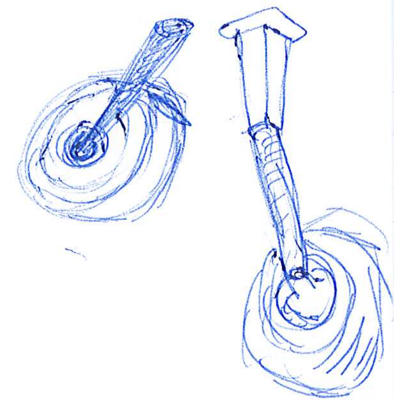
ϕ^4 theory:

$$N_P = \frac{4N_V - N_E}{2} + N_E = 2N_V + \frac{1}{2}N_E$$

$N_{\delta\text{-functions}}: \; N_V - 1$ ← overall

$$N_L = \underbrace{\frac{4N_V - N_E}{2}}_{\text{internal prop}} - N_V + 1 = N_V - \frac{N_E}{2} + 1$$

$\frac{1}{\hbar}(-\partial^2 + m^2)\, G = \delta$

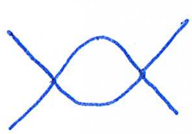


$$\frac{1}{\hbar}\left(\frac{1}{2}\partial\phi\partial\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\right)$$

$S/\hbar \to \qquad \Delta \to \hbar \qquad V \to \hbar^{-1}$

$$\hbar^{+N_P - N_V} = \hbar^{-N_V + \left(\frac{4N_V - N_E}{2}\right)} = \hbar^{+N_V - N_E/2} = \hbar^{+N_L - 1}$$

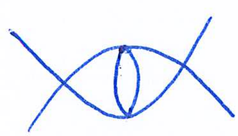
ignoring external legs $\hbar^{N_L - 1}$; $\hbar$ loop counting parameter.



$N_V = 2 \quad N_E = 4$

$N_L = 2 - 2 + 1$

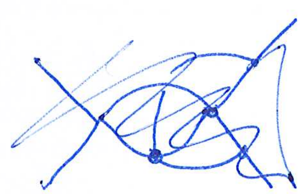
$= 1$ ✓



$N_V = 4 \quad N_E = 4$

$N_L = 4 - 2 + 1$

$= 3$



$N_V =$

 ; 

ϕ^3 $\qquad$ ϕ^4

N_3 ; $\qquad$ N_4

$$N_p^I = \frac{3N_3 + 4N_4 - N_E}{2} \qquad N_p^E = N_E$$

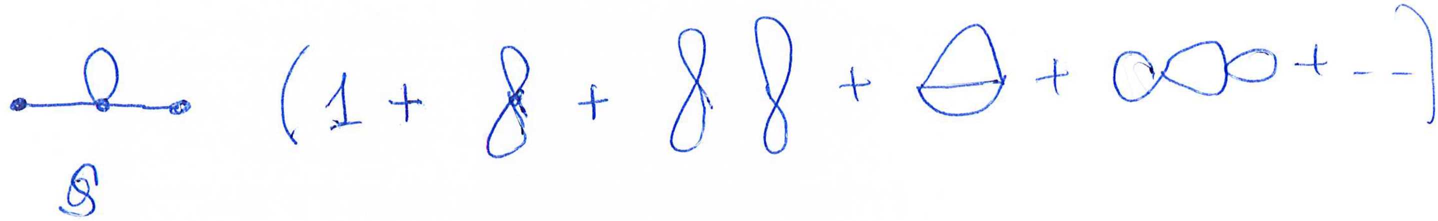
$$N_L = \frac{3N_3 + 4N_4 - N_E}{2} - (\overbrace{N_3 + N_4 - 1}^{\#\text{ of }\delta}) = N_p^I - N_V + 1$$

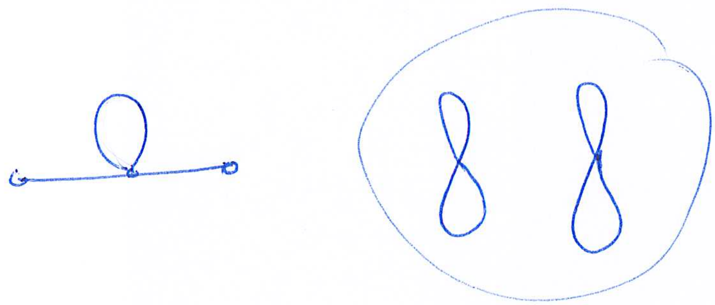
$$= \frac{3}{2}N_3 + 2N_4 - \frac{1}{2}N_E - N_3 - N_4 + 1$$

$$\boxed{N_L = \frac{1}{2}N_3 + N_4 - \frac{1}{2}N_E + 1}$$

h_1 : $N_p^I - N_V = N_L - 1$

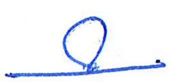
Vacuum ~~bubbles~~ diagrams.





S-factor = product.

$$S = S_1 \times S_2$$

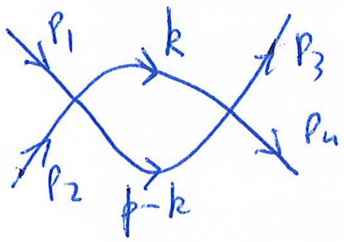


$$\times \quad \langle 0 | T \left\{ e^{-i\int V(t)\,dt} \right\} | 0 \rangle$$

$$\langle 0_H | \hat{T} \{ \phi_H(x_1) \ldots \phi_H(x_n) \} | 0 \rangle = \langle 0 | T \left\{ \phi(x_1) \ldots \phi(x_n)\, e^{-i\int_{-\infty}^{\infty} V_I(t)\,dt} \right\} | 0 \rangle$$

No vacuum ~~bubbles~~ diagrams

Removing external legs and overall $\delta^{(4)}$



$p = p_1 + p_2$

$$\frac{1}{2}(-i\lambda)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2 - m^2 + i\varepsilon} \frac{i}{(p-k)^2 - m^2 + i\varepsilon}$$

Schwinger parameters
$$\frac{i}{p^2 - m^2 + i\varepsilon} = \int_0^\infty d\alpha\, e^{i\alpha(p^2 - m^2 + i\varepsilon)}$$

makes integral converge.

$$A = \int \frac{dk}{2\pi} e^{i\alpha k^2} = \frac{e^{i\pi/4}}{\sqrt{4\pi\alpha}} \quad ; \quad \int \frac{d^dk}{(2\pi)^d} e^{iak^2 + ibk^2} = \frac{e^{i\frac{\pi}{4}(2-d)}}{(4\pi\alpha)^{d/2}} e^{-ib^2/4a}$$

