662, Homework I, (3 problems)

Problem 1

Consider two scalar fields of the same mass $\phi_{1,2}$. They can be combined in a complex field

$$\phi(\vec{x},t) = \frac{1}{\sqrt{2}}\left(\phi_1(\vec{x},t) + i\phi_2(\vec{x},t)\right) \tag{0.1}$$

The Lagrangian (density) is

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{0.2}$$

The Lagrangian has a symmetry $\phi \to e^{i\alpha}\phi$ for any constant real α.

a) Write the equations of motion, Hamiltonian and spatial momentum $\vec{P}$.

b) Use the symmetry described above $\phi \to e^{i\alpha}\phi$ to find a conserved Noether current and conserved charge Q. Use the equations of motion to check that it is indeed conserved.

c) Quantize the complex field, that is, write appropriate expressions for $\phi(\vec{x})$ and $\Pi(\vec{x})$ in terms of momentum creation and annihilation operators and check the canonical commutation relations. Notice that you now need two sets of annihilation (and creation) operators, either for the real fields $\phi_{1,2}$, namely $a_{\vec{k},\ell}$, with $\ell = 1,2$ or for the complex field:

$$\phi(\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{k}}}}(e^{i\vec{k}\vec{x}}a_{\vec{k}} + e^{-i\vec{k}\vec{x}}b^\dagger_{\vec{k}}) \tag{0.3}$$

where $b^\dagger_{\vec{k}}$ is not the hermitian conjugate of $a_{\vec{k}}$ since $\phi(x)$ is not hermitian.

d) Write the Hamiltonian H, momentum $\vec{P}$ and conserved charge Q in terms of annihilation and creation operators and show that Q is conserved (namely commutes with H).

Problem 2

Consider the Feynman propagator for a massive scalar field in momentum space:

$$\Delta_F(p) = -\frac{i}{p^2 - m^2 + i\epsilon} \tag{0.4}$$

a) Perform a Fourier transform and write it explicitly in space-time coordinates in terms of Bessel functions discussing separately its form outside and inside the light cone.

b) Discuss its behavior in the limit of large space-like separation, time-like separation and near the light-cone.

Problem 3

(see https://arxiv.org/pdf/hep-th/0306133.pdf)

Consider the $O(N)$ scalar theory in 3 dimensions (you can also consider arbitrary dimension d) with a magnetic field H coupled to σ.

$$S = \int d^3x \left[\frac{1}{2}(\nabla\phi_a)^2 + \frac{1}{2} r \phi_a^2 + \frac{u}{4!}(\phi_a^2)^2 + H\sigma \right] \tag{0.5}$$

in the large N limit. As done in class write $\phi = (\sigma, \pi_{\tilde{a}})$, and set $\rho = \frac{1}{N}\phi_a^2$ using a delta function

$$1 = N \int d\rho \delta(\phi_a^2 - \rho N) = \frac{N}{4\pi i} \int d\rho d\lambda \; e^{\frac{1}{2}\lambda(\phi_a^2 - N\rho)} \tag{0.6}$$

and use the saddle point approximation for the resulting action.

a) Write the equations of motion for the action for the case of constant fields. Define $\lambda = m^2$ and proceed as we did in class for the case $H = 0$.

b) From the equations of motion compute the susceptibility χ for $r > r_c$ and determine the critical exponent γ in $\chi \sim t^{-\gamma}$. Here $t \sim (r - r_c)$

c) Consider the case $t = 0$, namely $r = r_c$ and compute m^2 as a function of H. From there determine the critical exponent δ in $\sigma \sim H^{\frac{1}{\delta}}$.