Symmetries.

Charge.

$$Q_{ab} = i\int d^3x\,(\Pi_a \phi_b - \phi_a^* \overline{\Pi}_b) \qquad ((Q_{ab}^\dagger = Q_{ba}))$$

Take $Q_\epsilon = \epsilon^{ab} Q_{ab}$

$$[Q_\epsilon, \phi_c(\vec y)] = i\int d^3x\,(-i)\,\delta_{ac}\,\delta^{(3)}(\vec x - \vec y)\,\phi_b\,\epsilon_{ab} =$$

$$= \epsilon_{cb}\,\phi_b(\vec y) = -i\delta_\epsilon\, \phi_b(\vec y)$$

recall $\delta_\epsilon \phi_a = i\epsilon_{ab}\phi_b$

~~What about the currents?~~

$$j^\mu_\epsilon =$$

$$[Q_{ab}, Q_{cd}] = -\int d^3x\,d^3y\,[\Pi_a\phi_b - \phi_a^*\overline{\Pi}_b,\ \Pi_c\phi_d - \phi_c^*\overline{\Pi}_d]$$

$$= -\int d^3x\,\Big(-i\delta_{ad}\,\phi_b\Pi_c + i\delta_{bc}\,\Pi_a\phi_d + i\,\delta_{ad}\,\overline{\Pi}_b\phi_c^\dagger - i\delta_{bc}\,\phi_c^\dagger\overline{\Pi}$$

$$= i\delta_{ad}\int d^3x\,(\phi_b\Pi_c - \overline{\Pi}_b\phi_c^\dagger) + i\delta_{bc}\int d^3x\,(-\Pi_a\phi_d + \phi_a^\dagger\overline{\Pi}_d)$$

$$= \delta_{ad}\, Q_{cb} - \delta_{bc}\, Q_{ad}$$

same algebra as the group SU(N))

Commutator of charges follow group algebra.

What about currents?

$$j^{\mu}_{ab} = i\left(\partial_\mu \phi^*_a \phi_b - \phi^*_a \partial_\mu \phi_b\right)$$

$$[j^k_{ab}, j^l_{cd}] = 0$$ since no time-derivatives.

(But in general is not zero)

(interacting theory)

$\vec{x}$, $\vec{y}$,

$$[j^0_{ab}, j^0_{cd}] = -[\pi_a \phi_b - \phi^*_a \bar{\pi}_b, \pi_c \phi_d - \phi^*_c \bar{\pi}_d]$$

$$= -(-i)\delta_{ad}\delta^{(3)}(\vec{x}-\vec{y})\phi_b \pi_c - i\delta_{bc}\delta^{(3)}(\vec{x}-\vec{y})\pi_a \phi_d$$

$$- \delta_{bc}(-i)\delta^{(3)}(\vec{x}-\vec{y})\phi^*_a \bar{\pi}_d - i\,\delta_{ad}\delta^{(3)}(x-y)\bar{\pi}_b \phi^*_c$$

$$= i\delta_{ad}\delta^{(3)}(\vec{x}-\vec{y})(\phi_b \pi_c - \bar{\pi}_b \phi^*_c) - i\delta_{bc}\delta^{(3)}(\vec{x}-\vec{y})(\pi_a \phi_d - \phi^*_a \bar{\pi}_d)$$

$$= \delta^{(3)}(\vec{x}-\vec{y})\left(\delta_{ad}\, j^0_{cb} - \delta_{bc}\, j^0_{ad}\right)$$ Same as charge algebra.

$\vec{x}$ $\vec{y}$

$$[j^0_{ab}, j^l_{cd}] = -[\pi_a \phi_b - \phi^*_a \bar{\pi}_b, \nabla\phi^*_c \phi_d - \phi^*_c \nabla\phi_d]$$

$$= i\delta_{ad}\left(\delta(\vec{x}-\vec{y})\, \underset{x}{\phi_b} \underset{y}{\nabla\phi^*_c} - \nabla_y \delta(x-y)\, \underset{x}{\phi_b} \underset{y}{\phi^*_c}\right)$$

$$+ i\delta_{bc}\left(-\nabla_y \delta(x-y)\, \underset{x}{\phi^*_a} \underset{y}{\phi_d} + \underset{x}{\phi^*_a} \underset{y}{\nabla\phi_d}\, \delta(x-y)\right)$$

$$\nabla_y \delta(x-y)\phi_b(x)\phi^*_c(y) = -\nabla_x \delta(x-y)\phi_b(x)\phi^*_c(y) \underset{\varphi}{=} \nabla_x \phi_b(x)\phi^*_c(x)\delta(x-y) - \nabla_x \delta(x-y)\phi_b(y)\phi^*_c(y)$$