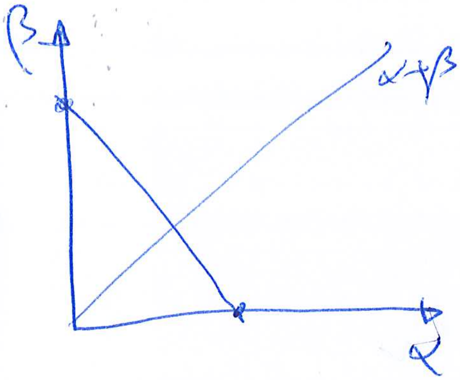
$$\frac{1}{2}(-i\lambda)^2 \int_0^\infty d\alpha \int_0^\infty d\beta \int \frac{d^dk}{(2\pi)^d} e^{i\alpha(k^2 - m^2 + i\varepsilon) + i\beta((p-k)^2 - m^2 + i\varepsilon)}$$

$$i[(\alpha+\beta)k^2 - 2\beta pk + \beta p^2 - (\alpha+\beta)m^2 + i(\alpha+\beta)i\varepsilon]$$

$$= \frac{1}{2}(-i\lambda)^2 \int_0^\infty d\alpha\, d\beta \frac{e^{i\frac{\pi}{4}(2-d)}}{(4\pi(\alpha+\beta))^{d/2}} e^{-i\frac{\cancel{4}\beta^2 p^2}{\cancel{4}(\alpha+\beta)} + i\beta p^2 - i(\alpha+\beta)m^2 - \varepsilon(\alpha+\beta)}$$

$$\frac{i(\alpha\beta)p^2}{\alpha+\beta} - i(\alpha+\beta)m^2 - \varepsilon(\alpha+\beta)$$

(16)



$\mu = \alpha+\beta : 0 \to \infty$

$\lambda = \alpha - \beta : -\mu \to \mu$

$$\frac{\partial(\mu,\lambda)}{\partial(\alpha\,\beta)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \qquad |\,| = 2.$$

$$\frac{1}{2}(-i\lambda)^2 \frac{1}{2} \int_0^\infty d\mu \int_{-\mu}^{\mu} d\lambda \, \frac{e^{i\frac{\pi}{4}(2-d)}}{(4\pi\mu)^{d/2}} \, e^{i\frac{\alpha\beta}{\mu}p^2 - i\mu m^2 - \varepsilon\mu}$$

$\alpha = \frac{\mu+\lambda}{2} \qquad \beta = \frac{\mu-\lambda}{2}$

$\lambda = (2\xi - 1)\mu \qquad \xi : 0 \to 1$

$\alpha\beta = \frac{\mu^2-\lambda^2}{4} = \frac{\mu^2 - (2\xi-1)^2\mu^2}{4}$

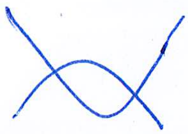
$$\frac{1}{2}(-i\lambda)^2 \int_0^\infty d\mu \int_0^1 d\xi \; \mu^{1-d/2} \frac{e^{i\frac{\pi}{4}(2-d)}}{(4\pi)^{d/2}} e^{i\mu\xi(1-\xi)p^2 - i\mu m^2 - \varepsilon\mu}$$

$1-(2\xi-1)^2 = \cancel{1} - 4\xi^2 + 4\xi - \cancel{1} = 4\xi(1-\xi)$

$= (i)^{(1-d/2)} \quad \cancel{= ((-i)^{1-d/2}}$

$$= \frac{1}{2}(-i\lambda)^2 \int_0^1 d\xi \;\; \Gamma(2-d/2) \frac{e^{i\frac{\pi}{4}(2-d)}}{(4\pi)^{d/2}} \left(\varepsilon + im^2 - i\xi(1-\xi)p^2\right)^{-2+d/2}$$

$$\frac{(-i)}{(-i)^{2-d/2}} \frac{1}{(\varepsilon + im^2 - i\xi(1-\xi)p^2)^{2-d/2}} = \frac{-i}{(m^2 - \xi(1-\xi)p^2 - i\varepsilon)^{2-d/2}}$$

 $= \frac{1}{2}(-i\lambda)^2 \frac{(-i)\,\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 d\xi \, \frac{1}{(m^2 - \xi(1-\xi)p^2 - i\varepsilon)^{2-d/2}}$

—— o ——

Feynman parameters.

$$\frac{1}{AB} = \int_0^1 d\alpha \, \frac{1}{(\alpha A + (1-\alpha)B)^2}$$

$$\frac{1}{2}(-i\lambda)^2 \underbrace{(-1)}_{i^2} \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[\alpha(p-k)^2 \cancel{-\alpha m^2 + \alpha i\varepsilon} + (1-\alpha)k^2 - (1-\alpha)m^2 + (1-\alpha)i\varepsilon\right]^2}$$

$$-\frac{1}{2}(-i\lambda)^2 \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 - 2\alpha pk + \alpha p^2 - m^2 + i\varepsilon\right]^2}$$

$$(k-\alpha p)^2 + \alpha(1-\alpha)p^2 - m^2 + i\varepsilon$$

$k \to k + \alpha p$.

$\partial_{p^2} \to$ finite

$$-\frac{1}{2}(-i\lambda)^2 \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 + \alpha(1-\alpha)p^2 - m^2 + i\varepsilon\right]^2}$$

$$= -\frac{1}{2}(-i\lambda)^2 \int_0^1 d\alpha \, \frac{i\,\Gamma(2-d/2)}{(4\pi)^{d/2}\,\underbrace{\Gamma(2)}_{1}} \frac{1}{\left[m^2 - \alpha(1-\alpha)p^2 - i\varepsilon\right]^{2-d/2}}$$

$\boxed{\text{agrees}}$

(18)

Feynman parameters

$$\frac{1}{\Delta_1 \cdots \Delta_n} = \int_0^\infty d\alpha_1 \cdots d\alpha_n \, e^{-\alpha_1 \Delta_1 - \cdots - \alpha_n \Delta_n} \qquad (\mathrm{Re}\,\Delta_i > 0)$$

α: Schwinger param.

$$= \int_0^\infty d\lambda \int_0^\infty d\alpha_1 \cdots d\alpha_n \, \delta(\lambda - \Sigma \alpha_i) \, e^{-\alpha_1 \Delta_1 - \cdots - \alpha_n \Delta_n}$$

$\alpha_i \to \lambda \alpha_i$

$$= \int_0^\infty d\lambda \int_0^\infty d\alpha_1 \cdots d\alpha_n \, \lambda^n \, \underbrace{\delta(\lambda(1 - \Sigma \alpha_i))}_{\frac{1}{\lambda}\delta(1-\Sigma \alpha_i)} \, e^{-\lambda(\Sigma \alpha_i \Delta_i)}$$

$$= \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n \, \delta(1 - \Sigma \alpha_i) \int_0^\infty d\lambda \, \lambda^{n-1} e^{-\lambda(\Sigma \alpha_i \Delta_i)}$$

$$\boxed{= \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_n \, \frac{\Gamma(n) \, \delta(1 - \Sigma \alpha_i)}{(\Sigma \alpha_i \Delta_i)^n}}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta + i\varepsilon)^n} = \int_0^\infty \frac{d\alpha}{\Gamma(n)} \int \frac{d^d k}{(2\pi)^d} \, e^{i\alpha k^2 - i\alpha\Delta - \alpha\varepsilon} \, \alpha^{n-1} \, (-i)^n$$

$$\frac{1}{(\varepsilon + i\Delta + ik^2)^n} = \frac{e^{-\frac{i\pi}{2}n}}{\Gamma(n)} \int_0^\infty d\alpha \, \alpha^{n-1} \frac{e^{i\frac{\pi}{4}(2-d)}}{(4\pi\alpha)^{d/2}} \, e^{-\alpha\varepsilon - i\alpha\Delta}$$

$$= \frac{e^{-i\frac{\pi}{2}n}}{\Gamma(n)} \frac{e^{i\frac{\pi}{4}(2-d)}}{(4\pi)^d} \frac{\Gamma(n - d/2)}{(\varepsilon + i\Delta)^{n-d/2}} = \frac{e^{-\frac{i\pi}{2}n + \frac{i\pi}{2} - \cancel{\frac{i\pi d}{4}}} \, e^{-\frac{i\pi}{2}n + \cancel{\frac{i\pi d}{2\,2}}} \, \Gamma(n - d/2)}{\Gamma(n) \, (4\pi)^d} = \frac{i(-)^n \Gamma(n - d/2)}{\Gamma(n) \, (4\pi)^d}$$

We still have.

$$= \frac{1}{2} \frac{i\lambda^2 \,\Gamma(2-d/2)}{(4\pi)^{d/2}} \int_0^1 d\alpha \, \frac{1}{(m^2 - \alpha(1-\alpha)p^2 - i\varepsilon)^{2-d/2}}$$

$d = 4-\varepsilon$ dim. reg.

