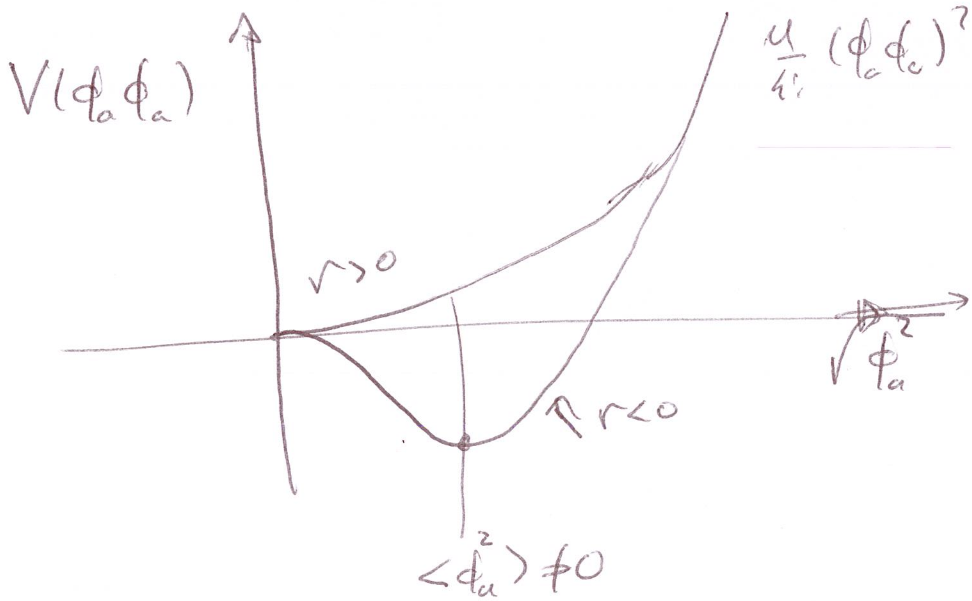
Consider an n-d. SO(N) field theory (scalar):

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_a)(\partial^\mu \phi^a) + \frac{1}{2} r\, \phi_a \phi_a + \frac{u}{4!}(\phi_a \phi_a)^2$$

(m^2 before). sometimes called λ.

$\phi_a \in \mathbb{R}$

this is the same SU(N/2) theory where we explicitly write imaginary and real part.



Classically there is a phase transition when the sign of r changes.

To understand quantum corrections or fluctuation we compute.

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

$$\boxed{S[\phi] = \int d^d x \, \mathcal{L}}$$

②

As a simplification we consider the limit $N \to \infty$

Let's change variables to $\rho = \frac{1}{N} \langle \phi_a^2 \rangle$

A good way to "integrate in" new fields is to use a δ-function:

$$1 = N \int d\rho \; \delta(\phi_a^2 - N\rho) = \frac{N}{4\pi i} \int d\rho \, d\lambda \; e^{-\frac{1}{2}\lambda(\phi_a^2 - N\rho)}$$

over imaginary axis $\lambda = i\xi$ gives usual rep. of δ-function.

$$Z = \int \mathcal{D}\phi \, \mathcal{D}\rho \, \mathcal{D}\lambda \; e^{-S[\phi] - \frac{1}{2}\int \lambda(\phi_a^2 - N\rho)\, d^d x}$$

$\lambda(x)$, $\rho(x)$ new fields

Separate $\phi_1 = \sigma\sqrt{N}$ $\qquad \phi_a = \pi_a \quad a = 2 \ldots N$

we choose direction 1 along ϕ

$\phi^2 \sim N \Rightarrow \phi_1^2 \sim N$ if ϕ_1 is the main component.

π_a represents fluctuations around that value.

$$Z = \int \mathcal{D}\sigma \, \mathcal{D}\pi_a \, \mathcal{D}\lambda \, \mathcal{D}\rho \; e^{-S[\sigma, \pi_a] - \int d^d x \, \frac{1}{2}\lambda(\phi_a^2 - N\rho)}$$

$$S[\sigma,\pi_a] = \frac{N}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{N}{2}r\rho + \frac{uN^2}{4!}\rho^2$$

$$\rho = (N\sigma^2 + \pi_a^2)/N$$

$$Z = \int \mathcal{D}\sigma\,\mathcal{D}\rho\,\mathcal{D}\lambda\,\mathcal{D}\pi_a\, e^{-\int \frac{1}{2}\partial_\mu\pi_a\partial^\mu\pi_a + \frac{1}{2}\lambda\pi_a^2} \times$$

$$\times\, e^{-N\int \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}r\rho + \frac{uN}{4!}\rho^2 + \frac{1}{2}\lambda\sigma^2 - \frac{1}{2}\lambda\rho}$$

$uN = \bar{u}$ that means we take $N\to\infty$ $u\to 0$ keeping $\bar{u} = uN$ fixed.

$$Z = \int \mathcal{D}\sigma\,\mathcal{D}\rho\,\mathcal{D}\lambda\, e^{-N\int \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}r\rho + \frac{\bar{u}}{4!}\rho^2 + \frac{1}{2}\lambda\sigma^2 - \frac{1}{2}\lambda\rho}\,.$$

$$\cdot\, e^{-\frac{N-1}{2}\mathrm{Tr}\ln(-\partial^2+\lambda)}$$

Now N is a parameter, instead of the # of fields

$N\to\infty$ is a classical limit for this action. The path integral is dominated by the solution to the equations of motion.

$\rho)$ $$\frac{\bar{u}}{12}\rho - \frac{1}{2}\lambda + \frac{r}{2} = 0$$

$$\boxed{\rho = \frac{6}{\bar{u}}(\lambda - r)}$$

$\sigma)$ $$\partial^2 \sigma = \lambda \sigma$$

$\lambda)$ $$\frac{1}{2}\sigma^2 - \frac{1}{2}\rho + \frac{1}{2}\frac{\delta}{\delta\lambda(x)} \mathrm{Tr}\ln(-\partial^2 + \lambda) = 0$$

To look at the ground state we consider constant fields $\Rightarrow \partial^2\sigma = 0$

$\lambda\sigma = 0$ $\nearrow$ $\lambda = 0 \quad \sigma \neq 0 \leftarrow$ ferromagnetic phase, $\langle d_a^2 \rangle \neq 0$

$\searrow$ $\lambda \neq 0 \quad \sigma = 0 \leftarrow$ paramagnetic phase.

$$\mathrm{Tr}\ln(-\partial^2+\lambda) = V\int \frac{d^dk}{(2\pi)^d} \ln(k^2+\lambda)$$

volume

$\int d^dx$ equation for $\lambda \Rightarrow \cancel{V}\left(\frac{1}{2}\sigma^2 - \frac{1}{2}\rho + \frac{1}{2}\frac{\delta}{\delta\lambda}\int\frac{d^dk}{(2\pi)^d}\ln(k^2+\lambda)\right) = 0$

$$\frac{\delta}{\delta\lambda}\int\frac{d^dk}{(2\pi)^d}\ln(k^2+\lambda)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+\lambda}$$

o) Ferromagnetic phase: $\lambda=0$.

$$\sigma^2-\underbrace{\frac{6}{\bar{u}}(-r)}_{\rho}+\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}=0$$

$$\sigma^2=-\frac{6r}{\bar{u}}-\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}=-\frac{6(r-r_c)}{\bar{u}}$$

$$r_c=-\frac{\bar{u}}{6}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}$$

But $\underline{\sigma^2>0}$

$\Rightarrow$ $\underline{r<r_c}$ this phase is possible and actually has lower energy.

if $\underline{r>r_c}$ " " " not possible and one have $\lambda\neq0$, $\sigma=0$.

$$r_c=-\frac{\bar{u}}{6}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}=-\frac{\bar{u}}{6}\frac{\Omega_d}{(2\pi)^d}\int_0^{\overset{\text{U.V. cut-off.}}{\Lambda}}k^{d-3}dk=-\frac{\bar{u}}{6}\frac{\Omega_d}{(2\pi)^d}\frac{k^{d-2}}{d-2}\Bigg|_0^{\Lambda}$$

$$=-\frac{\bar{u}}{6}\frac{\Omega_d}{(2\pi)^d}\frac{\Lambda^{d-2}}{d-2}\;;$$ Ω_d: Volume of $(d-1)$-sphere. S_{d-1}

if $d<2$ $\Rightarrow$ diverges as $k\to0$. IR effects remove the phase transition for $d=2$ $\leftarrow$ logarithmically divergent.

