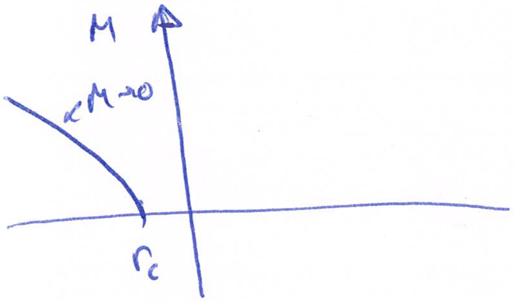
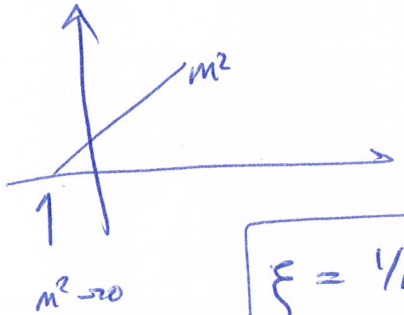
Critical behavior

①

At the critical point of a second order phase transition the correlation length diverges. $m^2 \to 0$





$$\boxed{\xi = 1/m}$$

$$\langle \sigma(\vec{r})\sigma(0)\rangle \underset{r\to\infty}{\sim} \frac{e^{-mr}}{r^p} = \frac{e^{-r/\xi}}{r^p}$$

$\xi \to \infty$; $m^2 \to 0$ $\quad \langle \sigma(\vec{r})\,\sigma(0)\rangle \sim 1/r^p$ power law.

scale invariance

$$\boxed{\Delta_\sigma = p/2}$$

Critical exponents

$Z = e^{-\beta(A)}$; $A = -\frac{1}{\beta}\ln Z$ ← includes magnetic field H.

$M = -\frac{\partial A}{\partial H}$ magnetization

$\chi = \frac{1}{V}\frac{\partial M}{\partial H}$ susceptibility $\qquad C = -T\frac{\partial^2 A}{\partial T^2}$ specific heat

$A = E - TS \qquad dE = TdS - pdV + \mu dN \qquad dA = -SdT - pdV + \mu dN - MdH$

MdH

adimensional temperature $t = \frac{T-T_c}{T_c}$ ← critical temp.

$t \to 0$ critical point

Critical exponents:

$$\langle M(r) M(0) \rangle \sim \frac{e^{-r/\xi}}{r^p} \qquad p = d-2+\eta$$

$$\xi \sim t^{-\nu} \qquad (m \sim t^{\nu})$$

$$\frac{M}{V} = m \sim |t|^{\beta} \qquad \chi \sim |t|^{-\gamma}$$

$$C \sim |t|^{-\alpha} \qquad \frac{M}{V} \sim H^{1/\delta} \quad (t=0)$$

Scaling hypothesis : ξ: only length scale.
(near the conformal point.)

βA : adimensional

$$a = \frac{A}{V} \sim \xi^{-d} = t^{\nu d}$$

$m \sim \xi^{-p/2}$ from correlation function. $m \sim t^{\nu p/2} \to \boxed{\beta = \frac{\nu}{2}(d-2+\eta)}$

$$m \sim -\frac{\partial a}{\partial H} \qquad \xi^{-p/2} \sim \frac{\xi^{-d}}{H} \qquad H \sim \xi^{-d+p/2}$$

$$\chi \approx \partial m / \partial H \sim \xi^{-p/2+d-p/2} = \xi^{d-p} = \xi^{\cancel{d}-\cancel{d}+2-\eta} = t^{\nu(\eta-2)}$$

$$\boxed{\gamma = \nu(2-\eta)}$$

③

$$\frac{C}{V} \simeq -\frac{T}{V}\frac{\partial^2 A}{\partial T^2} \sim t^{\nu d-2} \Rightarrow \boxed{\alpha = 2-\nu d}$$

$$m \sim H^{1/\delta} \qquad \left(\xi^{-p/2}\right)^{\delta} \sim \xi^{-d+p/2}$$

$$-p\frac{\delta}{2} = -d+p/2 \Rightarrow \delta = -\frac{2}{p}\left(\frac{p}{2}-d\right) = \frac{2d-p}{p} = \frac{d+2-\eta}{d-2+\eta}$$

$$\boxed{\alpha = 2-\nu d} \qquad \boxed{\gamma = \nu(2-\eta)}$$

$$\boxed{\beta = \frac{\nu}{2}(d-2+\eta)} \qquad \boxed{\delta = \frac{d+2-\eta}{d-2+\eta}}$$

We only need $\boxed{\nu, \eta}$

For $(\phi_a^2)^2$ in dimension d: (Large N-limit)

$$\langle\sigma(\vec{r})\sigma(0)\rangle = \int d^dk \, \frac{e^{i\vec{k}\vec{r}}}{k^2+m^2} \underset{r\to\infty}{\sim} \frac{e^{-mr}}{r^{d-2}}$$

dimensional analysis.