$2 - d/2 = \varepsilon/2$

$$= \frac{i\lambda^2}{2} \frac{\Gamma(\varepsilon/2)}{(4\pi)^{d/2}} \int_0^1 d\alpha \, (m^2 - \alpha(1-\alpha)p^2 - i\varepsilon)^{-\varepsilon/2}$$

$$\frac{\varepsilon}{2}\Gamma(\varepsilon/2) = \Gamma(1+\varepsilon/2) = 1 + \Gamma'(1)\,\varepsilon/2 + \mathcal{O}(\varepsilon^2)$$

$$\Gamma(\varepsilon/2) = \frac{2}{\varepsilon} + \Gamma'(1) + \mathcal{O}(\varepsilon)$$

$$(4\pi)^{-d/2} = (4\pi)^{-2+\varepsilon/2} = \frac{1}{(4\pi)^2}\left(1 + \frac{\varepsilon}{2}\ln(4\pi) + \ldots\right)$$

$$\lambda^2 \to (\mu^\varepsilon \lambda)^2 = (\mu^\varepsilon \lambda)\,\lambda\,(1 + \varepsilon \ln\mu + \ldots)$$

$$= -\frac{1}{2}(-i\lambda\mu^\varepsilon)\frac{\lambda}{(4\pi)^2}\left(1 + \frac{\varepsilon}{2}\ln(4\pi)\right)(1+\varepsilon\ln\mu)\left(\frac{2}{\varepsilon} + \gamma + \ldots\right)$$

$$\int_0^1 d\alpha \left(1 - \frac{\varepsilon}{2}\ln(m^2 - \alpha(1-\alpha)p^2 - i\varepsilon) + \ldots\right)$$

$$= -\frac{1}{2}(-i\lambda\mu^{\varepsilon})\frac{\lambda}{(4\pi)^2}\Big(\overbrace{\underset{\uparrow MS}{\frac{2}{\varepsilon}} + \ln(4\pi)}^{\overline{MS}} + \underset{\ln\mu^2}{2\ln\mu} + \gamma -$$

$$- \int_0^1 \ln(m^2 - \alpha(1-\alpha)p^2 - i\varepsilon)\,d\alpha + \dots \mathcal{O}(\varepsilon)\Big)$$

$$= (-i\lambda\mu^{\varepsilon})\Big[1 - \frac{1}{2}\frac{\lambda}{(4\pi)^2}\Big(\frac{2}{\varepsilon} + \ln(4\pi) + \gamma\Big) -$$

$$-\frac{1}{2}\frac{\lambda}{(4\pi)^2}\int_0^1 \ln\Big(\frac{m^2}{\mu^2} - \alpha(1-\alpha)\frac{p^2}{\mu^2} - i\varepsilon\Big)d\alpha$$

$$\frac{p^2}{\mu^2}\Big(\frac{m^2}{p^2} - \alpha(1-\alpha)\Big)$$

$\ln p \quad -i\pi$

$p\,e^{-i\pi}$

$$(-i\lambda\mu^{\varepsilon})\Big[1 - \frac{1}{2}\frac{\lambda}{(4\pi)^2}\Big(\frac{2}{\varepsilon} + \ln(4\pi) + \gamma\Big) -$$

$$-\frac{1}{2}\frac{\lambda}{(4\pi)^2}\ln\Big(\frac{p^2}{\mu^2}\Big) - \frac{1}{2}\frac{\lambda}{(4\pi)^2}\int_0^1 d\alpha\,\ln\Big(\underbrace{\frac{m^2}{p^2} - \alpha(1-\alpha)}_{-i\pi}\ - i\varepsilon\Big)\Big]$$

if $(1-\alpha(1-\alpha)\,p^2/m^2) > 0 \quad \forall\ 0<\alpha<1 \Rightarrow$ real & eliminate $i\varepsilon$

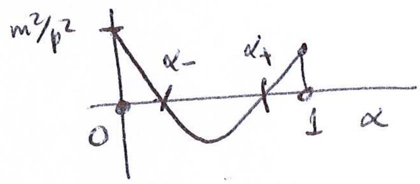
if $(1-\alpha(1-\alpha)\,p^2/m^2) < 0 \quad \rightarrow \quad -i\pi$ contribution to Log.

$\alpha^2 - \alpha + m^2/p^2 > 0 \qquad \underline{\alpha = 0}$.

$$\alpha_{1,2} = \frac{1 \pm \sqrt{1 - 4m^2/p^2}}{2}$$

if $\frac{4m^2}{p^2} > 1 \Rightarrow$ no real roots & $\alpha^2 - \alpha + m^2/p^2 > 0 \quad \forall \alpha$

$\boxed{p^2 < 4m^2}$



if $\frac{4m^2}{p^2} < 1 \Rightarrow \alpha_{1,2}$ real and $\underline{0 < \alpha_{1,2} < 1}$

$\Rightarrow$ imag. part. $(\alpha^2 - \alpha + m^2/p^2 - i\epsilon)$

$$-i\pi \to -\frac{1}{2}\frac{\lambda}{(4\pi)^2}\int_{\alpha_2}^{\alpha_1} d\alpha\,(-i\pi) = \frac{i\pi\lambda}{2(4\pi^2)}(\alpha_1 - \alpha_2)$$

$$= \frac{i\pi\lambda}{8\pi^2}\, 2\sqrt{1 - 4m^2/p^2} = \boxed{\frac{i\lambda}{4\pi}\sqrt{1 - \frac{4m^2}{p^2}}}$$ imag. part for $p^2 > 4m^2$ $\boxed{|p| > 2m}$

Case $p^2 < 4m^2 \Rightarrow \int$ real.

$$-\frac{1}{2}\frac{\lambda}{(4\pi)^2}\int_0^1 d\alpha \ln\left(\frac{m^2}{p^2} - \alpha(1-\alpha)\right)$$

$\ln\left((\alpha-\alpha_1)(\alpha-\alpha_2)\right)$

$$\partial \quad \frac{1}{\alpha-\alpha_1}+\frac{1}{\alpha-\alpha_2}=\frac{\alpha-\alpha_2+\alpha-\alpha_1}{(\alpha-\alpha_1)(\alpha-\alpha_2)}$$

$$(\alpha-\alpha_1)\ln(\alpha-\alpha_1)\Big|_0^1+(\alpha-\alpha_2)\ln(\alpha-\alpha_2)\Big|_0^1-2\alpha\Big|_0^1$$