$\Omega_d = ?$

Trick:

$$\int d^dk\, e^{-\alpha k^2} = \left(\int dk\, e^{-\alpha k^2}\right)^d = \frac{\pi^{d/2}}{\alpha^{d/2}}$$

$$= \Omega_d \int k^{d-1} dk\, e^{-\alpha k^2} = \Omega_d \frac{\alpha^{-d/2}}{2} \int_0^\infty x^{\frac{d-1}{2}-\frac{1}{2}} dx\, e^{-x}$$

$x = \alpha k^2 \quad k = \sqrt{x}/\sqrt{\alpha}$

$dk = \frac{1}{2\sqrt{x}} dx$

$$= \frac{1}{2} \cdot \frac{\Omega_d}{\alpha^{d/2}} \Gamma(d/2)$$

$$\boxed{\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}}$$

$$r_c = -\frac{\bar{u}}{6} \frac{2\pi^{d/2}}{2^d \pi^d} \frac{1}{\Gamma(\frac{d}{2})} \frac{\Lambda^{d-2}}{d-2} = -\frac{\bar{u}}{3} \frac{1}{2^d \pi^{d/2}} \frac{\Lambda^{d-2}}{\Gamma(\frac{d}{2})(d-2)}$$

$$\boxed{r_c = -\frac{\bar{u}}{3} \frac{1}{(4\pi)^{d/2}} \frac{\Lambda^{d-2}}{d-2} \frac{1}{\Gamma(d/2)}}$$

← the value of the critical r moves because of the fluctuations.

c) Para magnetic phase $\lambda \neq 0$ $\sigma = 0$

$$\cancel{\ }\ - \frac{6}{\bar{u}}(\lambda - r) + \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+\lambda} = 0$$

Saddle point (i.e. solutions to) is for λ real. $\boxed{\lambda = m^2}$ Define m

$$\cancel{\ }\ - \frac{6}{\bar{u}}(m^2 - r) + \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} = 0$$

$$S = \int \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma + \frac{1}{2} m^2 \sigma^2$$

σ is massive w/ mass given by

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} = \frac{\Omega_d}{(2\pi)^d} \int_0^\Lambda \frac{k^{d-1}\,dk}{k^2+m^2}$$

approximate for $\Lambda \gg m$

Example $d=3$

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2+m^2} = \frac{4\pi}{8\pi^3} \int_0^\Lambda \frac{k^2\,dk}{k^2+m^2} = \frac{4\pi}{8\pi^3}\left(\Lambda - m^2 \int_0^\Lambda \frac{dk}{k^2+m^2}\right)$$

$k^2 \to k^2+m^2-m^2$

$$= \frac{\Lambda}{2\pi^2} - \frac{m^{\cancel{2}}}{2\pi^2}\frac{1}{\cancel{m}} \operatorname{atan}(k/m)\Big|_0^\Lambda \underset{\Lambda \gg m}{\simeq} \frac{\Lambda}{2\pi^2} - \frac{m}{4\pi}$$

In general (i.e. any d).

$$\int_0^{\Lambda} \frac{k^{d-1}\,dk}{k^2+m^2} = \int_0^{\Lambda^2/m^2} (m^2 x)^{\frac{d-1}{2}} \frac{\cancel{m^2}}{2(m^2x)^{1/2}} \frac{dx}{\cancel{m^2}(1+x)} = \frac{(m^2)^{\frac{d}{2}-1}}{2} \int_0^{\Lambda^2/m^2} \frac{x^{\frac{d}{2}-1}\,dx}{1+x}$$

$$x = k^2/m^2$$

$$dx = \frac{2k\,dk}{m^2}$$

We need

$$\int_0^{\bar\Lambda} \frac{x^{\alpha}\,dx}{1+x} = I(\alpha,\bar\Lambda) \text{ define. } ; \quad \underline{\alpha > -1} \text{ otherwise divergence } x \to 0$$

$\bar\Lambda \to \infty$

$$I(\alpha,\bar\Lambda) = \int_0^{\bar\Lambda} \frac{x^{\alpha}\,dx}{1+x} = \int_0^{\bar\Lambda} \frac{x^{\alpha-1}(x+1-1)}{1+x}\,dx = \int_0^{\bar\Lambda} x^{\alpha-1}\,dx - I(\alpha-1,\bar\Lambda)$$

$$= \frac{\bar\Lambda^{\alpha}}{\alpha} - I(\alpha-1,\bar\Lambda)$$

$$I(\alpha,\bar\Lambda) = \frac{\bar\Lambda^{\alpha}}{\alpha} - \frac{\bar\Lambda^{\alpha-1}}{\alpha-1} + \frac{\bar\Lambda^{\alpha-2}}{\alpha-2} \cdots + (-)^{p} I(\alpha-p,\bar\Lambda)$$

p integer.

if $-1 < \alpha < 0$ $\qquad x \to \infty \quad \int_*^{\bar\Lambda} x^{\alpha-1}\,dx = \frac{x^{\alpha}}{\alpha} \to 0.$ finite

$$x \to 0 \quad \int_0^{*} x^{\alpha}\,dx = \left.\frac{x^{\alpha+1}}{\alpha+1}\right|_0 = 0 \text{ finite.}$$

For $-1<\alpha<0$ the integral converges if we take $\bar\Lambda \to \infty$

$$\int_0^\infty \frac{x^\alpha dx}{1+x} = \int_1^\infty \frac{(y-1)^\alpha dy}{y} = \int_0^1 \frac{(\frac{1}{u}-1)^\alpha}{1/u}\frac{du}{u^2} =$$

$y = 1+x$ $\quad$ $y = 1/u$

$$= \int_0^1 (1-u)^{\alpha+1-1} u^{-\alpha-1} du = B(1+\alpha, -\alpha) = \frac{\Gamma(1+\alpha)\Gamma(-\alpha)}{\Gamma(1)} = \Gamma(1+\alpha)\Gamma(-\alpha)$$

If $2<d<4 \Rightarrow 1<\frac{d}{2}<2 \Rightarrow 0<\frac{d}{2}-1<1 \Rightarrow 0<\alpha<1$

So we only need to subtract once:

$$2<d<4 \Rightarrow I(\alpha,\bar{\Lambda}) = \frac{\bar{\Lambda}^\alpha}{\alpha} - I(\alpha-1,\bar{\Lambda}) \simeq \frac{\bar{\Lambda}^\alpha}{\alpha} - \Gamma(\alpha)\Gamma(1-\alpha)$$

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2} = \frac{2\pi^{d/2}}{\Gamma(d/2)}\frac{1}{(2\pi)^d}\frac{(m^2)^{\frac{d}{2}-1}}{2}\left[\frac{1}{d/2-1}\left(\Lambda^2/m^2\right)^{\frac{d}{2}-1} - \Gamma\left(\frac{d}{2}-1\right)\Gamma\left(1-\frac{d}{2}+1\right)\right]$$

$$= \frac{1}{(4\pi)^{d/2}}\frac{m^{d-2}}{\Gamma(d/2)}\left[\frac{2}{d-2}\frac{\Lambda^{d-2}}{m^{d-2}} - \frac{\Gamma(d/2)}{\frac{d}{2}-1}\Gamma(2-d/2)\right]$$

$$= \frac{1}{(4\pi)^{d/2}}\frac{2}{d-2}\left[\frac{\Lambda^{d-2}}{\Gamma(d/2)} - \underbrace{m^{d-2}\Gamma\left(\frac{4-d}{2}\right)}_{\text{indep. of }\Lambda}\right]$$

definition of r_c

$$= -\frac{6 r_c}{\bar{u}} - \frac{2\Gamma(\frac{4-d}{2})}{(4\pi)^{d/2}(d-2)} m^{d-2}$$

$$-\frac{6}{\bar{u}}(m^2-r) + \frac{6r_c}{\bar{u}} - \frac{2\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{d/2}(d-2)}\, m^{d-2} = 0$$

$$-\frac{6m^2}{\bar{u}} + \frac{6}{\bar{u}}(r-r_c) - \frac{2\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{d/2}(d-2)}\, m^{d-2} = 0$$

