Quantum Field theory (Fields & particles)

Newton: particles + forces

Faraday → Maxwell → Einstein

forces → fields → exist independently of the particles (sources) and are dynamical (e.g. e.m. waves)

($\vec{E}(\vec{x})$, $\vec{B}(\vec{x})$ one, or more, degrees of freedom at each point)

In quantum mechanics

Fields → particles (photon, graviton, --)

Yukawa (mesons, --)

$\boxed{\text{Fields and particles are equivalent}}$

Quantum field theory also helps understand better the quantum mechanics of particles.

Initially it was developed to quantize the e.m. field and understand quantum relativistic theories.

It can be applied in many other contexts: many-body physics, condensed matter, statistical mechanics, finite temperature, etc.

In its simplest form it's a theory of weakly interacting particles (elementary excitation, in condensed matter they are sometimes called quasi-particles).

Typical setup for relativistic particle physics

•) identify single particle states & define the Hamiltonian H of those free particles. (bosons and/or fermions)

Single particle states are classified by the symmetries of the problem : Poincare Lorentz + internal symmetries

$$|\vec{k}, \sigma ; a\rangle$$

$\vec{k}$: momentum $\vec{p} = \hbar \vec{k}$

σ: $-s \dots s$ polarization along $\vec{k}$ (spin)

a: other quantum numbers (charge, ...)

••) Identify interactions

•••) Use perturbation theory to compute quantities that can be compared to experiment (~~mean lifes~~, mean-lifes, scattering cross sections).

Standard tool: Fermi Golden rule.

Finding bound states is difficult. (retardation effects, particle creation ...).

Application to statistical mechanics.

there is no relativistic invariance [fixed lattice, material, etc).
and many times not even translational invariance (atomic lattice).

Special case of great interest:
when the system undergoes a second order phase transition, at the transition the dynamic is dominated by long-wave length modes that are described by a scale invariant theory ⇒ translational & rotational invariance is recovered + scale invariance → conformal invariance.
correlation functions decay as power law:

$$\langle \mathcal{O}_\Delta(\vec{x})\ \mathcal{O}_\Delta(\vec{y}) \rangle = \frac{C}{|\vec{x}-\vec{y}|^{2\Delta}}$$

↖ scaling exponent

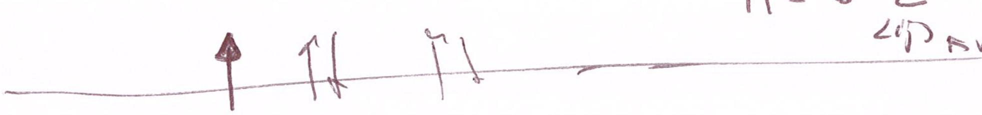
Typical calculations:
Identify operators of lowest conformal dimension and compute Δ.

Example $\langle S_z^{(i)}\ S_z^{(i+j)} \rangle$ in Ising model.

$$H = J \sum_{\langle ij \rangle} \sigma_i \sigma_j \qquad \sigma_i = \pm 1$$

$\langle ij \rangle$ nearest neighbours

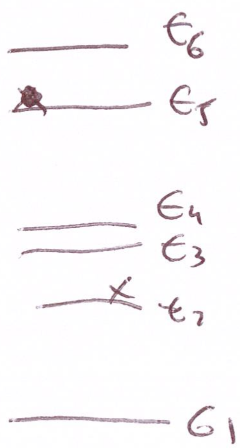
$$Z = \sum_{\{\sigma\}} e^{-\beta \sum_{\langle ij \rangle} \sigma_i \sigma_j}$$



↑ fix to be up & compute probability of others up/down.

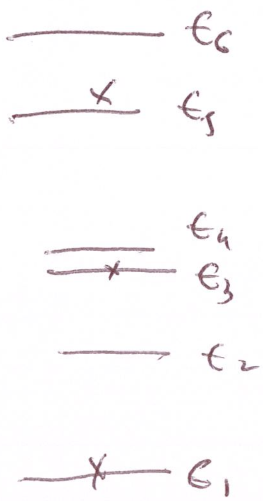
Second quantization (occupation number formalism).

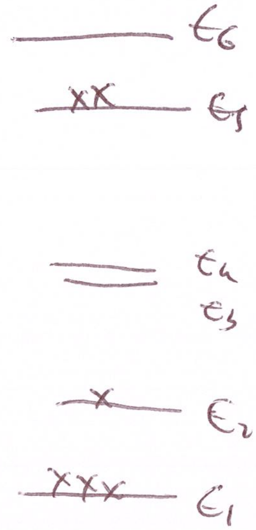
How to deal with bosons & fermions without having to symmetrize (or antisymmetrize) states.