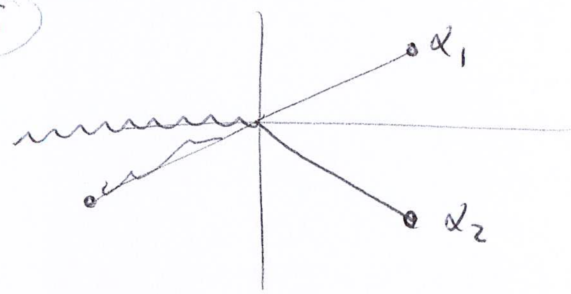
$$(1-\alpha_1)\ln(1-\alpha_1)+\alpha_1\ln(-\alpha_1)+(1-\alpha_2)\ln(1-\alpha_2)+\alpha_2\ln(-\alpha_2)-2$$

$$\alpha_2\ln\alpha_2+\alpha_1\ln(-\alpha_1)+\alpha_1\ln\alpha_1+\alpha_2\ln(-\alpha_2)-2$$

$$\boxed{\alpha_1\ln(-\alpha_1^2)+\alpha_2\ln(-\alpha_2^2)-2}$$

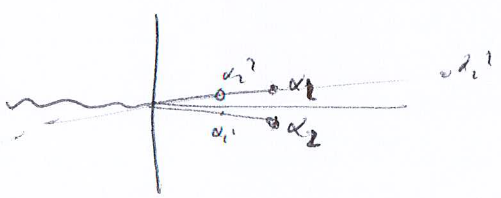
$+i\pi$ $\qquad$ $+i\pi$

$$\alpha=\frac{1\pm\sqrt{1-4m^2/p^2+i\varepsilon}}{2}$$



$\sqrt{a-i\varepsilon}$ $\quad \sqrt{a}\,(1-i\varepsilon$

~~$i\pi(\alpha_2-\alpha_1)$~~



$-(\alpha_1-\alpha_2)\,i\pi$ ✓

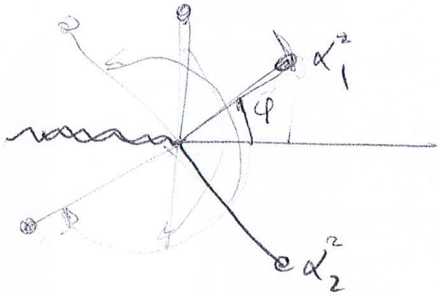
$$\mathrm{Re}: \boxed{2\alpha_1\ln|\alpha_1|+2\alpha_2\ln|\alpha_2|-2} \quad \checkmark$$

for $p^2>4m^2$ $\quad \displaystyle\int_0^1 d\alpha\,\ln\left(\frac{m^2}{p^2}-\alpha(1-\alpha)\right)=-i\pi(\alpha_1-\alpha_2)+2\left(\alpha_1\ln\alpha_1+\alpha_2\ln\alpha_2-1\right)$

$$\alpha_{1,2}=\frac{1}{2}\left(1\pm\sqrt{1-4m^2/p^2}\right)$$

$\alpha_1 = \alpha_+ \quad \alpha_2=\alpha_-$

$p^2 < 4m^2$ $\qquad \alpha_{1,2}$ complex.



$$\alpha_1 = \rho e^{i\varphi}$$
$$\alpha_2 = \rho e^{-i\varphi}$$

$$\rho e^{i\varphi} \ln(-\rho^2 e^{2i\varphi}) + \rho e^{-i\varphi} \ln(-\rho^2 e^{-2i\varphi}) - 2$$

$$\rho e^{i\varphi} \ln(\rho^2 e^{2i\varphi - i\pi}) + \rho e^{-i\varphi} \ln(\rho^2 e^{-2i\varphi + i\pi}) - 2$$

$$2\rho\cos\varphi \ln\rho^2 + \rho\left[e^{i\varphi}(2i\varphi - i\pi) + e^{-i\varphi}(-2i\varphi + i\pi)\right] - 2$$

$$4\rho\ln\rho\cos\varphi + \rho\, 2i s\varphi\,(2i\varphi - i\pi) - 2$$

$$\boxed{4\rho\ln\rho\cos\varphi - 2\rho s\varphi\,(2\varphi - \pi) - 2}$$

$$\alpha_1 = \frac{1}{2}\left(1 + i\sqrt{\frac{4m^2}{p^2} - 1}\right) \qquad \rho = m/p$$

$$\rho^2 = \frac{1}{4}\left(\cancel{1} + \frac{4m^2}{p^2} \cancel{-1}\right) = m^2/p^2 \qquad 1 + \tan^2\varphi = \frac{4m^2}{p^2} = \frac{1}{c^2\varphi}$$

$$\tan\varphi = \sqrt{\frac{4m^2}{p^2} - 1} \qquad c\varphi = \frac{p}{2m} \qquad s\varphi = \sqrt{1 - p^2/4m^2}$$

$p^2 < 4m$

$$\int_0^1 d\alpha \ln\left(\frac{m^2}{p^2} - \alpha(1-\alpha)\right) = 4\frac{m}{\cancel{p}} \ln\left(\frac{m}{p}\right) \frac{\cancel{p}}{2\cancel{m}} -$$

$$-\frac{2m}{p}\sqrt{1-p^2/4m^2}\,\left(2a\cos(p/2m) - \pi\right) - 2$$

$$= 2\ln m/p - \sqrt{\frac{4m^2}{p^2}-1}\,\left(2a\cos\left(p/2m\right) - \pi\right) - 2$$

$$= \ln\frac{m^2}{p^2} - \sqrt{\frac{4m^2}{p^2}-1}\; 2\,\underbrace{\left(\overset{a\sin(p/2m)}{a\cos(p/2m)} - \frac{\pi}{2}\right)}_{\sin\ \alpha} - 2$$

$\sin\alpha = \ldots$

$$\boxed{= \ln\frac{m^2}{p^2} + 2\sqrt{\frac{4m^2}{p^2}-1}\; a\sin\left(\frac{p}{2m}\right) - 2}$$

$$\int_0^1 d\alpha \ln\left(\frac{m^2}{p^2} - i\varepsilon - \alpha(1-\alpha)\right) =$$

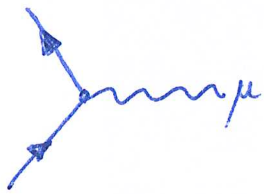
$\nearrow\ p^2 < 4m^2 \quad \ln\frac{m^2}{p^2} + 2\sqrt{\frac{4m^2}{p^2}-1}\; a\sin\left(\frac{p}{2m}\right) - 2$

$\searrow\ p^2 > 4m^2 \quad -i\pi(\alpha_1 - \alpha_2) + 2\left(\alpha_1 \ln\alpha_1 + \alpha_2\ln\alpha_2 - 1\right)$

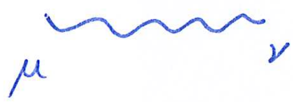
$$\alpha_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{1-\frac{4m^2}{p^2}}\right)$$