$p = d-2 \Rightarrow \boxed{\eta = 0}$

~~ma~~ m solution of $$-\frac{6m^2}{\bar{u}} + \frac{6}{\bar{u}}(r-r_c) - \frac{2\Gamma\left(\frac{4-d}{2}\right)}{(4\pi)^{d/2}(d-2)} m^{d-2} = 0$$

$r - r_c = r'(0)\, t$

if $\underline{d>4}$ $\Rightarrow$ $-\frac{6m^2}{\bar{u}} = \frac{6}{\bar{u}}(r-r_c)$ $\qquad m \sim (r_c-r)^{1/2}$ $\qquad \boxed{\nu = 1/2}$

if $\underline{d<4}$ $\Rightarrow$ $\frac{6}{\bar{u}}(r-r_c) = \frac{2\Gamma\left(\frac{4-d}{2}\right)}{} m^{d-2}$ $\Rightarrow m \sim (r_c-r)^{\frac{1}{d-2}}$ $\qquad \boxed{\nu = \frac{1}{d-2}}$

Also

$$\sigma^2 = -\frac{6(r-r_c)}{\bar{u}} \Rightarrow \quad \sigma \sim (r_c - r_0)^{1/2} \quad \Rightarrow \boxed{\beta = 1/2}$$

agrees!

But $\beta = \frac{\nu}{2}(d-2+\eta) = \frac{1}{\cancel{d-2}} \, \frac{1}{2} \, (\cancel{d-2}) = 1/2$

Also all dependence of A in temp is in r

$$\frac{\partial \ln Z}{\partial r} = -\frac{N}{2} \langle\langle \rho \rangle\rangle V$$

$$+\frac{\partial a}{\partial r} = +\frac{N}{2} \langle\langle \rho \rangle\rangle = \frac{N}{2} \rho = \frac{6N}{2} (\lambda - r)$$

$$\frac{\partial^2 a}{\partial r^2} = 3N \frac{\partial \lambda}{\partial r} - 3N$$

$$= 3N \left(\frac{\partial m^2}{\partial r} - 1 \right) \qquad \underset{\uparrow \text{definition}}{\lambda = m^2}$$

$$m^2 \sim (r_c - r)^{2/d-2} \qquad \frac{\partial m^2}{\partial r} \sim (r_c - r)^{\frac{2}{d-2} - 1} = (r_c - r)^{\frac{2-d+2}{d-2} = -\frac{d-4}{d-2}}$$

$$\frac{\partial^2 a}{\partial r^2} \sim t^{-\frac{d-4}{d-2}} \qquad \boxed{\alpha = \frac{d-4}{d-2}}$$

agrees!

But $\alpha = 2 - \nu d = 2 - \frac{d}{d-2} = \frac{2d-4-d}{d-2} = \frac{d-4}{d-2}$

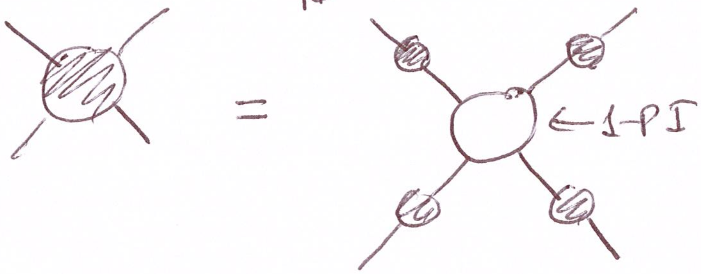
-) 1-PI functions

$\int (\partial\phi)^2 d^d x \quad [\phi] = M^{-1+d/2} = M^{\frac{d-2}{2}}$ ①

$$\int e^{-ip_1x_1 - \dots - ip_nx_n} d^dx_1 \dots d^dx_n \, G(x_1 \dots x_n) = \overset{M^{-d}}{(2\pi)^d \delta(p_1 + \dots + p_n)} \, G(p_1, \dots p_n)$$

$M^{-nd+n\frac{d-2}{2}}$ $\quad$ $M^{d-nd+n\frac{d-2}{2}}$

$G^{(2)}(p) = M^{d-\cancel{2d}+\cancel{d}-}$



$G^{(2)} \sim \frac{1}{x^{d-2+\eta}}$

$G^{(2)}(p) \sim p^{d-\cancel{2d}+\cancel{d}-2-}$

$$\boxed{G^{(2)}(p) \sim p^{\eta-2}\Lambda^{-\eta}}$$

$$G(p_1 \dots p_n) = G^{(2)}(p_1) \dots G^{(2)}(p_n) \, \overset{\text{1-PI}}{\Gamma^{(n)}(p_1 \dots p_n)}$$