$\int$ by parts against arbitrary function $F(x)$

$$[j^0_{ab}, j^l_{cd}] = i\delta_{ad}\delta(\vec{x}-\vec{y})\left(\underbrace{\phi_b \nabla\phi^*_c - \nabla\phi\,\phi^*_c}_{\vec{x}}\right) + i\delta_{bc}\left(-\nabla_x \phi^*_a \phi_d \delta(x-y) + \right.$$

$$\left. + \phi^*_a \nabla\phi_d \delta(x-y)\right) + i\delta_{ad}\nabla_x \delta(x-y)\phi_b(y)\phi^*_c(y) -$$

$$+ i\delta_{bc}\nabla_x \delta(x-y)\phi^*_a(y)\phi_d(y)$$

$$[j^0_{ab}(\vec{x}),\ j^l_{cd}(\vec{y})] = \delta_{ad}\,\delta^{(3)}(\vec{x}-\vec{y})\, j^l_{cb} - \delta_{bc}\,\delta^{(3)}(\vec{x}-\vec{y})\, j^l_{ad}$$

$$+ i\, \nabla_{\vec{x}}\delta^{(3)}(\vec{x}-\vec{y})\, \left(\delta_{ad}\, \phi_b(\vec{y})\phi_c^*(\vec{y}) + \delta_{cd}\, \phi_a^*(\vec{y})\phi_d(\vec{y}) \right)$$

Schwinger term.

total $\vec{x}$ derivative

$$[Q_{ab},\ j^l_{cd}(y)] = \int d^3x\, [j^0_{ab}(\vec{x}),\ j^l_{cd}(\vec{y})] =$$

$$= \delta_{ad}\, j^l_{cb}(\vec{y}) - \delta_{bc}\, j^l_{ad}(\vec{y}) + i \underbrace{\int d^3x\, \nabla_{\vec{x}}\delta^{(3)}(\vec{x}-\vec{y})}_{0} \cdot \underbrace{(\delta_{ad}\phi\phi - \delta_{cd}\phi\phi)}_{(\vec{y})}$$

Q_{ab} generates group transf. ✓.

Schwinger term is necessary. In general.

$$[j^0_{ab}(\vec{x}),\ j^l_{cd}(\vec{y})] = \delta_{ad}\,\delta^{(3)}(\vec{x}-\vec{y})\, j^l_{cb}(\vec{y}) - \delta_{bc}\,\delta^{(3)}(\vec{x}-\vec{y})\, j^l_{ad}(\vec{y})$$

$$+ i\, \nabla_{\vec{x}_n}\delta^{(3)}(\vec{x}-\vec{y})\, S^{ln}_{abcd}(\vec{y})$$

Schwinger term.

total derivative

exact form depends ~~on~~ field theory.

If $S=0$ there is a problem $\Rightarrow S \neq 0$.

Indeed, consider operator $\mathcal{O}$.

$$\dot{\mathcal{O}} = i[H, \mathcal{O}]$$

$$\langle 0|[\mathcal{O}, \dot{\mathcal{O}}]|0\rangle = \langle 0|\mathcal{O}\dot{\mathcal{O}} - \dot{\mathcal{O}}\mathcal{O}|0\rangle =$$

insert identity of energy eigenvectors

$$= \sum_n \langle 0|\mathcal{O}|n\rangle\langle n|i[H,\mathcal{O}]|0\rangle -$$

$$-\sum_n \langle 0|i[H,\mathcal{O}]|n\rangle\langle n|\mathcal{O}|0\rangle =$$

$$= 2i\sum_n E_n |\langle 0|\mathcal{O}|n\rangle|^2 \neq 0 \quad \text{unless } \langle 0|\mathcal{O}|n\rangle = 0 \quad \forall n$$

Now

$$[j^0_{ab}(\vec{x}), j^l_{cd}(\vec{y})] = \delta_{ad}\,\delta^{(3)}(\vec{x}-\vec{y})\, j^l_{cb}(\vec{y}) - \delta_{bc}\,\delta^{(3)}(\vec{x}-\vec{y})\, j^l_{ad}(\vec{y}) + i\,\nabla_{x_n}\delta^{(3)}(\vec{x}-\vec{y})\, S^{ln}_{abcd}(\vec{y})$$