Single particle states $\epsilon_1 \dots \epsilon_n$

multi-particle states





fermions

$|n_{\epsilon_1}=1\ \cancel{n_{\epsilon_2}=0}\ n_{\epsilon_3}=1, n_{\epsilon_5}=1\rangle$

all other zero.

or: $|1_{\epsilon_1}, 1_{\epsilon_3}, 1_{\epsilon_5}\rangle$

bosons

$|n_1 = 3, n_2 = 1, n_5 = 2\rangle$

all others zero

$|3_{\epsilon_1}, 1_{\epsilon_2}, 2_{\epsilon_5}\rangle$

Form a basis for states of the multi-particle system.

It only makes sense to define operators that act on this space: most basic ones, creation and annihilation ops:

Bosons

$$a_i^\dagger |n_i\rangle = \sqrt{n_i+1}\, |n_i+1\rangle$$

$$a_i |n_i\rangle = \sqrt{n_i}\, |n_i-1\rangle$$

$$\boxed{[a_i^\dagger, a_j] = -\delta_{ij}}$$

Fermions

sign to be determined

$$c_i^\dagger |0_i\rangle = (\pm) |1_i\rangle \qquad c_i |1_i\rangle = (\pm) |0_i\rangle$$

$$c_i^\dagger |1_i\rangle = 0 \qquad c_i |0_i\rangle = 0$$

they are defined such that

$$\{c_i, c_j^\dagger\} = \delta_{ij} \qquad \{A,B\} = AB+BA.$$

$$\{c_i, c_j\} = 0 = \{c_i^\dagger, c_j^\dagger\}$$

Sign cannot be + $\Rightarrow$ they commute.

We can choose, for example:

$$c_i^\dagger |n_1, n_2, \dots 0_i, \dots\rangle = (-)^{\sum_{j<i} n_j} |n_1, n_2, \dots 1_i \dots\rangle$$

$$c_i |n_1, n_2, \dots 1_i, \dots\rangle = (-)^{\sum_{j<i} n_j} |n_1, n_2, \dots 0_i \dots\rangle$$

$$|1_{i_1} \dots 1_{i_n} \dots\rangle = c_{i_n}^\dagger \dots c_{i_2}^\dagger c_{i_1}^\dagger |0\,0\,0 \dots 0\rangle$$

requires ordering the states.

$c_2^\dagger c_1^\dagger |00\rangle = -c_1^\dagger c_2^\dagger |00\rangle$

$c_2^\dagger |10\rangle = -|11\rangle$

$c_1^\dagger c_2^\dagger |00\rangle = c_1^\dagger |01\rangle = |11\rangle$

$c_1 c_1^\dagger |00\rangle = c_1 |10\rangle = |00\rangle$

$c_1 c_1^\dagger |10\rangle = 0$

$c_1 c_1^\dagger + c_1^\dagger c_1 = 1 \Rightarrow \begin{cases} c_1^\dagger c_1 |00\rangle = 0 \\ c_1^\dagger c_1 |10\rangle = |10\rangle \end{cases}$

$\boxed{n_i = c_i^\dagger c_i}$

Example of Hamiltonian. (e^-)

$$H = \sum_i \frac{P_i^2}{2m} + \sum_{i<j} V(\vec{r}_i - \vec{r}_j) + \sum_i U(\vec{r}_i)$$

all masses equal.

potential between particles e.g. $V = \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$

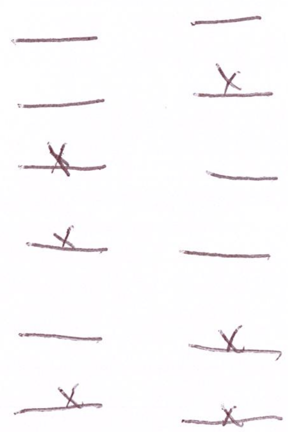
external potential e.g. lattice.

Single particle states: we can take $\psi = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}}$

$\vec{k} = \frac{2\pi}{L}\vec{n}$; $\epsilon_k = \frac{\hbar^2 k^2}{2m}$

or we can find eigenfunctions of $\frac{p^2}{2m} + U(\vec{r})$, depends on the problem.