$$G^{(2)}(p_1, p_2) = G^{(2)}(p_1) \, G^{(2)}(p_2) \, \Gamma^{(2)}(p_1, p_2)$$

$p_1 = p_2$

$$\boxed{\Gamma^{(2)}(p) = \left(G^{(2)}(p)\right)^{-1}}$$

Also $\int d^dx \, t \, \phi_a^2$ dependence

$\xi \sim t^{-\nu}$ $\quad$ $M^{-d} \, M^{1/\nu} \, M^{d_{\phi^2}}$

$-d + 1/\nu + d_{\phi^2} = 0 \qquad \boxed{d_{\phi^2} = d - 1/\nu}$

$\langle \phi^2 \phi^2 \rangle \sim \frac{1}{|x-y|^{2d_{\phi^2}}}$

$\Rightarrow \Gamma^{(2,0)} = \Lambda^{2/\nu - 4} q^{d-2/\nu}$

We renormalize $\Gamma^{(n)}$ since $-G-$ renormalized separately.

$\int \langle \phi^2 \phi^2 \rangle e^{iqx} e^{ipy} d^dx \, d^dy = \delta(q+p) \, \Gamma^{(2,0)}$

$$\langle \phi \dots \phi \rangle = \langle \phi\phi \rangle \dots \langle \phi\phi \rangle \, \Gamma$$

$$\Gamma_R^{(n)} = \frac{\langle \phi_R \dots \phi_R \rangle}{\langle \phi\phi \rangle_{RR} \dots \langle \phi\phi \rangle_{RR}} = Z_\phi^{n/2}(g_B, \Lambda) \, \Gamma_B (g_B(g_R, \Lambda); p_i, \Lambda)$$

$$\phi_R = Z^{-1/2} \phi_B$$

$$\Gamma_R^{(n)}(g_R, \mu, p_i) = Z_\phi^{n/2}(g_B(g_R, \mu); \Lambda)\, \Gamma_B^{(n)}(g_B(g_R, \mu); p_i, \Lambda)$$

$$\Lambda \frac{\partial}{\partial \Lambda} \Gamma_R = 0 = \frac{n}{2} \frac{1}{\Lambda} \frac{\partial Z_\phi}{\partial \Lambda} Z_\phi^{n/2 - 1} \Gamma_B +$$

$$+ Z_\phi^{n/2} \Lambda \frac{\partial \Gamma_B}{\partial g_B} \frac{\partial g_B}{\partial \Lambda} + Z_\phi^{n/2} \Lambda \frac{\partial \Gamma_B}{\partial \Lambda} = 0$$

$$\Lambda \frac{\partial \Gamma_B}{\partial \Lambda} + \underbrace{\Lambda \frac{\partial g_B}{\partial \Lambda}}_{\beta} \frac{\partial \Gamma_B}{\partial g_B} + \underbrace{\frac{n}{2} \frac{\partial \ln Z_\phi}{\partial \Lambda}}_{-\eta} \Gamma_B = 0$$

$$\Lambda \frac{\partial \Gamma_B}{\partial \Lambda} + \beta \frac{\partial \Gamma_B}{\partial g_B} - \frac{n}{2} \eta\, \Gamma_B = 0$$

Γ_B does not dep on μ.

$\beta(g_B, \frac{\Lambda}{\mu})$ no units

β indep. of μ

$$\boxed{\Lambda \frac{\partial \Gamma_B^{(n)}}{\partial \Lambda} + \beta(g_B) \frac{\partial \Gamma_B^{(n)}}{\partial g_B} - \frac{n}{2} \eta(g_B)\, \Gamma_B^{(n)} = 0}$$

Suppose $\beta(g_B^*) = 0$ $\Rightarrow$ $\Lambda \frac{\partial \Gamma_B^{(n)}}{\partial \Lambda} = \frac{n}{2} \eta(g_B^*) \Gamma_B^{(n)}$

$$\Gamma_B^{(n)} = \Lambda^{\frac{n}{2}\eta} \tilde{\Gamma}^{(n)}(p_i) \Rightarrow G^{(2)} = \Lambda^{-\eta} \tilde{G}^{(2)}(p^2) = \overset{\text{constant.}}{C} \Lambda^{-\eta} p^{\eta - 2}$$

$\boxed{\eta = \text{critical exp!!}}$

$[G^{(2)}] = M^{-2}$

Cut-off dependence.

Consider a theory

$$S = \int \underset{\substack{\downarrow \\ M^{-d}}}{d^dx} \; \frac{1}{2}(\underset{\substack{| \\ M}}{\partial}\phi_a)^2 + \frac{m^2}{2}(\phi_a^2) + \frac{\lambda}{4!}(\phi_a^2)^2$$

units:

$$[\phi] = M^{d/2-1} \quad ; [m] = M^{+\frac{d}{2}-\frac{d}{2}+\frac{2}{2}} = M \checkmark$$

$$[\lambda] = M^{d-\frac{4d}{2}+4} = M^{4-d}$$

in $\underline{d=4}$ λ has no units

if we set $d=4$, $m=0$ we do not have any mass unit.