$r \to r_c \qquad m \to 0.$

$2<d<4 \qquad 0<d-2<2 \qquad m^2 \ll m^{d-2} \qquad m \to 0.$

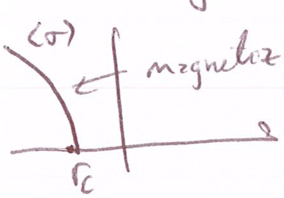
$$\Rightarrow \quad m^{d-2} = \frac{6}{\bar{u}}\,\frac{(4\pi)^{d/2}(d-2)}{2\Gamma\left(\frac{4-d}{2}\right)}\,(r-r_c)$$

we need $\underline{r>r_c}$ ✓.

$$m \sim (r-r_c)^{\frac{1}{d-2}}$$

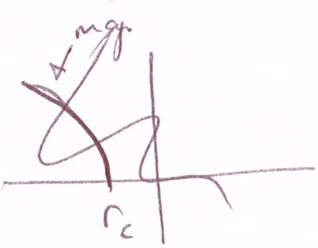
Ferromagnetic phase : $\underline{r<r_c}$ $\qquad \sigma^2 \sim (r_c-r)$

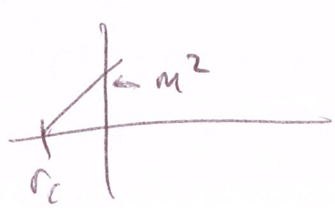
$\sigma \sim (r_c-r)^{1/2}$ ← critical exponent.



Paramagnetic phase : $\langle\sigma\rangle=0$, $\underline{r>r_c}$

σ: massive field $\qquad m \sim (r-r_c)^{\frac{1}{d-2}}$ ← critical exponent





ρ propagator

$\langle\langle \rho(x)\, \rho(y) \rangle\rangle$.

$$\lambda = r + \frac{1}{6}\bar{u}\rho$$

$$\bar{\lambda} + \eta = r + \frac{1}{6}\bar{u}\rho$$

fluctuations

lets compute $\langle\langle \eta(x)\, \eta(y) \rangle\rangle = \frac{\bar{u}^2}{36} \langle\langle \rho\, \rho \rangle\rangle$ up to constant.

$$\frac{1}{N} S = \int d^d x \left(\frac{1}{2}(\partial_\mu \bar{\sigma})^2 + \frac{1}{2}\lambda \sigma^2 - \frac{3}{2u}\lambda^2 + \frac{3}{\mu} r\lambda \right)$$

$$+ \frac{1}{2} \operatorname{Tr} \ln(-\nabla^2 + \lambda)$$

At first order in η we get 0 (minimization)

2nd order

$$\frac{1}{N} S^{(2)} = \int d^d x \left(-\frac{3}{2u}\eta^2 \right) + \underbrace{\frac{1}{2} \operatorname{Tr} (\ln(-\nabla^2 + \bar{\lambda} + \eta))}_{\text{at second order.}}$$

$$\ln(\underbrace{A + \epsilon}_{\text{matrices}}) = \ln(A + AA^{-1}\epsilon) = \ln A + \ln(1 + A^{-1}\epsilon) =$$

$$= \ln A + A^{-1}\epsilon - \frac{1}{2}(A^{-1}\epsilon)(A^{-1}\epsilon) + \cdots$$

We need $\frac{1}{4} \operatorname{Tr}\left((-\nabla^2 + \bar{\lambda})^{-1} \eta\, (-\nabla^2 + \bar{\lambda})^{-1} \eta \right)$

Easier in momentum space since $-\nabla^2+\lambda$ is diagonal.

Using Q.M. notation:

$$\frac{1}{4}\mathrm{Tr}\left((-\nabla^2+\bar\lambda)^{-1}\eta\,(-\nabla^2+\bar\lambda)^{-1}\eta\right) =$$

$$= \frac{1}{4}\int\frac{d^dk}{(2\pi)^d}\,\langle k|\,(-\nabla^2+\bar\lambda)^{-1}\,\eta\,(-\nabla^2+\bar\lambda)^{-1}\,\eta\,|k\rangle$$

$|y\rangle\langle y|$ $\quad |q\rangle\langle q|$

$$= \frac{1}{4}\int\frac{d^dk}{(2\pi)^d}\int d^dx\,\frac{1}{k^2+\bar\lambda}\,\langle k|\,\eta\,(-\nabla^2+\bar\lambda)^{-1}|x\rangle\langle x|\,\eta\,|k\rangle$$

$$= \frac{1}{4}\int\frac{d^dk\,d^dq}{(2\pi)^d(2\pi)^d}\int d^dx\int d^dy\,\frac{\eta(x)}{k^2+\bar\lambda}\,\frac{\eta(y)}{q^2+\bar\lambda}\,\langle k|y\rangle\langle y|q\rangle\langle q|x\rangle\langle x|k\rangle$$

$$e^{-iky}\,e^{iqy}\,e^{-iqx}\,e^{ikx}$$

Fourier Transform

$$q = p-k$$

$$= \frac{1}{4}\int\frac{d^dk\,d^dq}{(2\pi)^d(2\pi)^d}\,\frac{\tilde\eta_{q-k}\,\tilde\eta_{k-q}}{(k^2+\bar\lambda)(q^2+\bar\lambda)} = \frac{1}{4}\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dk}{(2\pi)^d}\,\frac{\tilde\eta_p\,\tilde\eta_{-p}}{(k^2+\bar\lambda)((p-k)^2+\bar\lambda)}$$

$$= \frac{1}{4}\int\frac{d^dp}{(2\pi)^d}\,\tilde\eta_p\,\tilde\eta_{-p}\,B(p) \qquad B(p) = \int\frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2+\bar\lambda)((p-k)^2+\bar\lambda)}$$

$$\frac{1}{N}S = \int\frac{d^dk}{(2\pi)^d}\left(-\frac{3}{2u}+\frac{1}{4}B(p)\right)\tilde\eta_p\,\tilde\eta_{-p}$$

$$\int\eta^2 = \int d^dx\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\,e^{ikx}\,\tilde\eta_k\,e^{iqx}\,\tilde\eta_q = \int\frac{d^dk}{(2\pi)^d}\,\tilde\eta_k\,\tilde\eta_{-k}$$

$$S = -\frac{3N}{2u}\int\frac{d^dk}{(2\pi)^d}\left(1+\frac{u}{6}B(p)\right)\tilde{\eta}_p\tilde{\eta}_{-p}$$

$$\int dz_i d\bar{z}_i\, z_i\bar{z}_j\, e^{-\bar{z}_i A_{ij} z_j} = ?$$

$$\int dz_i d\bar{z}_i\, e^{-\bar{z}_i A_{ij} z_j + \bar{J}_i \bar{z}_i + j_j z_j} = \mathcal{N} e^{\bar{J}_i A^{-1}_{ij} j_j}$$

$$\frac{\delta}{\delta\bar{J}_i}\frac{\delta}{\delta j_j} = (A^{-1})_{ij}$$

But $\tilde{\eta}^*_{-p} = \eta_p$ we need to eliminate half the p's $S = -\frac{3N}{u}\int_{k_1>0}\frac{d^dk}{(2\pi)^d}\left(1+\frac{u}{6}B(k)\right)\eta_k\eta_k^*$

$$\langle\langle \eta_p\bar{\eta}_{-p}\rangle = -\frac{u}{3N}\frac{1}{1+\frac{u}{6}B(p)}$$

$$\langle\langle \phi^2\phi^2\rangle\rangle = \langle\langle\rho\rho\rangle\rangle = -\frac{u}{3N}\frac{36}{u^2}\frac{1}{1+\frac{u}{6}B(p)}$$

$$\delta\rho = \frac{6}{u}\eta$$

$$= -\frac{12}{uN}\frac{1}{1+\frac{u}{6}B(p)}$$

agrees w/ bubble summation but mass is changed

Perturbation theory for $(\phi_a \phi_a)^2$

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_a)(\partial^\mu \phi_a) + \frac{1}{2} m^2 \phi_a \phi_a + \frac{\lambda}{4!} (\phi_a \phi_a)^2$$

$$\langle\langle \phi_a(x) \phi_b(y) \rangle\rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \phi_a(x) \phi_b(y) \, e^{\overset{\uparrow \text{ free theory.}}{-S_0} - \int \frac{\lambda}{4!} (\phi_c \phi_c)^2}$$

$$= \frac{\int \mathcal{D}\phi \, \phi_a(x) \phi_b(y) \sum_{n=0}^{\infty} \frac{(-)^n \lambda^n}{(4!)^n} \left(\int d^dx \, (\phi_c \phi_c)^2 \right)^n e^{-S_0}}{\int \mathcal{D}\phi \sum_{n=0}^{\infty} \frac{(-)^n \lambda^n}{(4!)^n} \left(\int d^dx \, (\phi_c \phi_c)^2 \right)^n e^{-S_0}}$$

$$= \frac{\sum_n \langle\langle \phi_a(x) \phi_b(y) \frac{(-)^n \lambda^n}{(4!)^n} \left(\int d^dx \, (\phi_c \phi_c)^2 \right)^n \rangle\rangle_0}{\langle\langle \sum_{n=0}^{\infty} \frac{(-)^n \lambda^n}{(4!)^n} \left(\int d^dx \, (\phi_c \phi_c)^2 \right)^n \rangle\rangle_0}$$

in free theory.