Feynman rules for photons + scalars + fermions

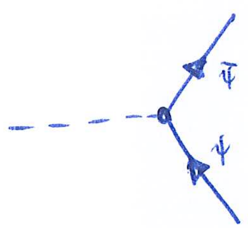
$$V_{int} = e\int d^3x \; \bar{\psi}\gamma^\mu \psi A_\mu$$



$$= -ie\gamma^\mu$$



$$= -i\eta_{\mu\nu}\frac{1}{q^2+i\varepsilon} \qquad \text{(Feynman gauge)}$$



$$-ig$$

$$i\left(\frac{\not{k}+m}{k^2-m^2+i\varepsilon}\right)_{ba}$$

External legs contractions

$\phi|q\rangle$  $\langle q|\phi$

$\psi\, \underbrace{|p,s\rangle}_{\text{fermion}}$  $\langle p,s|\bar{\psi}$

$\bar{\psi}\, \underbrace{|k,s\rangle}_{\text{antifermion}}$  $\langle k,s|\psi$

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left(e^{-ikx} a_k + e^{ikx} a_k^\dagger \right)$$

$$a_k|q\rangle = a_k \sqrt{2\omega_q}\, a^\dagger_{q,\sigma} |0\rangle = (2\pi)^3 \sqrt{2\omega_q}\, \delta^{(3)}(k-q)|0\rangle$$

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} e^{-ikx} a_k |q\rangle = \int \frac{d^3k}{(2\pi)^3} e^{-ikx} \delta^{(3)}(k-q)|0\rangle$$

$$= e^{-iqx}|0\rangle$$

$$\psi = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=1,2} \left(u^\sigma_{\vec{k}}\, e^{-ikx} c_{k,\sigma} + v^\sigma_{k}\, d^\dagger_{k,\sigma}\, e^{ikx} \right)$$

$$\psi\, |p,s\rangle = u^s_{\vec{p}}\, e^{-ipx}$$

$$\bar{\psi}\, |k,s\rangle = \bar{v}^s_k\, e^{-ikx}$$

$$\bar{\psi} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{\sigma=1} \left(\bar{u}^\sigma_{\vec{k}}\, e^{ikx} c^\dagger_{k,\sigma} + \bar{v}^\sigma_{\vec{k}}\, d_{k,\sigma}\, e^{-ikx} \right)$$

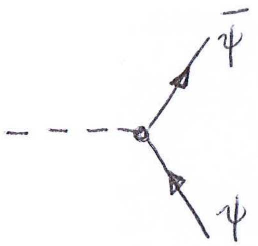
$$\langle q| = \langle 0|\, a_{q,\sigma} \sqrt{2\omega_q} \int \cdots = e^{iqx}$$

$$\underbrace{\langle p,s|}_{\text{fermion}}\, \bar{\psi} = \bar{u}^s_p\, e^{ipx}$$

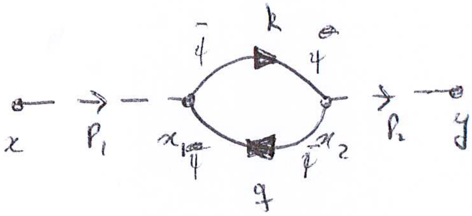
$$\langle k,s|\, \psi = v^s_k\, e^{ikx}$$

$$V = \int d^3x \; g\phi\bar{\psi}\psi$$

$\psi_b \bar{\psi}_a$



$$\left(i \frac{\not{k}+m}{k^2-m^2+i\varepsilon} \right)_{ba}$$



$$\frac{(-ig)^2}{2!} \hat{T}\left\{ \phi(x)\phi(y) \int d^3x_1 \int d^3x_2 \; \phi(x_1)\bar{\psi}_a(x_1)\psi_a(x_1)\,\phi(x_2)\bar{\psi}_b(x_2)\psi_b(x_2) \right\}$$

$2\times$

$$-(-ig)^2 \int_{x_1x_2} D_F(x-x_1)\,D_F(y-x_2)\; S_{ab}(x_1-x_2)\; S_{ba}(x_2-x_1)$$

$$-(-ig)^2 \int_{x_1x_2} \int \frac{d^dp_1}{(2\pi)^d} \frac{i e^{-ip_1(x-x_1)}}{p_1^2-m_b^2+i\varepsilon} \int \frac{d^dp_2}{(2\pi)^d} \frac{i e^{-ip_2(y-x_2)}}{p_2^2-m_b^2+i\varepsilon} \int \frac{d^dq}{(2\pi)^d} \frac{i e^{-iq(x_1-x_2)}}{q^2-m_f^2+i\varepsilon}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{i e^{-ik(x_2-x_1)}}{k^2-m_f^2+i\varepsilon} \, Tr\left((\not{q}+m_f)(\not{k}+m_f)\right)$$

$$= -(-ig)^2 \int \frac{d^dp_1}{(2\pi)^d} \int \frac{d^dp_2}{(2\pi)^d} e^{+ip_1x-ip_2y} \; i\Delta(p_1)\, i\Delta(p_2)\, (2\pi)^d\delta^{(d)}(-p_1-q+k)\,(2\pi)^d\delta^{(d)}(p_2+q-k)$$

$$\int \frac{d^dk}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{q^2-m^2+i\varepsilon} Tr\left((\not{q}+m)(\not{k}+m)\right)$$

p_2+k-p_1-k

$q = k - p_1$

$$= -(-ig)^2 \int \frac{d^d p_1}{(2\pi)^d} \int \frac{d^d p_2}{(2\pi)^d} e^{ip_1 x - ip_2 y} \, i\Delta_b(p_1)\, i\Delta_b(p_2)\, (2\pi)^d \delta^{(d)}(p_1 - p_2)$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m_f^2 + i\varepsilon} \frac{i}{(k-p_1)^2 - m_f^2 + i\varepsilon} \mathrm{Tr}\left[((\not k - \not p_1) + m_f)(\not k + m_f) \right]$$

$$-(-ig)^2 i^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 d\alpha \frac{\cancel{4}\,(k(k-p_1) + m_f^2)}{\left[\alpha (k-p_1)^2 + (1-\alpha)k^2 - m_f^2 + i\varepsilon\right]^2}$$

$$k^2 - 2\alpha k p_1 + \alpha p_1^2$$

$$(k - \alpha p_1)^2 + \alpha(1-\alpha) p_1^2$$

$$k^2 - (1-\alpha) k p_1 + \alpha k p_1 - \alpha(1-\alpha) p_1^2$$

$$-(-ig)^2 4 i^2 \int \frac{d^d k}{(2\pi)^d} \int_0^1 d\alpha \frac{(k+\alpha p_1)(k - (1-\alpha)p_1) + m_f^2}{\left[k^2 + \alpha(1-\alpha)p_1^2 - m_f^2 + i\varepsilon\right]^2}$$

$$\int k_\mu = 0 \qquad k^2 - \alpha(1-\alpha) p_1^2 + m_f^2$$

$$\Delta = m_f^2 - i\varepsilon - \alpha(1-\alpha) p_1^2$$

$$-(-ig)^2 4 i^2 \int_0^1 d\alpha \left[\frac{(-)\, i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-d/2)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{1-d/2} + \right.$$

$$\left. + \underbrace{(m_f^2 - \alpha(1-\alpha)p_1^2)}_{\Delta} \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-d/2} \right]$$

$$= \frac{-4(-ig)^2 i^{2+1}}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{1}{\Delta^{1-d/2}} \Gamma(1-d/2) \underbrace{\left[-\frac{d}{2} + 1 - \frac{d}{2} \right]}_{1-d} = \frac{4ig^2(d-1)}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{\Gamma(1-d/2)}{\Delta^{-d/2+1}}$$

 $$= \frac{4ig^2(d-1)}{(4\pi)^{d/2}}\,\Gamma\left(1-\frac{d}{2}\right)\int_0^1 d\alpha\,\left(m_f^2-\alpha(1-\alpha)p_1^2-i\varepsilon\right)^{d/2-1}$$