~~therefore~~ $\dfrac{\partial}{\partial y_l}$ $\quad [j^0_{ab}(\vec{x}), \underbrace{\partial_l j^l_{cd}(\vec{y})}_{=-\partial_0 j^0_{cd}(\vec{y})}]$

So, for $\vec{x} = \vec{y}$, $a=c$ $b=d$

we have $\langle 0|[j^0_{ab}(\vec{x}), \partial_0 j^0_{ab}(\vec{x})]|0\rangle \neq 0$ but on right hand side

we have $\langle 0|[j^0_{ab}(\vec{x}), \partial_l j^l_{ab}(\vec{x})]|0\rangle = \cancel{\delta_{ab}\langle 0|\frac{\partial}{\partial y_l}\delta^{(3)}(\vec{x}-\vec{y})\, j^l_{ab}(\vec{y})|0\rangle} -$

$$\cancel{-\delta_{bc}\frac{\partial}{\partial y_l}\left(\delta^{(3)}(x-y)\langle 0|j^l_{ab}(y)|0\rangle\right)} + i\frac{\partial}{\partial y_l}\left(\nabla_{x_n}\delta^{(3)}(\vec{x}-\vec{y})\langle 0|S^{ln}_{abab}|0\rangle\right) = 0$$

if $S=0$

example SU(2) ; $[Q_{ab}, Q_{cd}] = \delta_{ad} Q_{cb} - \delta_{bc} Q_{ad}$

Q_{ab} ; $\sum_a Q_{aa} = U(1) = Q.$; $Q_{11}, Q_{12}, Q_{21}, Q_{22}$

$[Q_{11}, Q_{12}] = -Q_{12}$ $\qquad$ $[Q_{12}, Q_{21}] = Q_{22} - Q_{11}$

$[Q_{11}, Q_{21}] = Q_{21}$ $\qquad$ $[Q_{12}, Q_{22}] = -Q_{12} \Rightarrow [Q_{22}, Q_{12}] = +Q_{12}$

$[Q_{11}, Q_{22}] = 0$ $\qquad$ $[Q_{21}, Q_{22}] = +Q_{21} \Rightarrow [Q_{22}, Q_{21}] = -Q_{21}$

$$[Q_{11} + Q_{22}, Q_{11}] = 0$$
$$Q_{22} = 0$$
$$Q_{12} = 0$$
$$Q_{21} = 0$$

$$[Q_{11} - Q_{22}, Q_{12}] = -2Q_{12}$$
$$[Q_{11} - Q_{21}, Q_{21}] = 2Q_{21}$$
$$[Q_{12}, Q_{21}] = -(Q_{11} - Q_{22})$$

$Q = Q_{11} + Q_{22}$ commutes with everything.

$[J_x, J_y] = iJ_z$

$[J_y, J_z] = iJ_x$

$[J_x, J_z] = -iJ_y$

$J_\pm = J_x \pm iJ_y$

$[J_z, J_+] = iJ_y + i(-J_x)i = J_x + iJ_y = J_+$

$[J_z, J_-] = -J_-$

$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y] =$

$\cancel{= -i\,iJ_z + iJ_z = -2iJ_z}$

$= -i\,iJ_z - i\,iJ_z = 2J_z$

$J_z = \frac{1}{2}(Q_{11} - Q_{22})$

$Q_{12} = J_-$ $\qquad$ $Q_{21} = J_+$

$[J_+, J_-] = [Q_{21}, Q_{12}] = 2J_z$ ✓.

$$\boxed{SU(2) \times U(1)}$$

$$j^0_{ab} = i\,(\pi_a \phi_b - \overline{\pi}_b \phi_a^{\dagger})$$

$$\frac{i\,i}{(2\pi)^3}\int d^3k\,d^3k' \Big(\frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\,(c^\dagger_{k,a}\,e^{-ikx} - d_{k,a}\,e^{ikx})(c_{k',b}\,e^{ik'x} + d^\dagger_{k',b}\,e^{-ik'x})$$

$$+\frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\,(c_{k,b}\,e^{ikx} - d^\dagger_{k,b}\,e^{-ikx})(c^\dagger_{k',a}\,e^{-ik'x} + d_{k',a}\,e^{ik'x})\Big)$$