Matrix elements



$\langle \psi_2 |$ $\quad | \psi_1 \rangle$

if two or more electrons are in different states then

$$\langle \psi_2 | U(\vec{r}_i) | \psi_1 \rangle = 0 .$$

only affects one electron.

$$\langle \psi^{k'} | U(r_i) | \psi^{k} \rangle = \frac{1}{V} \int d^3r \, e^{-i(\vec{k}'-\vec{k})\vec{x}} \, U(\vec{x}) = U(\vec{k}-\vec{k}')$$

$$H_{\substack{\text{single}\\ \text{particle}}} = \sum_{\vec{k}} \frac{\hbar^2 k^2}{2m} c_k^{\dagger} c_k + \sum_{\vec{k},\vec{k}'} U(\vec{k}-\vec{k}') \, c^{\dagger}_{\vec{k}'} c_{\vec{k}}$$

removes e^- from k

puts it in k'.

($\vec{k}$ can be equal to k')

$V(\vec{r}_i - \vec{r}_j)$ affects two particles.

$$\langle \vec{k}_1' \vec{k}_2' | V(r_1 - r_2) | k_1 k_2 \rangle = \frac{1}{V^2} \int d^3r_1 d^3r_2 \, e^{-i(\vec{k}_1'-\vec{k}_1)\vec{r}_1 - i(\vec{k}_2'-\vec{k}_2)\vec{r}_2} \, V(\vec{r}_1 - \vec{r}_2)$$

$$= V(\vec{k}_1', k_2'; k_1, k_2)$$

$$H = \sum_{k} \frac{\hbar^2 k^2}{2m} c_k^{\dagger} c_k + \sum_{\vec{k},\vec{k}'} U(\vec{k}-\vec{k}') c^{\dagger}_{k'} c_k + \sum_{\substack{k_1 k_2 \\ k_1' k_2'}} V_{k_1 k_2 k_1' k_2'} \, c^{\dagger}_{k_2'} c^{\dagger}_{k_1'} c_{k_1} c_{k_2}$$

We can also define

$$\psi^\dagger(x) = \sum_k \frac{e^{-i\vec{k}\vec{x}}}{\sqrt{V}} c_k^\dagger \quad ; \quad \psi(x) = \sum_k \frac{e^{i\vec{k}\vec{x}}}{\sqrt{V}} c_k$$

$$\int_0^L dx\, e^{-ik'x+ikx} = \left.\frac{e^{i(k'-k)x}}{i(k'-k)}\right|_0^L = 0 \qquad \left(k'-k = \frac{2\pi n}{L}\right)$$

$\uparrow_{k' \neq k}$; n integer

$$\boxed{\begin{gathered}\sum_k e^{ikx} = A\,\delta(x) = V\delta(x) \\ A\int_0^L \delta(x) = A = \sum_k \int e^{ikx} = V \\ \text{only } \vec{k}=0 \\ \boxed{\sum_k |k\rangle\langle k| = \mathbb{1}}\end{gathered}}$$

if $k'=k \rightarrow \int_0^L dx = L$

$$\frac{1}{V}\int d^3r\, e^{-i\vec{k}'\vec{r}+i\vec{k}\vec{r}} = \delta(\vec{k}-\vec{k}') = \langle k'|k\rangle$$

$$\frac{1}{\sqrt{V}}\int d^3r\, e^{+i\vec{k}'\vec{r}}\psi^\dagger(\vec{r}) = \frac{1}{\sqrt{V}}\int d^3r \sum_k e^{+i(k'-k)\vec{r}}\frac{c_k^\dagger}{\sqrt{V}} = c_{k'}^\dagger$$

$$\boxed{c_k^\dagger = \frac{1}{\sqrt{V}}\int d^3r\, e^{+i\vec{k}\vec{r}}\psi^\dagger(\vec{r})}$$

$$\{\psi^\dagger(x), \psi(y)\} = \sum_{kk'} \frac{1}{V} e^{+i\vec{k}\vec{x} - i\vec{k}'\vec{y}} \underbrace{\{c_k^\dagger, c_{k'}\}}_{\delta_{k-k'}} = \frac{1}{V}\sum_{\vec{k}} e^{+i\vec{k}(\vec{x}-\vec{y})}$$

$$= \cancel{}\ \delta(\vec{x}-\vec{y})$$

$$H_{s.p.} = \sum_{k} \frac{\hbar^2 k^2}{2m} \int d^3x\, d^3y \,\frac{1}{V}\, e^{i\vec{k}\vec{x}}\, \psi^\dagger(\vec{x})\, e^{-i\vec{k}\vec{y}}\, \psi(\vec{y})$$

$$+ \sum_{kk'} U(k-k') \int d^3x\, d^3y\, \frac{1}{V}\, e^{i\vec{k}'\vec{x}}\, \psi^\dagger(\vec{x})\, e^{-i\vec{k}\vec{y}}\, \psi(\vec{y})$$