Is theory conf. inv? <u>NO</u> there is cut-off. Λ
(introduce it for another scale μ)

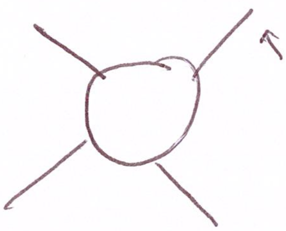
Cut-off dependence?

Use it explicitely as unit. (besides being cut-off)

$$\boxed{S = \int d^dx \quad \frac{1}{2}(\partial\phi_a)^2 + \frac{r}{2}\Lambda^2\phi_a^2 + \frac{\lambda}{4!}\Lambda^{2\varepsilon}(\phi_a^2)^2}$$

define $d=4-2\varepsilon$

Dependence on Λ from divergences



n_E : external legs

n_I : internal prop.

n_V : vertices

n_L : # of loops.

$$\boxed{4n_V = 2n_I + n_E} \qquad \boxed{n_L = n_I - n_V + 1}$$

↑ overall δ-function.

Superficial degree of divergence (scaling as $k \to \infty$)

~~d~~
$$\#_{div} = \underbrace{d . n_L}_{\prod_{i=1}^{n_L} d^d k_i} - \underset{\frac{1}{(k^2+m^2)^{n_I}}}{2n_I}$$

$$\#_{div} = dn_I - dn_V + d - 2n_I = (d-2)\left(2n_V - \frac{n_E}{2}\right) - dn_V + d$$

$$= d - \frac{(d-2)}{2} n_E + n_V(2d - 4 - d)$$

$$\boxed{\#_{div} = n_V(d-4) + d - \frac{(d-2)}{2} n_E}$$

If $d<4$ adding vertices improves convergence.

If $d=4$ " " does nothing

If $d>4$ " " makes diagram more divergent.

One can get divergences from subdiagrams.
We assume those are made convergent first.

Assume $\boxed{d=4}$

$$\#_{div} = d - \frac{d-2}{2} n_E \geq 0$$ (then it is divergent)

$$d \geq \frac{d-2}{2} n_E \qquad n_E \leq \frac{2d}{d-2} = \frac{8}{2} = 4$$

So only  and  are divergent!

Moreover taking derivative $\frac{\partial}{\partial p^2}$ ↑ external momenta

improves convergence $$\frac{\partial}{\partial p_i} \frac{1}{(p-k)^2+m^2} \sim \frac{2(p-k)}{((p-k)^2+m^2)^2} \underset{k\to\infty}{\sim} \frac{1}{k^3}$$

so

$$\#_{div}(n_E=4) = 4 - n_E = 0$$

$\frac{\partial}{\partial p_i}$  is finite : divergence is a constant! C_3

$\#_{div}(n_E=2) = 4 - n_E = 2$ $\frac{\partial}{\partial p^2}\frac{\partial}{\partial p^2}$ becomes finite

$= \boxed{\underset{\uparrow div}{C_0} + \underset{\uparrow div.}{C_1} p^2 + \text{finite}}$

C_0 can be absorbed by rescaling ϕ

C_1 is a redefinition of m^2 & C_3 of λ.

Divergent dependence in Λ is equiv to a renormalization of the parameters of the theory!

However we still need a mass scale.

$$\phi_R = Z_\phi^{-1/2}\,\phi$$

and an m_R^2 and λ_R can be defined by the value of $\Gamma^{(2)}(p^2=0)$ and $\Gamma^{(4)}$ at some arbitrary point.

For the critical theory we define:

$$\Gamma_R^{(2)}(p,\lambda_R,\mu,\Lambda)\Big|_{p^2=0} = 0 \qquad (m^2=0)$$

$$\frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p,\lambda_R,\mu,\Lambda)\Big|_{p^2=\mu^2} = 1 \qquad (\Gamma^{(2)})\ \text{~~}p^2+m^2+\text{~~}$$

$$\Gamma_R^{(4)}(p_i=\mu\,\underset{\uparrow}{\theta_i},\lambda_R,\mu,\Lambda) = \mu^{\epsilon}\lambda_R$$

constant vector

$$\Gamma_R^{(n)} = \underset{\uparrow}{Z_\phi^{n/2}}\,\Gamma^{(n)}$$

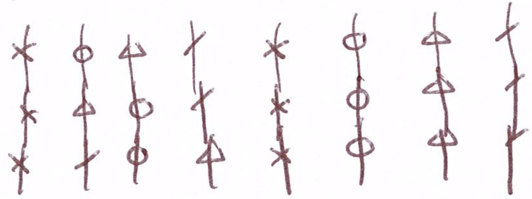
external legs.