$$\langle\langle \phi_a(x) \phi_b(y) \quad \phi_{a_1}(x_1) \phi_{a_1}(x_1) \phi_{b_1}(x_1) \phi_{b_1}(x_1) \ldots \phi_{a_n}(x_n) \phi_{a_n}(x_n) \phi_{b_n}(x_n) \phi_{b_n}(x_n) \rangle\rangle_0$$

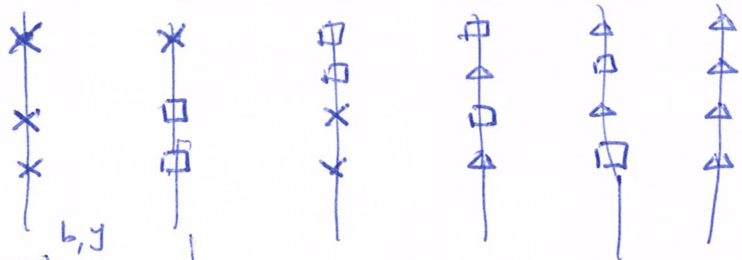
can be evaluated as before. All possible contractions.

Zero order

$$\langle\langle \phi_a(x) \phi_b(y) \rangle\rangle = \langle\langle \phi_a(x) \phi_b(y) \rangle\rangle_0 = \Delta(x, y)$$

Consider first order.

$$-\frac{\lambda}{4!}\int d^dx_1 \;\langle\langle \phi_a(x)\,\phi_b(y)\,\phi_{a_1}(x_1)\,\phi_{a_1}(x_1)\,\phi_{b_1}(x_1)\,\phi_{b_1}(x_1)\rangle\rangle_0$$

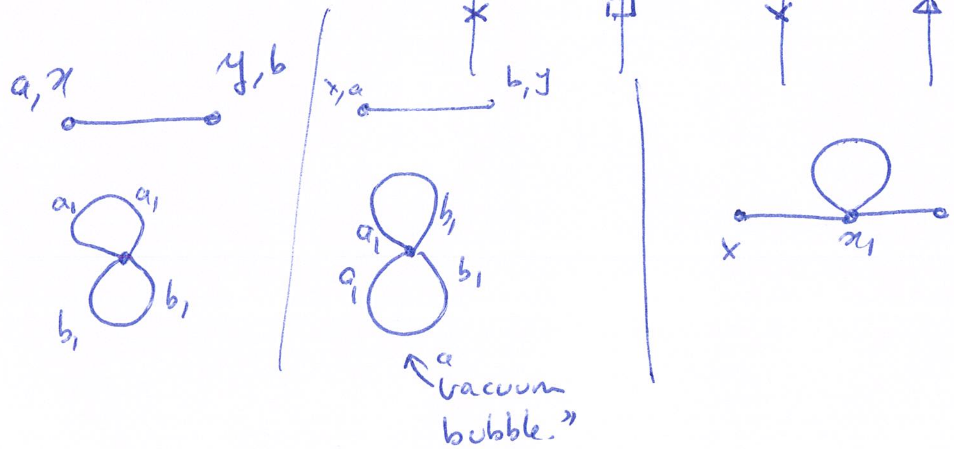


$\rightarrow 1\; N^2$

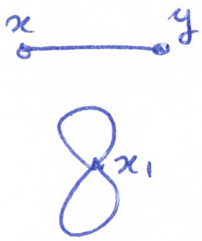
$\rightarrow 2\; N$

$\rightarrow 4\; N$

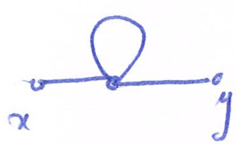
$\rightarrow 4\times 2\times$



$$\Delta(x,y) = \int\frac{d^dk}{(2\pi)^d}\,\frac{e^{ik(x-y)}}{k^2+m^2}$$



$$= -\frac{\lambda}{4!}\,\Delta(x,y)\int d^4x_1\,(\Delta(x_1,x_1))^2\;(N^2+2N)$$



$$= -\frac{\lambda}{4!}\int d^4x_1\quad \Delta(x,x_1)\,\Delta(x_1,x_1)\,\Delta(x_1,y)\quad(4N+8)$$

$$\Delta(x_1,x_1) = \int\frac{d^dk}{(2\pi)^d}\,\frac{1}{k^2+m^2} = \Delta(0) \quad \text{indep. of. } x_1$$

$$\int d^4x_1\;\Delta(x,x_1)\,\Delta(x_1,y) = \int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\;\underbrace{\int d^4x_1}_{(2\pi)^4\delta(k_1-k_2)}\;\frac{e^{ik_1(x-x_1)+ik_2(x_1-y)}}{(k_1^2+m^2)(k_2^2+m^2)} =$$

$$= \int\frac{d^4k_1}{(2\pi)^4}\,\frac{e^{ik_1(x-y)}}{(k_1^2+m^2)^3}$$

the denominator

Zero order : 1

First order $8 = -\frac{\lambda}{4!}\int d^4x_1 \; (N^2+2N)\;(\Delta(0))^2$

$$\Delta(x,y)\left(1-\frac{\lambda}{4!}\int d^4x_1\;(N^2+2N)\,(\Delta(0))^2\right) - \frac{\lambda}{4!}(4N+8)\,\Delta(0)\int\frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x-y)}}{(k_1^2+m^2)^2}$$

$$1-\frac{\lambda}{4!}\int d^4x_1\;(N^2+2N)\,(\Delta(0))^2$$

$\mathcal{O}(\lambda)$

$$= \Delta(x,y) - \frac{\lambda}{6}(N+2)\;\Delta(0)\int\frac{d^4k_1}{(2\pi)^4}\frac{e^{ik_1(x-y)}}{(k_1^2+m^2)^2}$$

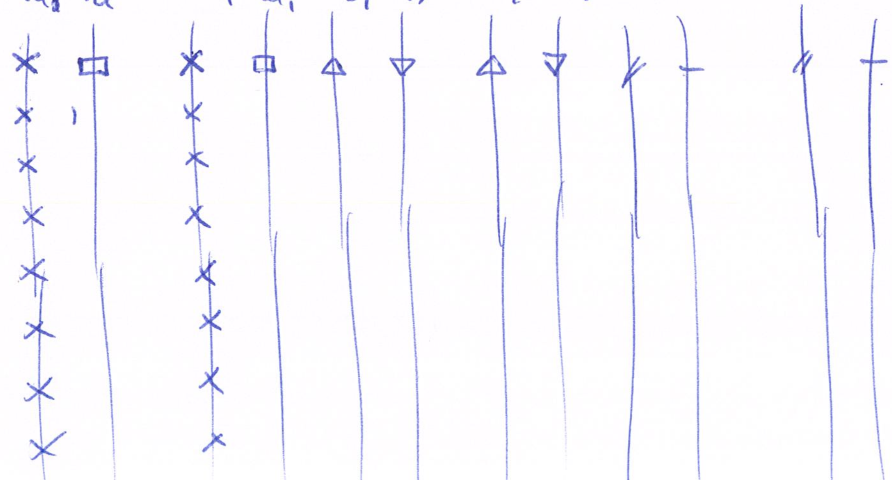
Vacuum Bubbles Cancel!

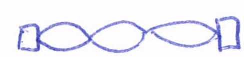
2^{nd} order. $\langle \phi_a^2 \; \phi_b^2 \rangle$ no vacuum bubbles.