$$j^0_{ab}|0\rangle = -\frac{1}{(2\pi)^3}\int d^3k\,d^3k'\,\frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\Big(c^\dagger_{k,a}c_{k',b}\,e^{i(k'-k)x} + c^\dagger_{k,a}d^\dagger_{k',b}\,e^{-i(k+k')x}$$

$$-d_{k,a}c_{k',b}\,e^{i(k+k')x} - d_{k,a}d^\dagger_{k',b}\,e^{i(k-k')x} + c_{k,b}c^\dagger_{k',a}\,e^{i(k-k')x}$$

$$+c_{k,b}d_{k',a}\,e^{i(k+k')x} - d^\dagger_{k,b}c^\dagger_{k',a}\,e^{-i(k+k')x} - d^\dagger_{k,b}d_{k',a}\,e^{-i(k-k')x}\Big)$$

$$:j^0_{ab}:|0\rangle = -\frac{1}{(2\pi)^3}\int d^3k\,d^3k'\,\frac{1}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\Big(c^\dagger_{k,a}d^\dagger_{k',b}\,e^{-i(k+k')x} -$$

$$- c^\dagger_{k',a}d^\dagger_{k,b}\,e^{-i(k+k')x}\Big)|0\rangle$$

$$= -\frac{1}{(2\pi)^3}\int d^3k\,d^3k'\,\frac{1}{2}\,c^\dagger_{k,a}d^\dagger_{k',b}\,e^{-i(k+k')x}\underbrace{\left(\sqrt{\frac{\omega_k}{\omega_{k'}}} - \sqrt{\frac{\omega_{k'}}{\omega_k}}\right)}_{\frac{\omega_k-\omega_{k'}}{\sqrt{\omega_k\omega_{k'}}}}|0\rangle$$

$$:j_{ab}:|0\rangle = -\frac{1}{(2\pi)^3}\frac{1}{2}\int d^3k\,d^3k'\,\frac{\omega_k-\omega_{k'}}{\sqrt{\omega_k\omega_{k'}}}\,e^{-i(k+k')x}\,c^\dagger_{k,a}d^\dagger_{k',b}|0\rangle \neq 0$$

$$\neq \alpha|0\rangle$$

Free theory w/ path integrals.

generating function:

$$Z(j) = \int \mathcal{D}\phi(x)\, e^{iS[\phi] + i\int j_a(x)\phi_a(x)\, d^4x.}$$

$$S[\phi] = \int d^4x\, (\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2)$$

Analytic continuation to Euclidean.

$$t = -i\tau = -ix_4$$

$$(\partial_t\phi)^2 \to -(\partial_\tau\phi)^2$$

$$\int d^3x\, dt = -i\int d^3x\, d\tau.$$

$$S_E(\phi) = i\int d^4x\, [(\partial_\tau\phi)^2 + (\nabla\phi)^2 + m^2\phi^2]$$

$$Z[j_a(x), j_a^\dagger(x)] = \int \mathcal{D}\phi_a \mathcal{D}\phi_a^* \, e^{-\int d^4x\, (\partial_\mu\phi_a \partial^\mu\phi_a^* + m^2\phi_a\phi_a^\dagger) + \int j_a(x)\phi_a + \int j_a^\dagger \phi_a^* d^4x.}$$

$$\partial_\mu\phi_a\partial^\mu\phi_a^\dagger = \nabla\phi_a \nabla\phi_a^*$$

$$\uparrow$$

$$x_1 \cdots x_n.$$

Gaussian integral.

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2+bx} = \int_{-\infty}^{\infty} dx\, e^{-a\left(x-\frac{b}{2a}\right)^2} e^{b^2/4a} = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

Multidimensional.