$d=4-\varepsilon \qquad d/2-1 = 1-\varepsilon/2$

$$\Gamma(1-d/2) = \Gamma(-1+\varepsilon/2) = \frac{\Gamma(\varepsilon/2)}{-1+\varepsilon/2} = \frac{\Gamma(1+\varepsilon/2)}{(\varepsilon/2)(\varepsilon/2-1)} \simeq -\frac{2}{\varepsilon}(1+\varepsilon/2)(1+\gamma\varepsilon/2)$$

$$\simeq -\frac{2}{\varepsilon}\left(1+(1+\gamma)\frac{\varepsilon}{2}+\dots\right)$$

$$\left(m_f^2-\alpha(1-\alpha)p_1^2-i\varepsilon\right)^{1-\varepsilon/2} = \left(m_f^2-\alpha(1-\alpha)p_1^2\right)\left(1-\frac{\varepsilon}{2}\ln\left(m_f^2-\alpha(1-\alpha)p_1^2-i\varepsilon\right)\right)$$

 $=$

$$= \frac{12ig^2}{16\pi^2}\left(-\frac{2}{\varepsilon}\right)\int_0^1 d\alpha\,\left(m_f^2-\alpha(1-\alpha)p_1^2\right)$$

divergent piece

$$\int_0^1 \alpha(1-\alpha)\,d\alpha = \frac{\alpha^2}{2}-\frac{\alpha^3}{3}\Big|_0^1 = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}$$

$\alpha-\alpha^2$

$$= \frac{3}{4}\frac{ig^2}{\pi^2}\left(-\frac{2}{\varepsilon}\right)\left(m_f^2-\frac{1}{6}p_1^2\right)$$

$$= -\frac{3}{2}\frac{ig^2}{\pi^2\varepsilon}m_f^2 + \frac{1}{4\varepsilon}\frac{ig^2}{\pi^2}p_1^2 + \dots = (-i)\left(\frac{3}{2}\frac{g^2}{\pi^2\varepsilon}m_f^2 - \frac{g^2 p_1^2}{4\pi^2\varepsilon}\right)$$

 $\frac{1}{\Delta(p)} = -i(p_1^2-m_b^2)$

 $\frac{i}{p^2-m^2}$ $+$  $\frac{i}{p^2-m^2}(-iM^2)\frac{i}{p^2-m^2}$ $+$  $= \frac{i}{p^2-m^2}\left(\frac{1}{1-\frac{-iM^2 i}{p^2-m^2}}\right)$

$$= \frac{i}{p^2 - m^2 - M^2}$$

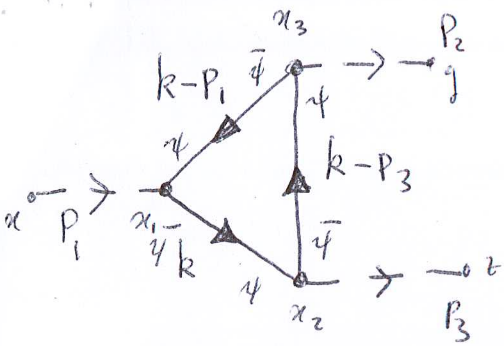
$$p^2 - m^2 - \frac{3}{2}\frac{g^2}{\pi^2 \varepsilon} m_f^2 - \frac{g^2}{4\pi^2 \varepsilon} p_1^2$$

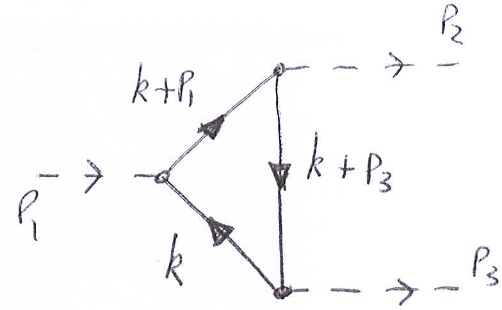
 $-i M_*^2$

$$= -\left(\frac{1}{1 - \bullet\!-}\right) = \frac{i}{(p^2 - m^2)\left(1 - \frac{(-iM_*^2)\, i}{p^2 - m^2}\right)} = \frac{i}{p^2 - m^2 - M_*^2 - i0}$$

$$M_*^2 + \frac{3}{2}\frac{g^2}{\pi^2 \varepsilon} m_f^2 + \frac{g^2}{4\pi^2 \varepsilon} p_1^2$$

$$\frac{1}{2}\int \partial\phi\, \partial\phi - \frac{1}{2}\int m^2 \phi^2$$





$$\bar\psi\psi\;\bar\psi\psi\;\bar\psi\psi$$

A fermion loop always has a $(-)$ sign.

$$\bar\psi_{x_1 a}\,\psi_{x_1 a}\,\bar\psi_{x_3 b}\,\psi_{x_3 b}\,\bar\psi_{x_2 c}\,\psi_{x_2 c}\quad e^{-ip_1(x_1-x)}\,e^{-ik_3(x_1-x_3)}\,e^{-ik_2(x_3-x_2)}\,e^{-ik_1(x_2-x_1)}$$

$$e^{-ip_2(y-x_3)}\quad e^{-ip_3(z-x_2)}$$

$-p_1-k_3+k_1=0$

$k_3-k_2+p_2=0$

$k_2-k_1+p_3=0$

$\cancel{k_1=k_3+p_1}$

$k_3=k_1-p_1$

$k_2=k_3+p_2$

$k_2=k_1-p_1+p_2$
$=k_1-p_3$

$p_3+p_2=p_1$

$$(S_F)_{ab}(x_1-x_3)\,(S_F)_{bc}(x_3-x_2)\,(S_F)_{ca}(x_2-x_1)$$

$$\mathrm{Tr}\left(\frac{i((\not k-\not p_1)+m_f)\,i((\not k-\not p_3)+m_f)}{(k-p_1)^2-m^2+i\varepsilon\quad(k-p_3)^2-m^2+i\varepsilon}\;\frac{i(\not k+m_f)}{k^2-m^2+i\varepsilon}\right)$$

$$-(-ig)^3(i)^3\int\frac{d^dk}{(2\pi)^d}\,\frac{\mathrm{Tr}\left[((\not k-\not p_1)+m_f)((\not k-\not p_3)+m_f)(\not k+m_f)\right]}{[(k-p_1)^2-m_f^2+i\varepsilon]\,((k-p_3)^2-m_f^2+i\varepsilon)\,(k^2-m^2+i\varepsilon)}$$

$$4i(-ig)^3\int\frac{d^dk}{(2\pi)^d}\,\frac{((k-p_1)(k-p_3)+(k-p_1)k+(k-p_3)k+m_f^3)}{((k-p_1)^2-m_f^2+i\varepsilon)\,((k-p_3)^2-m_f^2+i\varepsilon)\,(k^2-m^2+i\varepsilon)}$$

$$4i(-ig)^3\int\frac{d^dk}{(2\pi)^d}\,\frac{((k+p_1)(k+p_3)+k(k+p_1)+k(k+p_3)+m_f^3)}{((k+p_1)^2-m_f^2+i\varepsilon)\,((k+p_3)^2-m_f^2+i\varepsilon)\,(k^2-m^2+i\varepsilon)}$$