 $2N(\Delta(x-y))^2$

 $\frac{4\times 2}{4!}(-)\lambda N^2$

$$\phi_{a_2}\phi_a \quad \overbrace{\phi_{a_1}\phi_{a_1}\phi_{b_1}\phi_{b_1}}^{x_1}\overset{2}{\leftrightarrow}\overbrace{\phi_{a_2}\phi_{a_2}\phi_{b_2}\phi_{b_2}}^{x_2}\quad \phi_b\phi_b$$





2×4

$\times 4$

$\times 2$ $\quad \frac{N^3}{2}\;\frac{8\times 8}{4!\;4!}\;\lambda^2(-)^2 =$

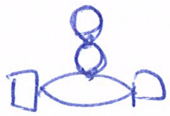
$$= 2\left(\frac{\lambda}{6}\right)^2 N^3$$

Zero order  N  $(\Delta(x,y))^2$

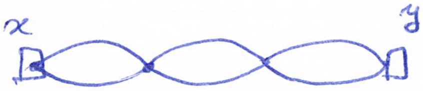
first order  

 N^2  N^2

2^{nd} order  N^3  N^3  N^3

 N^3  N^3  N^2  N^3

N^2



 +  +  + —

$$= 2N\left(\Delta(x-y)\right)^2 - \frac{\lambda}{6}2N^2 \int d^dx_1 \left(\Delta(x-x_1)\right)^2\left(\Delta(x_1-y)\right)^2 +$$

$$+ 2\left(\frac{\lambda N}{6}\right)^2 N \int d^dx_1 \int d^dx_2 \left(\Delta(x-x_1)\right)^2 \left(\Delta(x_1-x_2)\right)^2 \left(\Delta(x_2-y)\right)^2 + \cdots$$

$$\left(\Delta(x-x_1)\right)^2 = \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d} \frac{e^{ik_1(x-x_1)+ik_2(x-x_1)}}{(k_1^2+m^2)(k_2^2+m^2)}$$

$$\int d^dx_1 \int \frac{d^dk_1}{(2\pi)^d} \int \frac{d^dk_2}{(2\pi)^d}\Big|_{3,4} \frac{e^{ik_1(x-x_1)+ik_2(x-x_1)+ik_3(x_1-y)+ik_4(x_1-y)}}{(k_1^2+m^2)(k_2^2+m^2)(k_3^2+m^2)(k_4^2+m^2)} =$$

(6)

$$\left(\prod_{i=1}^{4}\int\frac{d^dk_i}{(2\pi)^4}\right)\frac{\delta(k_1+k_2-k_3-k_4)\;e^{i(k_1+k_2)(x-y)}}{(k_1^2+m^2)(k_2^2+m^2)(k_3^2+m^2)(k_4^2+m^2)}$$

$$\int_x e^{-ipx}(\Delta(x))^2=\int_x e^{-ipx}\int\frac{d^dk_1\,d^dk_2}{(2\pi)^d(2\pi)^d}\frac{e^{ik_1(x-\cancel{y})+ik_2(x-\cancel{y})}}{(k_1^2+m^2)\,(k_2^2+m^2)}=$$

$$=\int\frac{d^dk_1}{(2\pi)^d}\frac{\cancel{e^{ipx}}\;1}{(k_1^2+m^2)((p-k_1)^2+m^2)}=B(p)$$

$$(\Delta(x))^2=\int_x\frac{d^dp}{(2\pi)^d}B(p)\,e^{ipx}$$

$$\int d^dx_1\,(\Delta(x-x_1))^2(\Delta(x_1-y))^2=\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dq}{(2\pi)^d}\int_{x_1}e^{ip(x-x_1)+iq(x_1-y)}B(p)B(q)$$

$$=\int\frac{d^dp}{(2\pi)^d}\,e^{ip(x-y)}(B(p))^2$$

 $=$

$$=\int\frac{d^dp}{(2\pi)^d}\,e^{ip(x-y)}\,2N\left(B(p)-\frac{\lambda N}{6}(B(p))^2+\left(\frac{\lambda N}{6}\right)(B(p))^3+\dots\right)$$

$$=2N\int\frac{d^dp}{(2\pi)^d}\,e^{ip(x-y)}\frac{B(p)}{1+\frac{\lambda N}{6}B(p)}$$

partial resummation of diagrams (bubble) dominates as $N\to\infty$

$$2N \int \frac{d^dp}{(2\pi)^d} e^{ip(x-y)} \frac{6}{\lambda N} \frac{\left(\frac{\lambda N}{6} B(p) + 1 - 1\right)}{1 + \frac{\lambda N}{6} B(p)} =$$

$$= \frac{12}{\lambda} \int \frac{d^dp}{(2\pi)^d} e^{ip(x-y)} - \frac{12}{\lambda} \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip(x-y)}}{1 + \frac{\lambda N}{6} B(p)}$$

$$\langle \phi^2_{aa}(x) \phi^2_{bb}(y) \rangle = \frac{12}{\lambda} \delta_{ab}\, \delta(x-y) - \frac{12}{\lambda} \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip(x-y)}}{1 + \frac{\lambda N}{6} B(p)}$$

$$\lambda N = \bar{u}$$

$$\langle \phi^2 \phi^2 \rangle = \frac{12N}{\bar{u}} \delta_{ab}\, \delta(x-y) - \frac{12N}{\bar{u}} \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip(x-y)}}{1 + \frac{\bar{u}}{6} B(p)}$$

$$\rho = \phi^2/N$$

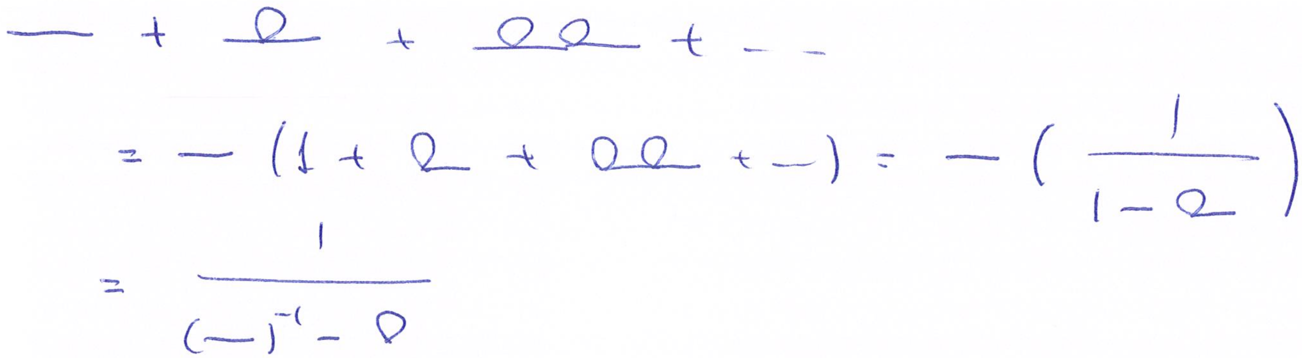
$$\langle\langle \rho(x) \rho(y) \rangle\rangle = \frac{12}{N\bar{u}} \delta_{ab}\, \delta(x-y) - \frac{12}{N\bar{u}} \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip(x-y)}}{1 + \frac{\bar{u}}{6} B(p)}$$

in momentum space

$$\int_x e^{ip(x-y)} \langle\langle \rho(x) \rho(y) \rangle\rangle = \frac{12}{N\bar{u}} - \frac{12}{N\bar{u}} \frac{1}{1 + \frac{\bar{u}}{6} B(p)}$$

large-N correction to the propagator

◦) Example 1-loop.



in formulas. $\quad — = \dfrac{1}{p^2+m^2}$

$$= \left(-\frac{\lambda}{6}\right)^2 \frac{1}{(p^2+m^2)^3}\left(\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\right)^2 = \frac{(\Delta(0))^2}{(p^2+m^2)^3}\left(-\frac{\lambda}{6}\right)^2$$

n bubbles

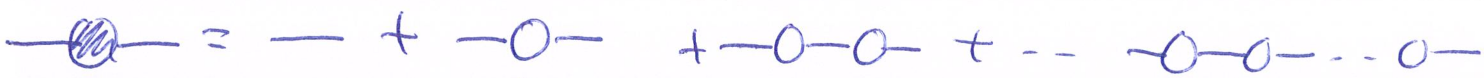
$$= \frac{(\Delta(0))^n}{(p^2+m^2)^{n+1}}\left(-\frac{\lambda}{6}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{(\Delta(0))^n}{(p^2+m^2)^{n+1}}\left(-\frac{\lambda}{6}\right)^n = \frac{1}{p^2+m^2}\,\frac{1}{1+\frac{\lambda}{6}\frac{\Delta(0)}{p^2+m^2}} = \frac{1}{p^2+m^2+\frac{\lambda}{6}\Delta(0)}$$

we can resum all these diagrams

More in general

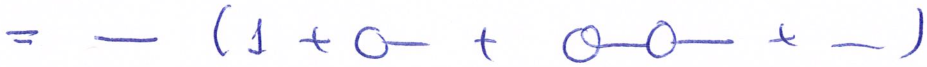
all connected diagrams

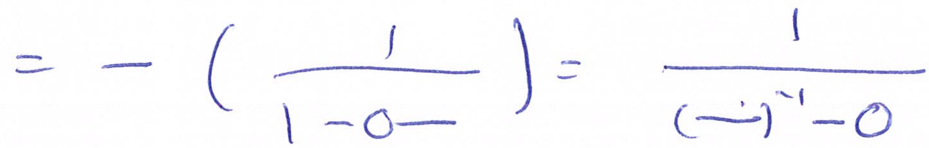


all 1-PI diagrams

1-PI : one particle irreducible.