$$\int_{-\infty}^{\infty} dx_i\, e^{-x_i A_{ij} x_j + b_i x_i} = \int_{-\infty}^{\infty} d\xi_\ell\, e^{-\xi_\ell^2 \lambda_\ell + b_i S_{\ell i} \xi_i}$$

$$\left(\begin{array}{l} A = S^t \underset{\uparrow\,\text{diagonal}}{\Lambda} S^{\cancel{-1}} \qquad x_i A_{ij} x_j = x_i S_{ji} \Lambda_{j\ell} \underbrace{S_{\ell m} x_m}_{\xi_\ell} \\ \text{orthogonal.} \\ \xi_\ell = S_{\ell m} x_m \qquad x_m = S_{\ell m} \xi_\ell \\ \dfrac{\partial \xi_\ell}{\partial x_m} = S_{\ell m} \qquad \det \dfrac{\partial \xi_\ell}{\partial x_m} = \det S = 1. \end{array} \right)$$

$$= \prod_\ell \sqrt{\frac{\pi}{\lambda_\ell}}\, e^{\frac{(b_i S_{\ell i})(b_j S_{\ell j})}{4\lambda_\ell}} = \frac{\pi^{n/2}}{(\det A)^{1/2}}\, e^{\frac{1}{4} b^t S^t \Lambda^{-1} S b}$$

$$\boxed{\int dx_i\, e^{-x^t A x + b^t x} = \frac{\pi^{n/2}}{(\det A)^{1/2}}\, e^{\frac{1}{4} b^t A^{-1} b}.}$$

$Z[j] =$

Complex variables.

$$\int dz\,d\bar z\; e^{-a\bar z z+\bar b z+b\bar z} = \int dx\,dy\; e^{-a(x^2+y^2)+(b_1+ib_2)(x+iy)+(b_1+ib_2)(x-iy)}$$

$$= \int dx\,dy\; e^{-a(x^2+y^2)+\cancel{(b_1 x+}\,2b_1 x+2b_2 y}$$

$$= \frac{\pi}{a}\, e^{\frac{4b_1^2+4b_2^2}{4a}} = \frac{\pi}{a}\, e^{\frac{b\bar b}{a}}$$

$$\int dz_i\,d\bar z_i\; e^{-z^\dagger A z+b^\dagger z+z^\dagger b} = \frac{\pi^n}{\det A}\, e^{b^\dagger A^{-1} b}$$

$$Z[j] = \int \mathcal{D}\phi_a\,\mathcal{D}\phi_a^*\; e^{-\int d^4x\,(-\phi_a^*\partial^2\phi_a+m^2\phi_a^*\phi_a)+\int(\bar j\phi_a+j\phi_a^*)}$$

$$= \mathcal{N}\left(\det(-\partial^2+m^2)\right)^{-N} e^{\int \bar j_a(-\partial^2+m^2)^{-1} j_a\, d^4x}$$

$$= e^{W[j]}$$

From the generating function we can compute generic Green functions. In Euclidean:

$$\langle 0| \phi^\dagger_{b_1}(y_1) \cdots \phi^\dagger_{b_n}(y_n)\, \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) |0\rangle =$$

$$= \frac{1}{Z[j,\bar{j}]} \frac{\delta^{(n)}}{\delta j_{a_1}(x_1) \cdots \delta j_{a_n}(x_n)} \frac{\delta^{(n)}}{\delta \bar{j}_{b_n}(y_n) \cdots \delta \bar{j}_{b_1}(y_1)} Z[j,\bar{j}] \Bigg|_{j,\bar{j}=0}$$

We need the same # of ϕ, and $\phi^\dagger$ because of global U(1) conservation.
It can also be seen from the calculation.

Minkowski:

$$\langle 0| \hat{T} \{ \phi^\dagger_{b_1}(y_1) \cdots \phi^\dagger_{b_n}(y_n)\, \phi_{a_1}(x_1) \cdots \phi_{a_n}(x_n) \} |0\rangle = \frac{(-i)^{2n}}{Z[j,\bar{j}]} \frac{\delta^{(2n)}}{\delta j_{a_n}(x_n) \cdots \delta \bar{j}_{b_1}(y_1)} Z[j,\bar{j}] \Bigg|_{j,\bar{j}=0}$$

$$Z[j,\bar{j}] = \langle 0| T \left\{ e^{i\int d^4x \left(j_a(x)\phi_a(x) + \bar{j}_a(x)\phi^\dagger_a(x) \right)} \right\} |0\rangle$$

For free field theory:

$$Z[j,\bar{j}] = \mathcal{N}\, e^{\int \bar{j}_a(x) K_{ab}(x,y)\, j_b(y)\, d^4x\, d^4y}$$

$$K_{ab}(x,y) = \delta_{ab} (-\partial_x^2 + m^2)^{-1} \delta^{(4)}(x-y)$$

$$K_{ab}(x,y) = \delta_{ab} \int d^4k \frac{e^{ik(x-y)}}{k^2+m^2} \quad ; \quad \int e^{+iq_1y_1 - ip_1x_1} K_{ab}(x,y) = \delta_{ab} \frac{\delta^{(4)}(p_1+q_1)}{p_1^2+m^2}$$

$$\frac{\delta Z}{\delta j_{a_1}(x_1)} = N \int \bar{J}_{\tilde{a}_1}(\tilde{x}_1)\, K_{\tilde{a}_1 a_1}(\tilde{x}_1, x_1) \quad e^{\int \bar{J} K j}$$