$k \to -k$ in 2^{nd} integral. gives 1^{st} integral.

$$8i(-ig)^3 \int \frac{d^dk}{(2\pi)^d} \int_0^1 d\alpha\, d\beta\, d\gamma\; \delta(1-\alpha-\beta-\gamma).\; \frac{\qquad}{\left[\alpha(k+p_1)^2 + \beta(k+p_3)^2 + \gamma k^2 - m_f^2 + i\varepsilon\right]^3}$$

$$k^2 + 2\alpha k p_1 + 2\beta k p_3 + \alpha p_1^2 + \beta p_3^2$$

$$(k + \alpha p_1 + \beta p_3)^2 + \alpha(1-\alpha)p_1^2 + \beta(1-\beta)p_3^2 - 2\alpha\beta p_1 p_3$$

$$k \to k + \alpha p_1 + \beta p_3$$

$$8i(-ig)^3 \int \frac{d^dk}{(2\pi)^d} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \;\; \begin{matrix} \cancel{(k+\alpha p_1+\beta p_3)(k+(\alpha-1)p_1+\beta p_3)} \\ \cancel{(k+(\alpha-1)p_1+\beta p_3)(k+(\beta-1)p_3+\alpha p_1)+} \end{matrix}$$

$$\cancel{3k^2 + k(p_3+p_1+p_1+p_3) + p_1 p_3 + m_f^3}$$

$$3k^2 + k(p_1+p_3+p_1+p_3) + p_1 p_3 + m_f^3$$

$$3k^2 - 2k(p_1+p_3) + p_1 p_3 + m_f^3$$

$$3(k+\alpha p_1+\beta p_3)^2 - 2(k+\alpha p_1+\beta p_3)(p_1+p_3) + p_1 p_3 + m_f^3$$

$$3k^2 + 6\alpha k p_1 + \cancel{6\beta k p_3} + 6\alpha\beta p_1 p_3 - \cancel{2k(p_1+p_3)} - 2(\alpha p_1+\beta p_3)(p_1+p_3)$$
$$+ p_1 p_3 + m_f^3$$

$$3k^2 + 6\alpha\beta p_1 p_3 - 2\alpha p_1^2 - 2\alpha p_1 p_3 - 2\beta p_1 p_3 - 2\beta p_3^2 + p_1 p_3 + m_f^3$$

$$8i(-ig)^3 \int \frac{d^dk}{(2\pi)^d} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, \frac{3k^2 + (6\alpha\beta - 2\alpha - 2\beta + 1)p_1p_3 - 2\alpha p_1^2 - 2\beta p_3^2 + m_f^3}{\left[k^2 + \alpha(1-\alpha)p_1^2 + \beta(1-\beta)p_3^2 - 2\alpha\beta p_1 p_3 - m_f^2 + i\varepsilon\right]^3}$$

$$\Delta = m_f^2 - i\varepsilon - \alpha(1-\alpha)p_1^2 - \beta(1-\beta)p_3^2 + 2\alpha\beta p_1 p_3$$

$$8i(-ig)^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left[\frac{3\, i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\overbrace{\Gamma(2-d/2)}^{\text{pole}}}{\Gamma(3)} \frac{1}{\Delta^{2-d/2}} + \right.$$

$$\left. + \left\{(6\alpha\beta - 2\alpha - 2\beta + 1)p_1p_3 - 2\alpha p_1^2 - 2\beta p_3^2 + m_f^3\right\} \frac{(-)\, i}{(4\pi)^{d/2}} \frac{\overbrace{\Gamma(3-d/2)}^{\text{finite!}}}{\Gamma(3)} \frac{1}{\Delta^{3-d/2}} \right]$$

$d = 4 - \varepsilon \qquad 2 - d/2 = \varepsilon/2$

$(-i)^3 = +i$

Divergence.

$$\cancel{8}i(-ig)^3 \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \, \frac{3i}{\cancel{16}\pi^2} \frac{\cancel{4}}{\cancel{2}} \frac{2}{\varepsilon} = -\frac{3(-ig)^3}{\pi^2} \frac{1}{\varepsilon} \int_0^1 d\alpha \, (1-\alpha)$$

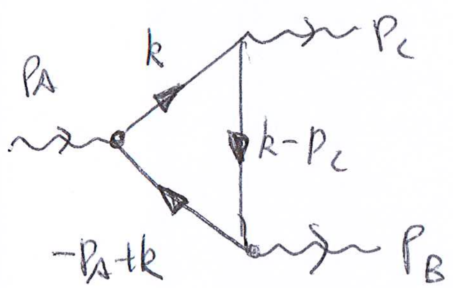
$$\alpha - \alpha^2/2 \Big|_0^1 \qquad 1 - 1/2 = 1/2$$

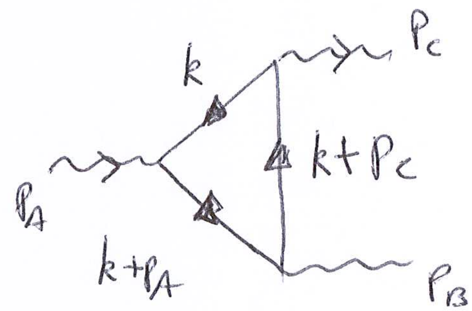
$$= -\frac{3i}{2\pi^2} \frac{g^3}{\varepsilon} + \text{finite}$$

 $-ig$  $-\frac{3}{2\pi^2}\frac{g^3}{\varepsilon}\phi^3$ (MS counterterm)

$$-\int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left(\gamma_C (\not k + m)\, \gamma_A\, (\not k - \not p_A \overset{+m}{)}\, \gamma_B\, (\not k - \not p_C \overset{+m}{)}\right)}{(k^2 - m^2 + i\epsilon)\,((k-p_A)^2 - m^2 + i\epsilon)\,((k-p_C)^2 - m^2 + i\epsilon)}$$

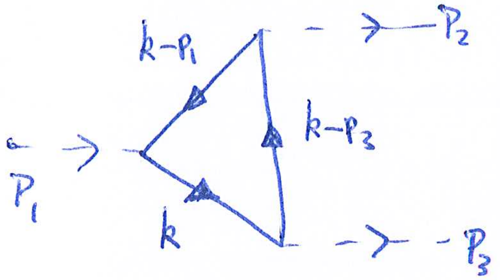
$$-\int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}\left(\gamma_C (\not k - \not p_C \overset{+m}{)}\, \gamma_B\, (\not k - \not p_A \overset{+m}{)}\, \gamma_A\, (\not k + m)\right)}{(k^2 - m^2 + i\epsilon)\,((k-p_A)^2 - m^2 + i\epsilon)\,((k-p_C)^2 - m^2 + i\epsilon)}$$