Cannot be separated in two by cutting just one line.



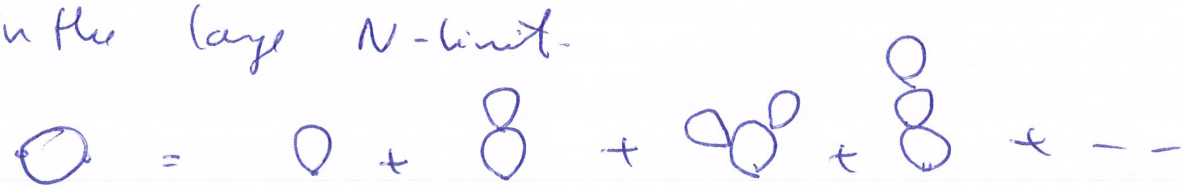


 $= \Sigma_1(p^2)$ is called self-energy

 $= \dfrac{1}{p^2+m^2-\Sigma(p^2)}$

The self energy can be computed at 1-loop 
and also at higher loops

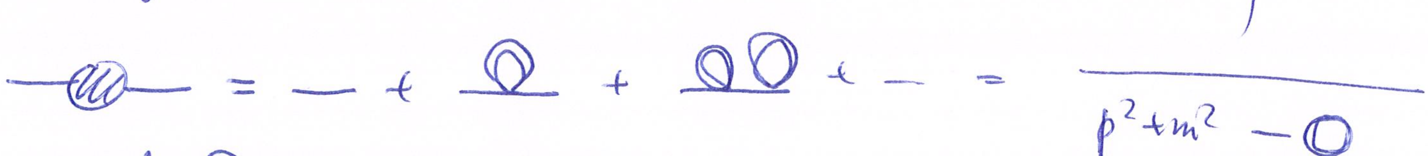
In the large N-limit.



We get

 ← full propagator.

At large N



But  is indep. of p^2.

 $= -\frac{\lambda}{6}\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+\bar{m}^2}$

renormalized mass

$$\cancel{p^2} + \bar{m}^2 = \cancel{p^2} + m^2 + \frac{\lambda}{6}\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+\bar{m}^2)}$$

We get a self consistent equation for $\bar{m}^2$:

$$\boxed{\bar{m}^2 = m^2 + \frac{\lambda}{6}\int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+\bar{m}^2}}$$

Same as before.

$$B(p) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+m^2)\left((p-k)^2+m^2\right)}$$

asymptotic behavior $\displaystyle \int \frac{k^{d-1}\,dk}{k^4}$

- $d>4$ divergent
- $d=4$ $\displaystyle\int \frac{dk}{k}$ log. divergent
- $d<4$ finite

Schwinger parameters

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha_1\, e^{-\alpha_1(k^2+m^2)}$$

$$B(p) = \int \frac{d^dk}{(2\pi)^d} \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2\, e^{-\alpha_1(k^2+m^2) - \alpha_2\left((p-k)^2+m^2\right)}$$

$$= \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \int \frac{d^dk}{(2\pi)^d}\, e^{-(\alpha_1+\alpha_2)k^2 - 2\alpha_2 p.k - (\alpha_1+\alpha_2)m^2 - \alpha_2 p^2}$$

$$= \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \frac{1}{(2\pi)^d} \frac{\pi^{d/2}}{(\alpha_1+\alpha_2)^{d/2}}\, e^{\frac{4\alpha_2^2 p^2}{4(\alpha_1+\alpha_2)} - \alpha_2 p^2 - (\alpha_1+\alpha_2)m^2}$$

$$\frac{\cancel{\alpha_2^2} - \alpha_1\alpha_2 - \cancel{\alpha_2^2}}{\alpha_1+\alpha_2}$$

$$= \int_0^\infty d\rho \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \frac{\delta(\rho-\alpha_1-\alpha_2)}{(4\pi)^{d/2}} \frac{1}{\rho^{d/2}}\, e^{-\frac{\alpha_1\alpha_2 p^2}{\rho} - \rho m^2}$$

$\alpha_1 = \rho\alpha$

$\alpha_2 = \rho\beta$

$0<\alpha<1$, $0<\beta<1$, $\alpha+\beta=1$

$$= \int_0^\infty \rho^2 d\rho \int_0^\infty d\alpha \int_0^\infty d\beta\, \frac{1}{\rho} \frac{\delta(1-\alpha-\beta)}{(4\pi)^{d/2}}\, \rho^{-d/2}\, e^{-\rho\,\alpha\beta p^2 - \rho m^2}$$

$$= \int_0^\infty \rho^{1-d/2} \frac{d\rho}{(4\pi)^{d/2}} \int_0^1 d\alpha\, e^{-\rho\,\alpha(1-\alpha)p^2 - \rho m^2}$$

$$B(p) = \frac{1}{(4\pi)^{d/2}} \int_0^1 d\alpha \, (\alpha(1-\alpha)p^2 + m^2)^{\frac{d}{2}-2} \, \Gamma(2-\tfrac{d}{2})$$

$$= \frac{1}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) \int_0^1 d\alpha \, (\alpha(1-\alpha)p^2 + m^2)^{\frac{d}{2}-2}$$

pole

$d=4 \to$ diverges

$d<4 \to$ ok

•) $m^2=0$

$$B(p) = \frac{1}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) (p^2)^{\frac{d-4}{2}} \underbrace{\int_0^1 d\alpha \, \alpha^{(\frac{d}{2}-1)-1} (1-\alpha)^{(\frac{d}{2}-1)-1}}_{B(\frac{d}{2}-1, \frac{d}{2}-1) = \frac{\Gamma(\frac{d}{2}-1)\Gamma(\frac{d}{2}-1)}{\Gamma(d-2)}}$$

$$B(p) = \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\frac{4-d}{2}) \left(\Gamma(\frac{d-2}{2})\right)^2}{\Gamma(d-2)} (p^2)^{\frac{d-4}{2}} = B_d \, p^{d-4}$$

•) $\underline{d=3}$

$$B(p) = \frac{1}{(4\pi)^{3/2}} \overbrace{\Gamma(\tfrac{1}{2})}^{\sqrt{\pi}} \int_0^1 \frac{d\alpha}{\sqrt{\alpha(1-\alpha)p^2+m^2}} = \frac{1}{8\pi\sqrt{p^2}} \int_0^1 \frac{d\alpha}{\sqrt{-\alpha^2+\alpha+m^2/p^2}}$$

$$= \frac{1}{4\pi\sqrt{p^2}} \arctan\left(\sqrt{\frac{p^2}{4m^2}}\right) = \frac{1}{4\pi p} \arctan\left(\frac{p}{2m}\right)$$

$$p = \sqrt{p^2}$$

$m^2=0$ critical theory. $\langle \phi^2 \phi^2 \rangle = \frac{12}{N\bar{u}} \delta(x-y) - \frac{12}{N\bar{u}} \int \frac{d^dp}{(2\pi)^d} \frac{e^{ip(x-y)}}{\frac{\bar{u}}{6} B_d \, p^{d-4}}$

$|x-y| \to \infty$ $\quad \langle \phi^2 \phi^2 \rangle \simeq \frac{c}{|x-y|^4}$ $\quad$ conf. dim. of $\phi^2 = 2$ $\quad \boxed{\Delta_{\phi^2} = 2}$ large N.