Massless ϕ^4 theory

$$S = \int d^dx \left(\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a + \frac{\lambda}{4!} (\phi_a^2)^2 \right)$$

$$\langle\langle \phi_{a_1} \phi_{a_2} \phi_{a_3} \phi_{a_4} \rangle\rangle_c = \int_{x_1} \langle\langle -\frac{\lambda}{24} \phi_{a_1} \phi_{a_1} \phi_{b_1} \phi_{b_1} \; \phi_{c_1}(x) \phi_{c_2}(x) \phi_{c_3}(x) \phi_{c_4}(x)$$

$y_1 \; y_2 \; y_3 \; y_4$



$$-\frac{8}{24} \lambda \, \delta_{a_1 c_1} \delta_{a_1 c_2} \delta_{b_1 c_3} \delta_{b_1 c_4}$$

4 x 1 x 2

4 x 2 x 1

4 x 2 x 1

$$-\frac{8}{24} \lambda \, \delta_{a_1 c_1} \delta_{a_1 c_3} \delta_{b_1 c_2} \delta_{b_1 c_4}$$

$$-\frac{8}{24} \lambda \, \delta_{a_1 c_1} \delta_{a_1 c_4} \delta_{b_1 c_2} \delta_{b_1 c_3}$$

$$\boxed{-\frac{\lambda}{3} \left(\delta_{c_1 c_2} \delta_{c_3 c_4} + \delta_{c_1 c_3} \delta_{c_2 c_4} + \delta_{c_1 c_4} \delta_{c_2 c_3} \right)}$$

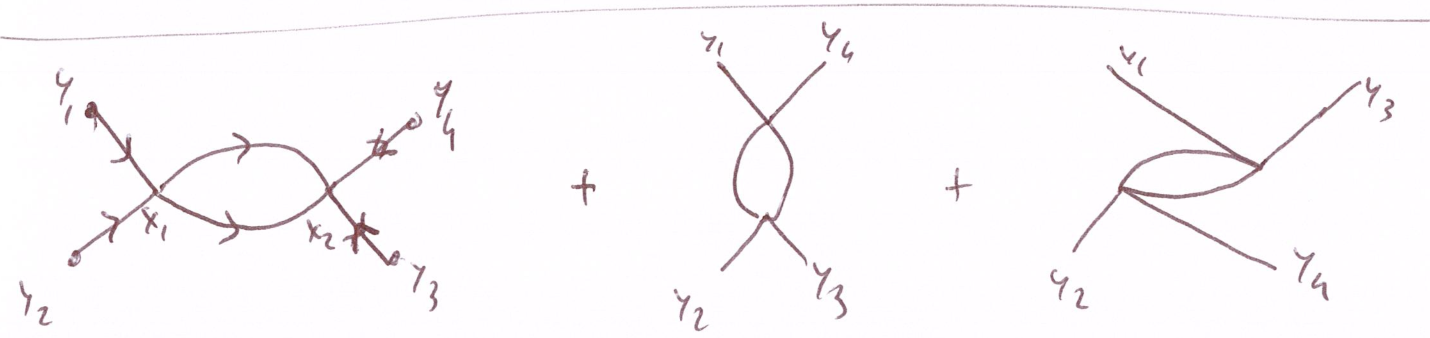
$$\int d^d x_1 \, \Delta(x_1 - y_1) \Delta(x_1 - y_2) \Delta(x_1 - y_3) \, \Delta(x_1 - y_4)$$

$\downarrow$

Fourier

$$\frac{1}{(p_1^2 + m)} \frac{1}{p_2^2} \frac{1}{p_3^2} \frac{1}{p_4^2} (2\pi)^d \delta(p_1 + \cdots + p_4$$

$$\boxed{-\frac{\lambda}{3} \left(\delta_{c_1 c_2} \delta_{c_2 c_4} + \delta_{c_1 c_3} \delta_{c_2 c_4} + \delta_{c_1 c_4} \delta_{c_2 c_3} \right)}$$



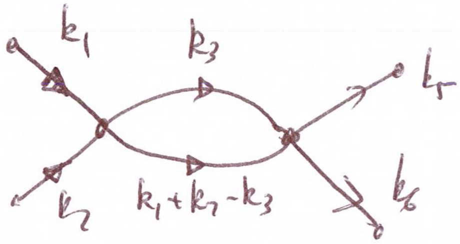
Prop. part.

$$\int \frac{d^dk_i}{(2\pi)^d} \int d^dx_1\, d^dx_2 \quad \frac{e^{ik_1(y_1-x_1)}}{k_1^2+m^2} \quad \frac{e^{ik_2(y_2-x_1)}}{k_2^2+m^2} \quad \frac{e^{ik_3(x_1-x_2)}}{k_3^2+m^2} \quad \frac{e^{ik_4(x_1-x_2)}}{k_4^2+m^2} \,.$$