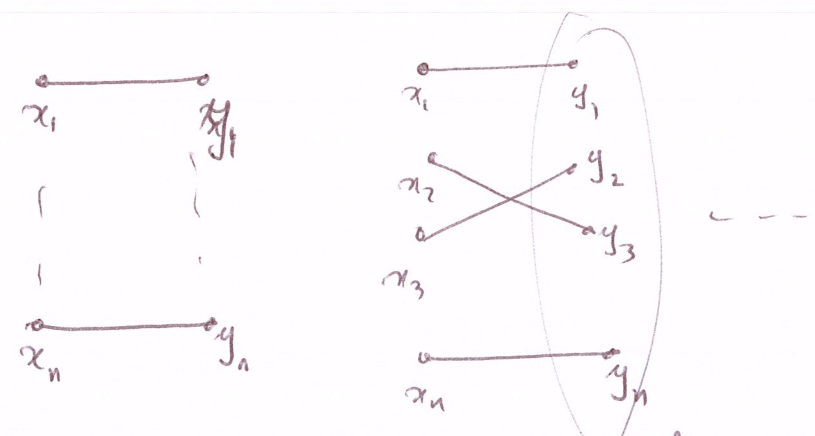
$$= \left(\int \bar{J}_{\tilde{a}_1}(\tilde{x}_1)\, K_{\tilde{a}_1 a_1}(\tilde{x}_1, x_1) \right) Z$$

If we do not put same # of $\delta/\delta \bar{j}$ $\Rightarrow$ when we set $j = \bar{J} = 0$ we get 0.

$$\frac{\delta^{(n)} Z}{\delta j_{a_n}(x_n) \cdots \delta j_{a_1}(x_1)} = \int \bar{J}_{\tilde{a}_1}(\tilde{x}_1)\, K_{\tilde{a}_1 a_1}(\tilde{x}_1, x_1) \cdots \int \bar{J}_{\tilde{a}_n}(\tilde{x}_n)\, K_{\tilde{a}_n a_n}(\tilde{x}_n, x_n)\; Z .$$

Now
$$\frac{\delta^{(n)}}{\delta \bar{J}_{b_n}(y_n) \cdots \delta \bar{J}_{b_1}(y_1)} \Big\downarrow = \sum_{\substack{\text{permutations} \\ \{\sigma_1 \cdots \sigma_n\}}} K_{b_{\sigma_1} a_1}(y_{\sigma_1}, x_1) \cdots K_{b_{\sigma_n} a_n}(y_{\sigma_n}, x_n)$$

$$\frac{1}{Z} \frac{\delta^{(2n)}}{\delta \bar{j}_{b_n}(y_n) \quad \delta j_{a_1}(x_1)} Z \Bigg|_{j_a = \bar{J}_b = 0} = \sum_{\substack{\text{perm.} \\ \{\sigma_1 \cdots \sigma_n\}}} K_{b_{\sigma_1} a_1}(y_{\sigma_1}, x_1) \cdots K_{b_{\sigma_n} a_n}(y_{\sigma_n}, x_n)$$



All diagrams.

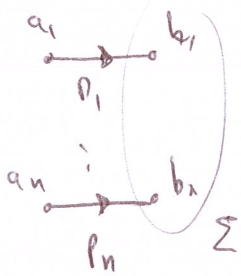
all permutations of these points.

Fourier transform.

$$G_{a_1\dots a_n\, b_1\dots b_n}(x_1\dots x_n, y_1\dots y_n) = \sum_{\substack{\text{perm.}\\ \{\sigma_1\dots\sigma_n\}}} K_{b_{\sigma_1}a_1}(y_{\sigma_1},x_1)\dots K_{b_{\sigma_n}a_n}(y_{\sigma_n},x_n)$$

$$\int e^{+iq_1y_1 - \dots + iq_ny_n - ip_1x_1 - \dots - ip_nx_n}\, d^4x_1 \dots d^4x_n\, d^4y_1 \dots d^4y_n =$$

$$= \sum_{\substack{\text{perm.}\\ \{\sigma_1\dots\sigma_n\}}} \frac{\delta_{b_{\sigma_1}a_1}\dots\delta_{b_{\sigma_n}a_n}\;\delta^{(4)}(q_{\sigma_1}-p_1)\dots\delta^{(4)}(q_{\sigma_n}-p_n)}{(p_1^2+m^2)\dots(p_n^2+m^2)}$$



Σ permutations.