$$Z = \int_{-\infty}^{\infty} d\phi_a \; e^{-\frac{1}{2}m^2\phi_a^2 - \frac{\lambda}{4!}(\phi_a^2)^2}$$

"Free propagator"

$$\langle\langle \phi_a \phi_b \rangle\rangle = \frac{\int d\phi_c \; \phi_a \phi_b \; e^{-\frac{1}{2}m^2\phi_c^2}}{\int d\phi_c \; e^{-\frac{1}{2}m^2\phi_c^2}} = \delta_{ab} \frac{\int d\phi \, \phi^2 e^{-\frac{m^2}{2}\phi^2}}{\int d\phi \; e^{-m^2/2 \, \phi^2}} =$$

$$= \delta_{ab} (-2) \frac{\partial}{\partial(m^2)} \ln \int_{-\infty}^{\infty} d\phi \; e^{-\frac{m^2}{2}\phi^2} = -2\delta_{ab} \frac{\partial}{\partial(m^2)} \ln \sqrt{\frac{2\pi}{m^2}} =$$

$$= \delta_{ab} \frac{\partial}{\partial m^2} \ln m^2 = \frac{\delta_{ab}}{m^2}$$

"Wick's theorem" $\qquad \phi = \sqrt{y} \quad d\phi = \frac{1}{2\sqrt{y}} dy$

$$\langle\langle \phi^{2n} \rangle\rangle = \sqrt{\frac{m^2}{2\pi}} \int_{-\infty}^{\infty} d\phi \, \phi^{2n} e^{-\frac{m^2}{2}\phi^2} = \cancel{2}\sqrt{\frac{m^2}{2\pi}} \int_0^\infty \frac{dy}{\cancel{2}\sqrt{y}} y^n e^{-\frac{m^2}{2}y} = \sqrt{\frac{m^2}{2\pi}} \left(\frac{m^2}{2}\right)^{-n-\frac{1}{2}} \Gamma(n+\tfrac{1}{2})$$

$$= \frac{\Gamma(n+1/2)}{\sqrt{\pi}} \left(\frac{2}{m^2}\right)^n \; ; \qquad \Gamma(n+\tfrac{1}{2}) = (n-1+\tfrac{1}{2})\Gamma(n-1+\tfrac{1}{2}) = (n-1+\tfrac{1}{2}) \cdots \tfrac{1}{2}\Gamma(\tfrac{1}{2})$$

$\Gamma(\tfrac{1}{2}) \to \sqrt{\pi}$

$$= \frac{(2n-1)\cdots 1}{2^n}\sqrt{\pi} = \frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

$$\langle\langle \phi^{2n} \rangle\rangle = \frac{(2n-1)!!}{(m^2)^n} \longleftrightarrow \checkmark$$

$$\overbrace{\phi\,\phi\,\phi\,\phi \cdots \phi}^{2n} \leftarrow \text{n-propagators} \rightarrow \langle\langle \phi^{2n} \rangle\rangle = \frac{(2n-1)!!}{m^{2n}}$$

$(2n-1) \quad 2n-3 \quad \cdots \quad 1 \quad = (2n-1)!!$

$Z = ?$

$$\int_{-\infty}^{\infty} d\sigma \; e^{-\alpha\sigma^2 + \beta\sigma} = \sqrt{\frac{\pi}{\alpha}}\; e^{\beta^2/4\alpha} \qquad \frac{\beta^2}{\cancel{4}\alpha} = -\frac{\lambda}{\cancel{24}\;6}(\phi_a^2)^2$$

$$\alpha = 1 \qquad \beta = i\sqrt{\frac{\lambda}{6}}\,\phi_a^2$$

$$Z = \int_{-\infty}^{\infty} d\sigma \int_{-\infty}^{\infty} d\phi_a \; e^{-\sigma^2}\; e^{-\frac{1}{2}m^2\phi_a^2 + i\sqrt{\frac{\lambda}{6}}\,\phi_a^2\sigma}$$

$$= \int_{-\infty}^{\infty} d\sigma \left(\frac{\sqrt{\pi}}{\sqrt{\frac{m^2}{2} - i\sqrt{\frac{\lambda}{6}}\,\sigma}}\right)^N e^{-\sigma^2} = \left(\frac{2\pi}{m^2}\right)^{N/2} \int_{-\infty}^{\infty} \frac{d\sigma\; e^{-\sigma^2}}{\left(1 - 2i\sqrt{\frac{\lambda}{6}}\frac{\sigma}{m^2}\right)^{N/2}}$$

λ - expansion

$$\left(1 - 2i\sqrt{\frac{\lambda}{6}}\frac{\sigma}{m^2}\right)^{-N/2} = \sum_{p=0}^{\infty}\left(-2i\sqrt{\frac{\lambda}{6}}\frac{\sigma}{m^2}\right)^p \binom{-N/2}{p} =$$

$$= \sum_{p=0}^{\infty}\left(-2i\sqrt{\frac{\lambda}{6}}\frac{1}{m^2}\right)^p \sigma^p \frac{\Gamma(1-N/2)}{p!\,\Gamma(-p-N/2+1)}$$

$$\int_{-\infty}^{\infty} d\sigma\; \sigma^p e^{-\sigma^2} = 0 \quad \text{if } p \text{ odd.}$$

$p = 2k$

$$Z = \left(\frac{2\pi}{m^2}\right)^{N/2} \int_{-\infty}^{\infty} e^{-\sigma^2} d\sigma \sum_{k=0}^{\infty} (-)^k \frac{\lambda^k}{6^k}\, 2^{2k}\frac{1}{m^{4k}}\; \frac{\Gamma(1-N/2)}{(2k)!\,\Gamma(1-2k-N/2)}$$

$$\Gamma(1-N/2) = \left(-\frac{N}{2}\right)\Gamma\left(-\frac{N}{2}\right) = \cancel{\emptyset}\left(-\frac{N}{2}\right)\left(-\frac{N}{2}-1\right)\Gamma\left(-\frac{N}{2}-1\right)$$

$$= \left(-\frac{N}{2}\right)\left(-\frac{N}{2}-1\right)\left(-\frac{N}{2}-2\right)\left(-\frac{N}{2}-3\right)\cdots\left(-\frac{N}{2}-(2k-1)\right)\Gamma\left(-\frac{N}{2}-2k\right)$$

$$\frac{\Gamma\left(1-\frac{N}{2}\right)}{\Gamma\left(1-2k-\frac{N}{2}\right)} = \frac{1}{2^{2k}}(N)(N+2)(N+4)(N+6)\cdots(N+2(2k-1))$$

$$= \frac{(N+2(2k-1))!!}{2^{2k}\,(N-2)!!}$$

$$Z = \left(\frac{2\pi}{m^2}\right)^{N/2}\sum_{k=0}^{\infty}\frac{(-)^k\lambda^k}{6^k}\,\frac{\cancel{2^{2k}}}{m^{4k}}\,\frac{(N+2(2k-1))!!}{\cancel{2^{2k}}(2k)!\,(N-2)!!}\cdot\frac{(2k-1)!!}{2^k}\sqrt{\pi}$$

$$= \left(\frac{2\pi}{m^2}\right)^{N/2}\sqrt{\pi}\left(1 - \frac{\lambda}{6}\,\frac{\cancel{12}}{m^4}\,\frac{N(N+2)}{\cancel{1}\cdot 2}\,\frac{1}{2} + \frac{\lambda^2}{6^2}\,\frac{1}{m^8}\,\frac{N(N+2)(N+4)(N+6)\,\cancel{3}}{\cancel{\emptyset}\,4!}\,\frac{1}{2^2} - + \right.$$

$$12\cancel{3}4$$

$$= \sqrt{\pi}\left(\frac{2\pi}{m^2}\right)^{\frac{N}{2}}\left(1 - \frac{\lambda}{2^3\times 3}\,\frac{N(N+2)}{m^4} + \frac{\lambda^2}{2^7\,3^2}\,\frac{N(N+2)(N+4)(N+6)}{m^8} + \cdots\right)$$

$$\ln(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + \cdots$$

$$\ln Z \simeq c - \frac{N}{2}\ln(m^2) - \frac{\lambda}{2^3\times 3}\,\frac{N(N+2)}{m^4} + \frac{\lambda^2}{2^7\,3^2}\,\frac{N(N+2)(N+4)(N+6)}{m^8} -$$