$\nearrow$ $m^2=0$ but we want general recipe

$$\frac{e^{-ik_5(x_2-y_4)}}{k_5^2+m^2} \qquad \frac{e^{-ik_6(x_2-y_3)}}{k_6^2+m^2}$$

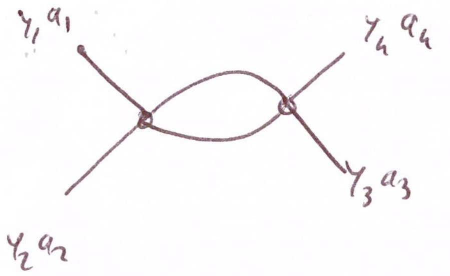
$$\int \frac{d^dk_i}{(2\pi)^d} \left((2\pi)^d\right)^2 \frac{\delta^{(d)}(\overset{-k_2}{-k_1}+k_3+k_4)\, \delta^{(d)}(\cancel{+}k_2-k_3-k_4+k_6+k_5)\, e^{ik_1y_1+ik_2y_2-ik_5y_4-ik_6}}{(k_1^2+m^2)(k_2^2+m^2)(k_3^2+m^2)(k_4^2+m^2)(k_5^2+m^2)(k_6^2+m^2)}$$

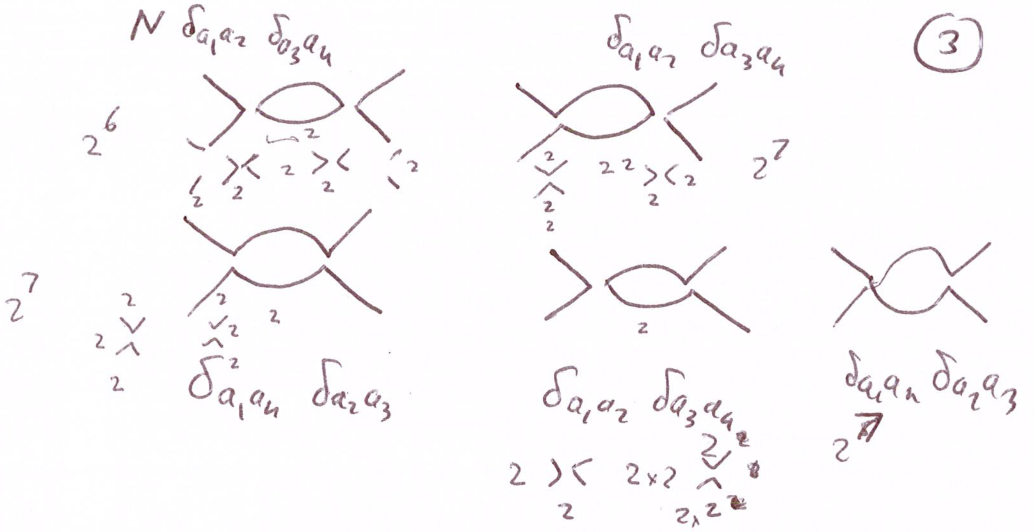


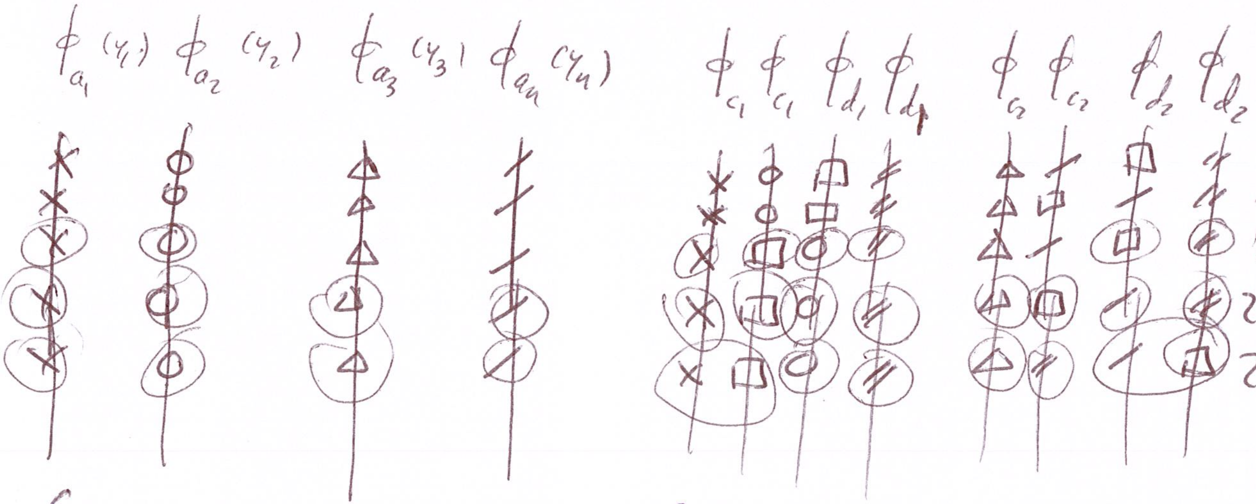
$$k_4 = k_1+k_2-k_3$$

$$\int d^dk_3 \int \frac{d^dk_1\, d^dk_2\, d^dk_5\, d^dk_6}{\left((2\pi)^d\right)^4} \frac{(2\pi)^d\, \delta^{(d)}(+k_6+k_5-k_1-k_2)\, e^{ik_1y_1+ik_2y_2-ik_5y_4-ik_6y_3}}{(k_1^2+m^2) \;-\;-\;-}$$

$$\underbrace{\int \frac{d^dk_i}{(2\pi)^d} \frac{e^{ik_1y_1+ik_2y_2-ik_5y_4-ik_6y_3}}{(k_1^2+m^2)(k_2^2+m^2)(k_5^2+m^2)(k_6^2+m^2)}}_{\text{Fourier transf. \& external propagators}} \quad \underbrace{\int d^dk_3 \frac{1}{(k_3^2+m^2)\left((k_1+k_2-k_3)^2+m^2\right)}}_{B(k_1+k_2)} (2\pi)^d\,\delta(k_1+k_2+k_5+k_6)$$







2x4x1x4x1x2

2x4x1x4x2x2

2x4x2x4x1x2

2x4x2x4x2x1

2x4x2x4x2x1

$$\frac{1}{2\cdot(2)3(4)\,(2)3(4)} = \frac{1}{9\times 2^7}$$

$$2^6 \quad \delta_{a_1a_2}\, N\, \delta_{a_3a_4} \quad \frac{\lambda^2}{2(4!)^2} \quad = \frac{1}{18}\lambda^2 (N+4)\, \delta_{a_1a_2}\delta_{a_3a_4}$$