$$\frac{1}{V}\int d^3r\, e^{-i(\vec{k}'-\vec{k})\vec{r}}\, U(\vec{r})$$

$$= \int d^3x\, d^3y\, \frac{1}{V} \sum_{k} \psi^\dagger(\vec{x})\, e^{i\vec{k}\vec{x}} \left(-\frac{\hbar^2 \nabla_y^2}{2m}\right) e^{-i\vec{k}\vec{y}}\, \psi(\vec{y})$$

$$+ \frac{1}{V^2} \int d^3x\, d^3y\, d^3r \sum_{kk'} e^{ik(r-y)}\, e^{ik'(\vec{x}-\vec{r})}\, U(\vec{r})\, \psi^\dagger(x)\, \psi(y)$$

$$= \int d^3x\, \psi^\dagger(\vec{x}) \left(-\frac{\hbar^2 \nabla^2}{2m}\right) \psi(\vec{x}) + \int d^3x\, U(\vec{x})\, \psi^\dagger(x)\, \psi(x)$$

$$H_{int} = \sum_{\substack{k_1 k_2 \\ k_1' k_2'}} \frac{1}{V^2} \int d^3r_1\, d^3r_2\, e^{-i(k_1'-k_1)r_1 - i(k_2'-k_2)r_2}\, \frac{V(\vec{r}_1-\vec{r}_2)}{V^2} \int d^3x_1 \ldots d^3x_4\, e^{ik_1'x_1}\, e^{ik_2'x_2}\, \cdot$$

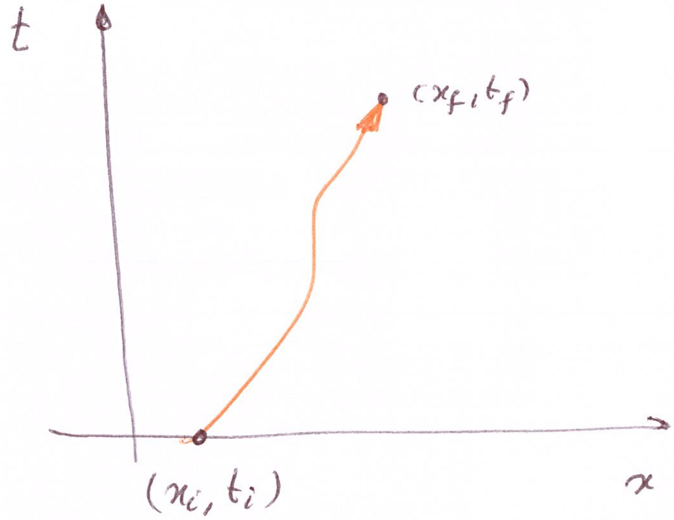
$$\times e^{-ik_1x_3 - ik_2x_4}\, \psi^\dagger(x_2)\, \psi^\dagger(x_1)\, \psi(x_3)\, \psi(x_4)$$

$$= \frac{\cancel{V^4}}{\cancel{V^4}} \int d^3r_1\, d^3r_2\, V(\vec{r}_1-\vec{r}_2) \int d^3x_1 \ldots d^3x_4\, \psi^\dagger(x_2)\, \psi^\dagger(x_1)\, \psi(x_3)\, \psi(x_4)\, \delta(r_1-x_1)\, \delta(r_1-x_3)\, \delta(x_2-r_2)\, \delta(r_2-$$

$$= \int d^3r_1\, d^3r_2\, V(\vec{r}_1-\vec{r}_2)\, \psi^\dagger(r_2)\, \psi^\dagger(r_1)\, \psi(r_1)\, \psi(r_2) \quad \leftarrow \text{normal order}$$

$$= + \int d^3r_1\, d^3r_2\, \underbrace{\psi^\dagger(r_2)\psi(r_2)}_{\rho(r_1)}\, V(r_1-r_2)\, \underbrace{\psi^\dagger(r_1)\psi(r_1)}_{\rho(r_2)}\, \underbrace{\cancel{+} \int d^3r_1\, V(0)\, \psi^\dagger(r_1)\psi(r_1)}_{\text{extra term}}$$

Path integral in Quantum mechanics



What is the probability of finding a particle at position x_f at time t_f if it was at x_i at time t_i?