$$- \frac{1}{2}\,\frac{\lambda^2}{2^6\,3^2}\,\frac{N(N+2)\,N(N+2)}{m^8} + \cdots$$

$$\ln Z = \overset{\text{constant}}{C} - \frac{N}{2}\ln(m^2) - \frac{\lambda}{3\times 2^3}\,\frac{N(N+2)}{m^4} +$$

$$+ \frac{\lambda^2}{2^7\times 3^2 m^8}\,N(N+2)\left(\cancel{N^2}+10N+24-\cancel{N^2}-2N\right)$$

$$8N+24 = 8(N+3)$$

$$\boxed{\ln Z = C - \frac{N}{2}\ln m^2 - \frac{\lambda}{3\times 2^3}\,\frac{N(N+2)}{m^4} + \frac{\lambda^2}{2^4\times 3^2 m^8}\,N(N+2)(N+3) + \cdots}$$

$$\frac{\partial \ln Z}{\partial \lambda} = -\frac{1}{24}\langle\langle (\phi_a^2)^2\rangle\rangle$$

$$\frac{\partial \ln Z}{\partial (m^2)} = -\frac{1}{2}\langle\langle \phi_a^2\rangle\rangle \quad ; \quad \frac{1}{Z}\frac{\partial^2 Z}{(\partial m^2)^2} = \frac{1}{4}\langle\langle \phi_a^2\phi_a^2\rangle\rangle$$

$$\frac{\partial^2 \ln Z}{(\partial m^2)^2} = \frac{\partial}{\partial m^2}\left(\frac{1}{Z}\frac{\partial Z}{\partial m^2}\right) = -\frac{1}{Z^2}\frac{\partial Z}{\partial m^2}\frac{\partial Z}{\partial m^2} + \frac{1}{Z}\frac{\partial^2 Z}{(\partial m^2)^2}$$

$$\frac{1}{4}\langle\langle \phi_a^2\phi_a^2\rangle\rangle = \frac{\partial^2 \ln Z}{(\partial m^2)^2} + \left(\frac{\partial \ln Z}{\partial m^2}\right)^2$$

$$\langle\langle (\phi_a^2)^2\rangle\rangle = 4\left(\frac{\partial^2 \ln Z}{(\partial m^2)^2} + \left(\frac{\partial \ln Z}{\partial m^2}\right)^2\right) = -24\,\frac{\partial \ln Z}{\partial \lambda}$$

gives an equation. (alternative way to get Z).

$$\ln Z = -\frac{N}{2}\ln m^2 + \ln f(\lambda/m^4)$$

$\partial \ln Z / \partial \lambda \rightarrow$ reduces order of pert. theory.

$\partial \ln Z / \partial m^2 \rightarrow$ better.

$$\frac{\partial \ln Z}{\partial m^2} = -\frac{N}{2m^2} + \frac{\lambda}{3\times 4}\frac{N(N+2)}{m^6} - \frac{\lambda^2}{2^2\times 3^2\times m^{10}} N(N+2)(N+3) + \cdots$$

$$\frac{\partial^2 \ln Z}{(\partial m^2)^2} = \frac{N}{2m^4} - \frac{\lambda}{4}\frac{N(N+2)}{m^8} + \frac{5\lambda^2}{2^2\times 3^2 m^{12}} N(N+2)(N+3) + \cdots$$

$$4\left[\left(\frac{\partial^2 \ln Z}{(\partial m^2)^2}\right) + \left(\frac{\partial \ln Z}{\partial m^2}\right)^2\right] = 4\left[\frac{N}{2m^4} - \frac{\lambda}{4}\frac{N(N+2)}{m^8} + \frac{5\lambda^2}{2^2\times 3^2 m^{12}} N(N+2)(N+3)\right.$$

$$+ \frac{N^2}{4m^4} + \frac{\lambda^2}{3^2\times 2^4}\frac{N^2(N+2)^2}{m^{12}} - \frac{N}{m^2}\frac{\lambda}{12}\frac{N(N+2)}{m^6} +$$

$$\left. + \frac{N}{m^2}\frac{\lambda^2}{2^2\times 3^2\times m^{10}} N(N+2)(N+3) + \cdots\right]$$

$$= 4\left[\frac{N}{4m^4}(2+N) - \frac{\lambda N(N+2)}{12 m^8}(N+3) + \frac{\lambda^2}{m^{12}}\frac{N(N+2)}{2^2 3^2}\left(5(N+3) + \frac{N(N+2)}{4} + N(N+3)\right)\right]$$

$$= \frac{N(N+2)}{m^4} - \frac{\lambda N(N+2)(N+3)}{3m^8} + \frac{4\lambda^2}{m^{12}}\frac{N(N+2)}{36}\left(5N+15+\frac{N^2}{4}+\frac{N}{2}+\right.$$

$$\left.+N^2+3N\right)$$

$$= \frac{1}{m^4}N(N+2) - \frac{1}{3}\frac{\lambda}{m^8}N(N+2)(N+3) + \frac{\lambda^2 4 N(N+2)}{m^{12}\;36}\left(\frac{17}{2}N+\frac{5}{4}N^2+15\right)$$

$$= \frac{1}{m^4}N(N+2) - \frac{1}{3}\frac{\lambda}{m^8}N(N+2)(N+3) + \frac{\lambda^2}{m^{12}}\frac{N(N+2)}{36\times 4}\left(34N+5N^2+60\right)$$

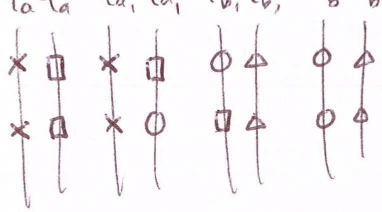
$$\langle\langle(\phi_a^2)^2\rangle\rangle = \frac{1}{m^4}N(N+2) - \frac{\lambda}{3m^8}N(N+2)(N+3) + \frac{\lambda^2}{m^{12}}\frac{N(N+2)}{36}\left(34N+5N^2+60\right)+\ldots$$

Diagrams

$$= \frac{1}{m^4}(2N+N^2) = \frac{1}{m^4}N(N+2)$$

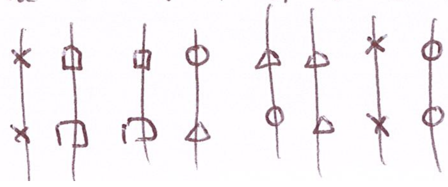
$$= -\frac{\lambda}{4!}8N(N+2)\frac{1}{m^8}$$

$\phi_a\phi_a \quad \phi_{a_1}\phi_{a_1} \quad \phi_{b_1}\phi_{b_1} \quad \phi_b\phi_b$



$$\left.\begin{array}{l} 4\times 1\times 2\,N^2 \\ 4\times 2\times 2\cdot N \end{array}\right\} 8(N^2+2N) = 8N(N+2)$$

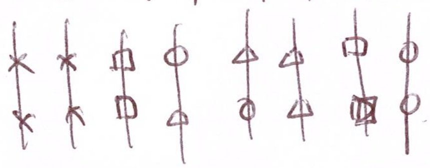
$\phi_a\phi_a \quad \phi_{a_1}\phi_{a_1} \quad \phi_{b_1}\phi_{b_1} \quad \phi_b\phi_b \rightarrow -\frac{\lambda}{4!}16N(N+2)\frac{1}{m^8}$



$$\left.\begin{array}{l} 4\times 4\times 1\times N^2 \\ 4\times 4\times 2\times N \end{array}\right\} 16N(N+2)$$

 $\longrightarrow \quad -\dfrac{\lambda}{4!}\, 4N^2(N+2)\, \dfrac{1}{m^8}$

$\phi_a\,\phi_a \quad \phi_{a_1}\phi_{a_1} \quad \phi_{b_1}\phi_{b_1} \quad \phi_b\,\phi_b$

 $\left.\begin{matrix} 4\times 1\times N^3 \\ 4\times 2\times N^2 \end{matrix}\right\} \; 4N^2(N+2)$

 $\quad 4N^2(N+2) \;\rightarrow\; -\dfrac{\lambda}{4!}\, 4N^2(N+2)\, \dfrac{1}{m^8}$

 $=$

$$= -\frac{\lambda}{4!}\,\frac{8N(N+2)}{m^8}\left(1+2+N\right) = -\frac{\lambda}{3m^8}\,N(N+2)(N+3) \quad \checkmark$$

1 2 3 4