$$2^7 \quad \delta_{a_1a_2} \quad \delta_{a_3a_4} \quad \frac{\lambda^2}{2(4!)^2}$$

$$2^7 \quad \delta_{a_1a_2}\ \delta_{a_3a_4} \quad \frac{\lambda^2}{2(4!)^2}$$

$$2^7 \quad \delta_{a_1a_3}\ \delta_{a_2a_4} \quad \frac{\lambda^2}{2(4!)^2} \qquad \frac{1}{9}\lambda^2\, \delta_{a_1a_3}\delta_{a_2a_4}$$

$$2^7 \quad \delta_{a_1a_4}\ \delta_{a_2a_3} \quad \frac{\lambda^2}{2(4!)^2} \qquad \frac{1}{9}\lambda^2\, \delta_{a_1a_4}\delta_{a_2a_3}$$

$B(p_1+p_2)$

$$\frac{\lambda^2}{18} B(p_1+p_2)\left((N+4)\,\delta_{a_1a_2}\delta_{a_3a_4} + 2\,\delta_{a_1a_3}\delta_{a_2a_4} + 2\delta_{a_1a_4}\delta_{a_2a_3}\right)$$ 

$$\frac{\lambda^2}{18} B(p_1+p_4)\left((N+4)\,\delta_{a_1a_4}\delta_{a_3a_2} + 2\,\delta_{a_1a_3}\delta_{a_2a_4} + 2\delta_{a_1a_2}\delta_{a_4a_3}\right)$$ 

$$\frac{\lambda^2}{18} B(p_1+p_3)\left((N+4)\,\delta_{a_1a_3}\delta_{a_2a_4} + 2\delta_{a_1a_2}\delta_{a_3a_4} + 2\delta_{a_1a_4}\delta_{a_2a_3}\right)$$ 

$$\frac{\lambda^2}{18}\delta_{a_1a_2}\delta_{a_3a_4}\left((N+4)B(p_1+p_2)+2B(p_1+p_4)+2B(p_1+p_3)\right)$$

$$+\frac{\lambda^2}{18}\delta_{a_1a_3}\delta_{a_2a_4}\left((N+4)B(p_1+p_3)+2B(p_1+p_2)+2B(p_1+p_4)\right)$$

$$+\frac{\lambda^2}{18}\delta_{a_1a_4}\delta_{a_2a_3}\left((N+4)B(p_1+p_4)+2B(p_1+p_2)+2B(p_1+p_3)\right)$$

Mandelstam variables: $s=(p_1+p_2)^2$ $t=(p_1+p_3)^2$ $u=(p_1+p_4)^2$

$B(p)$ is the bubble function from before.