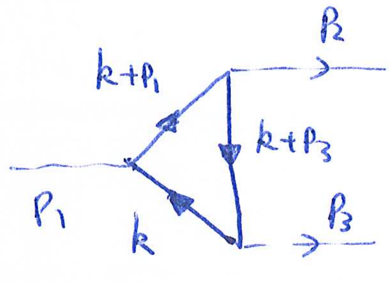




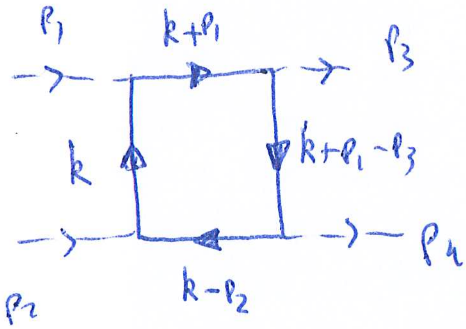
$$\mathrm{Tr}\left(\Gamma_C\, (\not k \pm m)\, \Gamma_A\, (\not k - \not p \pm m)\, \Gamma_C\, (\not k - \tilde{\not p} \pm m) \right)$$

$$C \quad C^{-1}C \quad C^{-1}C \quad C^{-1}$$





Same as $p_2 \leftrightarrow p_3$



$$-(-ig)^4 \int_0^1 d\alpha_1 \dots d\alpha_4 \int \frac{d^dk}{(2\pi)^d} \delta\left(\sum \alpha_i - 1\right) \frac{Tr\left[(\not{k}+m_f)(\not{k}-\not{p}_2+m_f)(\not{k}+\not{p}_1-\not{p}_3+m_f)(\not{k}+\not{p}_1+m_f)\right]}{\left[\alpha_1(k^2-m_f^2)+\alpha_2((k-p_2)^2-m_f^2)+\alpha_3((k+p_1-p_3)^2-m_f^2)+\alpha_4((k+p_1)^2-m_f^2)\right]^4}$$

Divergent piece $\int \frac{d^4k}{k^8} k^4$ $\quad Tr(\not{k}\not{k}\not{k}\not{k}) = 4k^4$

$\alpha_1 k^2 = k$ $\quad d_4\left[k^2 - m_f^2 - 2\alpha_2 kp_2 + \alpha_2 p_2^2 + 2\alpha_3 k(p_1-p_3) + \alpha_3(p_1-p_3)^2 + 2\alpha_4 k\cdot p_1 + \alpha_4 p_1^2\right]^4$

$(k - \alpha_2 p_2 + \alpha_3(p_1-p_3) + \alpha_4 p_1)^2 - m_f^2 + \alpha_2(1-\alpha_2)p_2^2 + \alpha_3(1-\alpha_3)(p_1-p_3)^2 \dots \Delta$

$$-(-ig)^4\,4\int_0^1 d\alpha_1 \dots d\alpha_4\, \delta(\Sigma\alpha_i - 1) \int \frac{d^d k}{(2\pi)^d} \frac{k^4}{(k^2-\Delta)^4}$$

$$= -(-ig)^4\,4 \int d\alpha_1 \dots d\alpha_4 \frac{\delta(\Sigma\alpha_i - 1)}{(4\pi)^{d/2}} \frac{i\,d(d+2)}{4} \frac{\Gamma(2-d/2)}{\Gamma(4)} \frac{1}{\Delta^{2-d/2}}$$

$d = 4-\varepsilon$

$$= -(-ig)^4\,4 \int d\alpha_1 \dots d\alpha_4 \frac{\delta(\Sigma\alpha_i - 1)}{(4\pi)^{d/2}} \frac{i\,d(d+2)}{4} \frac{\Gamma(\varepsilon/2)}{6} \frac{1}{\Delta^{\varepsilon/2}}$$

pole: $$-4\frac{(-ig)^4\, i}{16\pi^2} \int d\alpha_1 \dots d\alpha_4\, \delta(\Sigma\alpha_i - 1) \frac{\cancel{4\times 6}}{\cancel{4\times 6}} \frac{2}{\varepsilon}$$

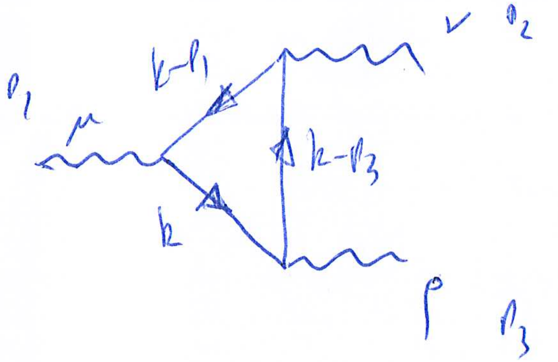
$$= -\frac{i}{2\pi^2}(-ig)^4 \frac{1}{\varepsilon} \underbrace{\int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2 \int_0^{1-\alpha_1-\alpha_2} d\alpha_3 \int_0^{1-\alpha_1-\alpha_2-\alpha_3} d\alpha_4\, \delta(\Sigma\alpha_i - 1)}_{1} = -\frac{i g^4}{12\pi^2}\frac{1}{\varepsilon}$$

$$\int_0^{1-\alpha_1} (1-\alpha_1-\alpha_2)\, d\alpha_2$$

$$(1-\alpha_1)\alpha_2 - \frac{\alpha_2^2}{2}\Big|_0^{1-\alpha_1}$$

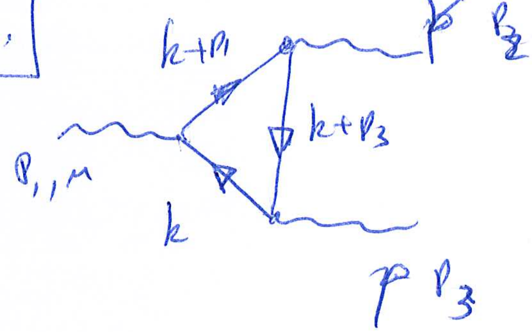
$$(1-\alpha_1)^2 - \frac{(1-\alpha_1)^2}{2} = \frac{(1-\alpha_1)^2}{2}$$

$$\int_0^1 \frac{(1-\alpha_1)^2}{2} d\alpha_1 = \frac{1}{2}\int_0^1 \alpha_1^2 d\alpha_1 = \frac{1}{2}\frac{\alpha_1^3}{3}\Big|_0^1 = 1/6$$



$\boxed{\text{cancel.}}$

$\longleftrightarrow$



$$Tr[((\not{k}-\not{p}_1)+m_f)\gamma^\nu((\not{k}-\not{p}_3)+m_f)\gamma^\rho((\not{k}+m)\gamma^\mu)$$

$$Tr[\overset{C}{(}(-\not{k}+\not{p}_1)+m_f)\overset{CC^{-1}}{\gamma^\mu}\overset{CC^{-1}}{(}-\not{k}+m_f)\overset{CC^{-1}}{\gamma^\rho}\overset{CC^{-1}}{(}(-\not{k}+\not{p}_3)+m_f)\overset{CC^{-1}}{\gamma^\nu}]$$

$$(-)\,Tr[[((\not{k}-\not{p}_1)+m_f)]^t(\gamma^\mu)^t(\not{k}+m_f)^t(\gamma^\rho)^t((\not{k}-\not{p}_3)^t+m_f)^t(\gamma^\nu)^t]$$

$$-Tr[\gamma^\nu[(\not{k}-\not{p}_3)+m_f]\gamma^\rho[(\not{k}+m_f)]\gamma^\mu[(\not{k}-\not{p}_1)+m_f)]])$$

Furry's Theorem.