Standard QM: $\langle x_f, t_f | x_i, t_i \rangle = \langle x_f | e^{-i\frac{H}{\hbar}(t_f - t_i)} | x_i \rangle$

$$\text{probability} = |\langle x_f, t_f | x_i, t_i \rangle|^2$$

Define $K(x_f\, t_f ; x_i\, t_i) = \langle x_f | e^{-i\frac{H}{\hbar}(t_f - t_i)} | x_i \rangle$

↑ propagator. Determines time evolution:

$$\langle \psi_f, t_f | \psi_i, t_i \rangle = \int dx_i\, dx_f \; \langle \psi_f \cancel{t_f} | x_f \rangle \langle x_f | e^{-i\frac{H}{\hbar}(t_f - t_i)} | x_i \rangle \langle x_i | \psi_i \rangle$$

$$= \int dx_i\, dx_f \; \psi_f^*(x_f)\, \psi_i(x_i)\, K(x_f\, t_f ; x_i\, t_i)$$

example: free particle $H = P^2/2m$

$$K = \int_{-\infty}^{\infty} \frac{dp}{\cancel{2\pi\hbar}} \langle x_f | e^{-\frac{ip^2}{2m\hbar}(t_f - t_i)} | p \rangle \langle p | x_i \rangle = \int \frac{dp}{2\pi\hbar}\, e^{-\frac{ip^2}{2m\hbar}\Delta t}\, e^{-\frac{ip(x_f - x_i)}{\hbar}}$$

$$\boxed{K = e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi (t_f - t_i)\hbar}}\; e^{\frac{im}{2\hbar}\frac{(x_f - x_i)^2}{t_f - t_i}}}$$ free particle.

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{-\frac{ip^2\Delta t}{2m\hbar} - \frac{ip(x_f-x_i)}{\hbar}} = ?$$

$$-\frac{i\Delta t}{2m\hbar}\Big(\underbrace{p + \frac{2m\hbar}{2\Delta t}\frac{\Delta x}{\hbar}}_{\tilde p}\Big)^2 + \frac{i\Delta t}{2m\hbar}\left(\frac{2m\hbar}{\Delta t}\right)^2 \frac{1}{4}\frac{\Delta x^2}{\hbar^2} = -\frac{i\Delta t}{2m\hbar}\tilde p^2 + i\frac{2m\hbar}{\Delta t}\frac{\Delta x^2}{4\hbar^2}$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\tilde p\; e^{-\frac{i\Delta t}{2m\hbar}\tilde p^2}\, e^{\frac{i}{2}\frac{m}{\Delta t}\frac{\Delta x^2}{\hbar}}$$

$$\int_{-\infty}^{\infty} d\tilde p\; e^{-ia\tilde p^2} = \int_{-\infty}^{\infty} d\tilde p\, \cos(a\tilde p^2) - i\int_{-\infty}^{\infty} \sin(a\tilde p^2)$$

$$= 2\left(\int_0^{\infty} d\tilde p\, \cos(a\tilde p^2) - i\int_0^{\infty} \sin(a\tilde p^2)\right) = \frac{2}{2}\sqrt{\frac{\pi}{2a}}\,(1-i)$$

$a>0$ (table of integrals)

$$= \sqrt{\frac{\pi}{a}}\, e^{-i\pi/4}$$

Then

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{-\frac{ip^2\Delta t}{2m\hbar} - \frac{ip(x_f-x_i)}{\hbar}} = \sqrt{\frac{\pi\, 2m\hbar}{\Delta t}}\; e^{-\frac{i\pi}{4}}\; \frac{e^{\frac{i}{2}\frac{m}{\Delta t}\frac{\Delta x^2}{\hbar}}}{2\pi\hbar}$$

$$= \sqrt{\frac{m}{2\pi\hbar\Delta t}}\; e^{-\frac{i\pi}{4}}\; e^{\frac{1}{2}i\frac{m\Delta x^2}{\hbar\Delta t}}$$

Some comments:

◦) action $S = \int dt \left(\frac{1}{2}mv^2 - V(x)\right)$

Here $V=0$ $\rightarrow S = \int dt \frac{1}{2}mv^2$

a free particle moves at constant velocity $v = \frac{x_f - x_i}{t_f - t_i}$

$$S = \frac{1}{2} \cancel{\Delta t}\, m \frac{\Delta x^2}{\Delta t^{\cancel{2}}} \quad \Rightarrow \quad K \sim e^{i\frac{S}{\hbar}}$$

◦◦) $t_f \rightarrow t_i$ $\quad K \rightarrow \delta(x_f - x_i) = \langle x_f | x_i \rangle$

$$\frac{e^{-i\pi/4}\, e^{ix^2/\sigma}}{\sqrt{\pi\sigma}} \xrightarrow[\sigma \to 0]{} \delta(x) \qquad \left(\text{e.g. } \int_{-\infty}^{\infty} \frac{e^{-i\pi/4}\, e^{ix^2/\sigma}}{\sqrt{\pi\sigma}} f(x)\,dx = \right.$$

$$\left. = \int_{-\infty}^{\infty} \frac{e^{-i\pi/4}\, e^{ix^2}}{\sqrt{\pi}} \underbrace{f(x\sqrt{\sigma})}_{\rightarrow f(0)}\, dx \underset{\sigma \to 0.}{=} f(0) \right)$$

Harmonic oscillator.