These formulas are called Wick's theorem.

Connected Green functions

$$Z[j,\bar{j}] = e^{W[j,\bar{j}]} \quad ; \quad W[j,\bar{j}] = \ln Z[j,\bar{j}]$$

We only need $W[\bar{j},j] = \int d^4x\, d^4y\, \bar{j}_b(y)\, K_{ba}(y,x)\, j_a(x)$

$$\frac{\delta Z}{Z\,\delta j_a(x)} = \langle \phi_a(x) \rangle_j = \frac{\delta W}{\delta j_a(x)}$$

Free fields:

$$\langle \phi_a(x) \rangle_j = \int d^4y\, \bar{j}_b(y)\, K_{ba}(y,x) \qquad (=0 \text{ if } \bar{j}=0)$$

Effective action

$$\Gamma[\phi_{cl}(x), \bar{\phi}] = -W[j,\bar{j}] + \int d^4x \left(j_a(x)\, \phi_a + \bar{j}_a\, \phi_a^{\dagger} \right)$$

where $j, \bar{J}$ should be set such that $\langle \phi_a \rangle = \phi_a^{cl}$

$\langle \phi_a^{\dagger} \rangle = \bar{\phi}_a^{cl}.$

$$-\frac{\delta \Gamma}{\delta \phi_{cl}(x)} = \cancel{\int_y \underbrace{\frac{\delta W}{\delta j_a(y)}}_{\langle \phi_a(y) \rangle = \phi_a} \frac{\delta j_a(y)}{\delta \phi(x)}} - j_a(x) - \cancel{\int_y \phi_a(y) \frac{\delta j(y)}{\delta \phi(x)}} = -j_a(x)$$

if $j=0$ $\quad \boxed{\delta \Gamma / \delta \phi_{cl}(x) = 0}$ $\quad \langle \phi(x) \rangle = \phi_c(x)$ extremizes the effective action.

For free fields:

$$\Gamma = -\int d^4x\, d^4y\; \bar{J}_b(y) K_{ba}(y,x)\, j_a(x) + \int j_a \phi_a + \bar{J}_a \bar{\phi}_a$$

we have to write it in terms of.

$$\langle \phi_a \rangle_j = \int d^4y\; \bar{J}_b(y) K_{ba}(y,x) = \int d^4y\; J_a(y)\, \Delta(y-x)$$

$$(-\partial^2 + m^2) \langle \phi_a(x) \rangle_j = \bar{J}_a(x) \qquad \left((-\partial^2+m^2)\Delta(y,x) = \delta^{(4)}(y-x) \right)$$

$$(-\partial^2 + m^2) \langle \phi_a^*(x) \rangle = j_a(x)$$

$$\Gamma = -\int d^4x\, d^4y\; (-\partial_y^2+m^2)\langle \phi_b(y)\rangle \underbrace{K_{ba}(y,x)}_{\delta_{ba}(\partial^2-m)^{-1}} (-\partial_x^2+m^2)\,\phi_a^*(x) +$$

$$+ \int (-\partial^2+m^2)\langle \phi_a^* \rangle \phi_a + (-\partial^2+m^2)\phi\, \phi_a^*$$

$$= + \int d^4x\; \phi_a (-\partial_x^2+m^2)\, \phi_a^* = + \int d^4x \left(\partial_\mu \phi_a \partial^\mu \phi_a^* + m^2 \phi_a \phi_a^* \right)$$

For free fields

$$\Gamma_{eff} = S_{cl}$$

More generically.