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \quad ; \qquad |n\rangle \ , \quad E = \frac{1}{2}\hbar\omega(2n+1)$$

$$\langle x_f | e^{-i\frac{H}{\hbar}(t_f - t_0)} | x_i \rangle = \sum_n \langle x_f | n \rangle \langle n | x_i \rangle \, e^{-i\omega(n+\frac{1}{2})\Delta t}$$

Hermite polynomials $\times$ gaussian.

~~Another way~~

$$\langle n | x \rangle = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x \right)$$

$$H_n(x) = (-)^n e^{x^2} \left(\frac{\partial}{\partial x} \right)^n e^{-x^2} = (-)^n e^{x^2} \frac{1}{\sqrt{\pi}} \left(\frac{\partial}{\partial x} \right)^n \int_{-\infty}^{\infty} e^{-s^2} e^{2isx}\, ds = \frac{(-)^n}{\sqrt{\pi}} e^{x^2} \int_{-\infty}^{\infty} (2is)^n e^{-s^2} e^{2isx}\, d[illegible]$$

$$K = \sum_n \frac{1}{2^n n!} \sqrt{\frac{m\omega}{\pi\hbar}}\, e^{-\frac{m\omega}{2\hbar}(x_i^2 + x_f^2)}\, e^{-i\omega n \Delta t}\, e^{-\frac{i\omega\Delta t}{2}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_i \right) H_n\left(\sqrt{\frac{m\omega}{\hbar}} x_f \right)$$

$$\sum_{n\geq 0} \frac{\alpha^n}{n!} H_n(x) H_n(y) = \int \frac{ds\,dt}{\pi} e^{x^2+y^2} \sum_{n\geq 0} \frac{\alpha^n}{n!} (2is)^n (2it)^n e^{2isx + 2ity} e^{-s^2-t^2}$$

$$= \iint \frac{e^{x^2+y^2}}{\pi} \sum_n (-)^n 4^n \frac{(st)^n \alpha^n}{n!} e^{2isx + 2ity - s^2 - t^2} = \frac{e^{x^2+y^2}}{\pi} \int_{-\infty}^{\infty} ds\,dt\, e^{-s^2-t^2+2isx+2ity} e^{-4st\alpha}$$

$$\int_{-\infty}^{\infty} ds\,dt\, e^{-s^2-t^2+2isx+2ity-4st\alpha} = \int d\eta\, d\xi\, \frac{1}{4}\, e^{-(\_\ \_\_)}$$

$$s = (\xi+\eta)/2 \qquad \xi = s+t \quad \eta = s-t \qquad \frac{\partial(s,t)}{\partial(\xi,\eta)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2$$

$$t = (\xi-\eta)/2$$

$$s^2+t^2+4st\alpha = \frac{1}{4}(\xi+\eta)^2+\frac{1}{4}(\xi-\eta)^2+4\,\frac{1}{4}(\xi+\eta)(\xi-\eta)\,\alpha =$$

$$= \frac{1}{4}\left(\xi^2+\eta^2+\cancel{2\xi\eta}+\xi^2+\eta^2-\cancel{2\xi\eta}\right)+(\xi^2-\eta^2)\alpha$$

$$= \frac{1}{2}\xi^2+\frac{1}{2}\eta^2+\xi^2\alpha-\eta^2\alpha = \left(\frac{1}{2}+\alpha\right)\xi^2+\left(\frac{1}{2}-\alpha\right)\eta^2$$

$$\boxed{|\alpha|<1/2}$$

$$2i(sx+ty) = \frac{2i}{2}\left((\xi+\eta)x+(\xi-\eta)y\right)$$

$$= \int \frac{d\eta\, d\xi}{2}\; e^{-\left(\frac{1}{2}+\alpha\right)\xi^2-\left(\frac{1}{2}-\alpha\right)\eta^2}\; e^{i\xi(x+y)+i\eta(x-y)}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{\frac{1}{2}+\alpha}}\sqrt{\frac{\pi}{\frac{1}{2}-\alpha}}\; e^{-\frac{(x+y)^2}{4\left(\frac{1}{2}+\alpha\right)}}\; e^{-\frac{(x-y)^2}{4\left(\frac{1}{2}-\alpha\right)}}$$