$$Z[j] = e^{W[j]} = \int \mathcal{D}\phi \, e^{-S[\phi] + \int j\phi}$$

$j /\ \langle \phi \rangle = \phi_{cl}$

take $\phi = \phi_{cl} + \eta \leftarrow \langle \eta \rangle = 0$

$$Z[j] = \int \mathcal{D}\phi \; e^{-S[\phi_{cl}] - \frac{\partial S}{\partial \phi}\eta - \frac{\partial^2 S}{\partial\phi\partial\phi}\eta^2 - \frac{\partial^3 S}{\partial\phi\partial\phi\partial\phi}\eta^3 + \int j\phi_{cl} + \int j\eta}$$

$$= e^{-S[\phi_{cl}] + \int j\phi_{cl}} \int \mathcal{D}\eta \; e^{\int \eta (j - \partial S/\partial\phi) - \int \frac{\partial^2 S}{\partial\phi\partial\phi}\eta^2 - \int \frac{\partial^3 S}{\partial\phi\partial\phi\partial\phi}\eta^3 - \cdots}$$

$$= e^{-S[\phi_{cl}] + \int j\phi_{cl}} \; \tilde{Z}[j]$$

$$W = -S[\phi_{cl}] + \int j\phi_{cl} + \ln \tilde{Z}[j]$$

$$\Gamma_{eff}(\phi_{cl}) = \underset{\substack{\uparrow \\ \text{classical action}}}{S} - \ln \tilde{Z}[j]$$

↑ Quantum correction.

lowest order

$$\int \mathcal{D}\eta \; e^{\int \eta \underbrace{(j - \partial S/\partial\phi)}_{\substack{\text{we should choose it} \\ \text{to be } 0 \,\to\, \langle\eta\rangle = 0}} - \int \frac{\partial^2 S}{\partial\phi\partial\phi}\eta^2} = \det{}^{-1/2}\left(\frac{\partial^2 S}{\partial\phi(x)\partial\phi(y)}\right)$$

Interactions

$$\mathcal{L}_E = (\partial_\mu \phi_a^*)(\partial^\mu \phi_a) + m^2 \phi_a^* \phi_a + \frac{1}{4}(\phi_a^* \phi_a)^2$$

$$\langle\langle \phi_a(x)\phi_b^*(y)\rangle\rangle = \int \mathcal{D}\phi_c\, \mathcal{D}\phi_d^*\; \phi_a(x)\phi_b^*(y)\, e^{-S}$$

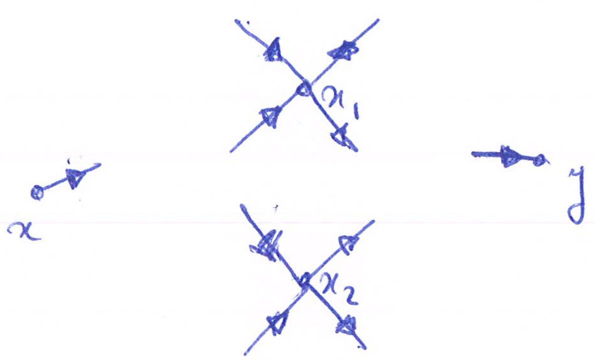
$$= \int \mathcal{D}\phi_c\, \mathcal{D}\phi_d^*\; e^{-\frac{1}{4}\int (\phi_c\phi_c^*)^2 d^4x}\; \phi_a(x)\phi_b^*(y)\, e^{-S_{free}}$$

$$= \sum_{n=0}^{\infty} \frac{(-)^n}{4^n n!} \int d^4x_1\, \phi_{a_1}(x_1)\phi_{a_1}^*(x_1)\phi_{b_1}(x_1)\phi_{b_1}^*(x_1).$$

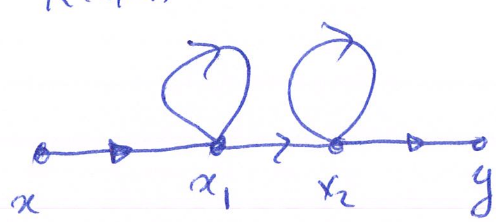
$$\ldots \int d^4x_n\, \phi_{a_n}(x_n)\phi_{a_n}^*(x_n)\phi_{b_n}(x_n)\phi_{b_n}^*(x_n)\; \phi_a(x)\phi_b^*(y)\, e^{-S}.$$

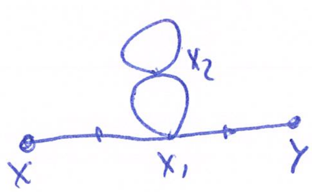
e.g.

Order 2



$K(x,x_1)$





$K(x,x_1)\, K(x_1,x_2)^3\, K(x_2,y)$