$$\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}H_n(x)H_n(y) = \frac{1}{\cancel{\pi}}\,\frac{1}{2}\,\frac{\cancel{\pi}}{\sqrt{\frac{1}{4}-\alpha^2}}\; e^{x^2+y^2-\frac{(x+y)^2}{2(1+2\alpha)}-\frac{(x-y)^2}{2(1-2\alpha)}}$$

$$K = \sqrt{\frac{m\omega}{\hbar\pi}}\; e^{-\frac{m\omega}{2\hbar}(x_i^2+x_f^2)-\frac{i\omega\Delta t}{2}}\;\frac{1}{2}\,\frac{e^{\frac{m\omega}{\hbar}(x_i^2+x_f^2)-\frac{m\omega}{2\hbar}\frac{(x_i+x_f)^2}{(1+2\alpha)}-\frac{m\omega}{\hbar}\frac{(x_i-x_f)^2}{2(1-2\alpha)}}}{\sqrt{\frac{1}{4}-\frac{1}{4}e^{-2i\omega\Delta t}}} \qquad \alpha=\frac{1}{2}e^{-i\omega\Delta t}$$

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exponent: $$\frac{1}{2}\frac{m\omega}{\hbar}(x_i^2+x_f^2)-\frac{m\omega}{2\hbar}\,\frac{(x_i+x_f)^2(1-2\alpha)+(x_i-x_f)^2(1+2\alpha)}{(1-4\alpha^2)}$$

$$= \frac{m\omega}{2\hbar}\left[x_i^2+x_f^2-\frac{2(x_i^2+x_f^2)-8\alpha x_i x_f}{1-4\alpha^2}\right] = \frac{m\omega}{2\hbar(1-4\alpha^2)}\left[(-1-4\alpha^2)(x_i^2+x_f^2)+8\alpha x_i x_f\right]$$

$$= -\frac{m\omega}{2\hbar(1-4\alpha^2)}\left((1+4\alpha^2)(x_i^2+x_f^2) - 8\alpha x_i x_f\right)$$

$$\alpha = \frac{1}{2}e^{-i\omega\Delta t}$$

$$1-4\alpha^2 = 1-e^{-2i\omega\Delta t} = 2ie^{-i\omega\Delta t}\left(\frac{e^{i\omega\Delta t}-e^{-i\omega\Delta t}}{2i}\right) = 2ie^{-i\omega\Delta t}\sin(\omega\Delta t)$$

$$1+4\alpha^2 = 1+e^{-2i\omega\Delta t} = 2e^{-i\omega\Delta t}\cos(\omega\Delta t)$$

$$= -\frac{m\omega}{2\hbar\, 2i\,\cancel{e^{-i\omega\Delta t}}\sin(\omega\Delta t)}\left[2\cancel{e^{-i\omega\Delta t}}\cos(\omega\Delta t)(x_i^2+x_f^2) - 4\cancel{e^{-i\omega\Delta t}}x_i x_f\right]$$

$$= \frac{im\omega}{2\hbar\sin(\omega\Delta t)}\left[(x_i^2+x_f^2)\cos(\omega\Delta t) - 2x_i x_f\right] \leftarrow \text{exponent}$$

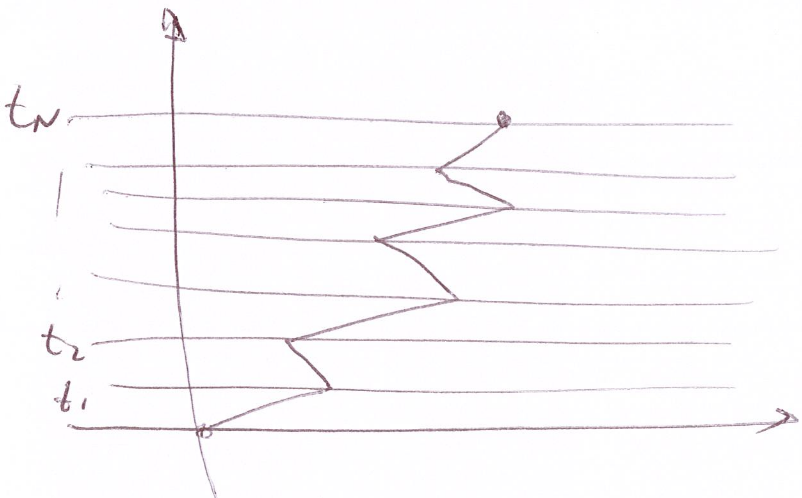
$$K = \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{e^{-i\omega\Delta t/2}}{\cancel{2}\sqrt{1-e^{-2i\omega\Delta t}}}\,e^{\frac{im\omega}{2\hbar\sin(\omega\Delta t)}\left[(x_i^2+x_f^2)\cos\omega\Delta t - 2x_i x_f\right]}$$

$$K = \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{e^{-i\pi/4}}{\cancel{2}\sqrt{2\sin(\omega\Delta t)}}\,e^{\frac{im\omega}{2\hbar\sin(\omega\Delta t)}\left[(x_i^2+x_f^2)\cos\omega\Delta t - 2x_i x_f\right]}$$

$$= \sqrt{\frac{m\omega}{2\pi\hbar\sin(\omega\Delta t)}}\,e^{-i\pi/4}\,e^{\frac{im\omega}{2\hbar\sin(\omega\Delta t)}\left[(x_i^2+x_f^2)\cos\omega\Delta t - 2x_i x_f\right]}$$

$\omega \to 0$ reduces to free particle ✓

Path integral approach to the propagator



$$\langle x_f | e^{-iH\frac{\Delta t}{\hbar}} | x_i \rangle = \int dx_1 .. dx_{N-1} \langle x_f | e^{-\frac{iH}{\hbar}\frac{\Delta t}{N}} | x_{N-1} \rangle \langle x_{N-1} | - -$$

$$- | x_1 \rangle \langle x_1 | e^{-\frac{iH}{\hbar}\frac{\Delta t}{N}} | x_i \rangle$$

$$= \int dx_1 -- dx_{N-1} \; K(x_f t_N ; x_{N-1} t_{N-1}) -- \; K(x_1 t_1 ; x_i t_i)$$

We can get the propagator K from convoluting many infinitesimal propagators (when $N \to \infty$)

For a very short $t_f - t_i$; $t_f - t_i \to 0$

$$K(x_f t_f ; x_i t_i) = \langle x_f | e^{-\frac{i p^2}{2m}\Delta t - iV(x)\Delta t} | x_i \rangle \simeq$$

$$\underset{\substack{\uparrow \\ \text{order } \Delta t}}{\simeq} \langle x_f | e^{-\frac{ip^2}{2m}\Delta t} e^{-iV(x)\Delta t} | x_i \rangle =$$

$$= e^{-iV(x_i)\Delta t} \underbrace{\langle x_f | e^{-\frac{ip^2}{2m}\Delta t} | x_i \rangle}_{\text{free prop}} =$$

$$= e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi(t_f-t_i)}} \; e^{\frac{im}{2\hbar}\frac{(x_f-x_i)^2}{t_f-t_i} - iV(x_i)\Delta t}$$

↑ for $\Delta t \to 0$.

action (in Δt)

$$K(x_f t_f; x_i t_i) \simeq e^{-\frac{i\pi}{4}} \sqrt{\frac{m}{2\pi\Delta t}} \; e^{i\frac{S}{\hbar}}$$

$t_f - t_i = \Delta t \to 0$

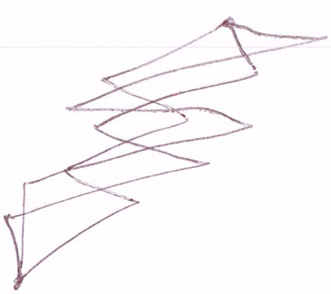
^ normalization indep. of x_i, x_f.

So, we can reinterpret

action of path $x_i \to x_1 \to \cdots \to x_N$

$$\langle x_f | e^{-\frac{iH\Delta t}{\hbar}} | x_i \rangle = \int dx_1 \cdots dx_{N-1} \, \mathcal{N} \, e^{i\frac{S}{\hbar}}$$

$$= \int \mathcal{D}x(t) \, e^{\frac{i}{\hbar}\int_{t_i}^{t_f} \left(\frac{1}{2} m \dot{x}^2 - V(x)\right) dt}$$



sum over all paths.

The differential $\mathcal{D}x(t)$ absorbs the normalization but it is not very well defined.

Euclidean continuation $t \to -i\tau$

$$\int \mathcal{D}x(\tau)\; e^{\frac{i}{\hbar}\int_{\tau_i}^{\tau_f}\left(-\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 - V(x)\right)(-i)\,d\tau}$$

$$= \int \mathcal{D}x(\tau)\; e^{-\frac{1}{\hbar}\int_{\tau_i}^{\tau_f}\underbrace{\left(\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right)}_{\text{energy!}}d\tau}$$

Better defined.

If $x(\tau)$ are interpreted as configurations, this defines a statistical mechanics of paths.