For massless theory we get

$$B(p)=\frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(\frac{4-d}{2}\right)\left(\Gamma\left(\frac{d-2}{2}\right)\right)^2}{\Gamma(d-2)}p^{d-4}$$

$d=4-2\varepsilon$

$$B(p)=\frac{1}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(\varepsilon)\,\Gamma(1-\varepsilon)^2}{\Gamma(2-2\varepsilon)}p^{-2\varepsilon}$$

Using cut-off + ε expansion. (we keep m to be more general).

$$B(p) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2)((p-k)^2+m^2)}$$

$$\frac{1}{ab} = \int_0^1 d\alpha \frac{1}{(\alpha a + (1-\alpha) b)^2} \quad \text{Feynmann parameters.}$$

$$B(p) = \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{((1\cancel{-\alpha})k^2 + (1\cancel{-\alpha})m^2 + \alpha(p-k)^2 + \cancel{\alpha m^2})^2}$$

$$= \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2-2\alpha pk + \alpha p^2)^2} \rightarrow (k-\alpha p)^2 + m^2 + \alpha(1-\alpha)p^2$$

shift → $k \to k + \alpha p$

$$= \int_0^1 d\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2+m^2+\alpha(1-\alpha)p^2)^2}$$

$d = 4-2\varepsilon$ ↗ logarithmically divergent.

Divergent piece:

$$B^{div}_\Lambda = \int^\Lambda \frac{\Omega_{d-1}}{(2\pi)^d} \frac{k^{d-1} dk}{k^4} =$$

$$= \int^\Lambda \frac{\Omega_{d-1}}{(2\pi)^d} k^{-1-2\varepsilon} dk \simeq \frac{\Omega_{d-1}}{(2\pi)^d} \ln\Lambda = \frac{2}{(4\pi)^{d/2}} \frac{\ln\Lambda}{\Gamma(d/2)}$$

↑ leading order $\varepsilon \to 0$

$d=4$ ↗ $= \frac{1}{8\pi^2} \frac{\ln\Lambda}{\cancel{8}} = \frac{\ln\Lambda}{8\pi^2}$

$$B = \frac{1}{8\pi^2}\ln\Lambda + f(p^2, m^2, \Lambda) \rightarrow \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{p^2} + \text{---}$$

↑ finite we can take $\Lambda \to \infty$.

$$= (\delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_2a_4} + \delta_{a_1a_4}\delta_{a_2a_3})\left(-\frac{\lambda}{3} + \frac{\lambda^2}{18}(N+8)\frac{\ln\Lambda}{8\pi^2}\right)$$

$$+ G^{(f)}_{a_1 \ldots a_4}(p_1 \ldots p_4, M^2; \Lambda)$$

↑ finite $\Lambda \to \infty$.

$$\Gamma^{(4)}_{R\, a_1=a_2=a_3=a_4}(p_i = \mu\theta_i ; \lambda_R ; \mu, \Lambda) = -\mu^{2\varepsilon}\lambda_R$$

$$\Gamma^{(4)}_R(p_i ; \lambda_R, \mu, \underset{\substack{\downarrow \\ \infty}}{\Lambda}) = Z_\phi^{n/2}(g, \Lambda/\mu)\,\Gamma^{(n)}(p_i ; g, \Lambda)$$

we can remove

Assume $Z_\phi = 1$ at this order.

$$\Gamma_{a_1=a_2=a_3=a_4} = -\lambda + \frac{\lambda^2}{6}(N+8)\frac{\ln\Lambda}{8\pi^2} + \Gamma^{(f)}(\mu, \lambda_R \ldots)$$

$$\lambda \to \lambda\Lambda^{2\varepsilon}$$

$$\mu^{2\varepsilon}\lambda_R = \Lambda^{2\varepsilon}\left(\lambda - \frac{\lambda^2}{48\pi^2}(N+8)\ln\frac{\Lambda}{\mu} + \text{finite}\right)$$

↑ by units has to come from finite part.

$$\mu^{2\varepsilon}\lambda_R = \lambda_B\left(1+2\varepsilon\ln\Lambda\right) - \frac{\lambda_B^2}{48\pi^2}(N+8)\ln\frac{\Lambda}{\mu} + -$$

$$\lambda_B = \frac{(1+2\varepsilon\ln\mu)}{(1+2\varepsilon\ln\Lambda)}\lambda_R + \frac{\lambda_R^2}{48\pi^2}(N+8)\ln\frac{\Lambda}{\mu}$$

$$\lambda_B = \lambda_R\left(1-2\varepsilon\ln\frac{\Lambda}{\mu}\right) + \lambda_R^2\,\frac{(N+8)}{48\pi^2}\ln\frac{\Lambda}{\mu}$$

$$\Lambda\frac{\partial\lambda_B}{\partial\Lambda} = -2\varepsilon\lambda_R + \lambda_R^2\,\frac{(N+8)}{48\pi^2}$$

$$= -2\varepsilon\lambda_B + \lambda_B^2\,\frac{(N+8)}{48\pi^2}$$

[Keeping lowest order in ε and for λ_B.

$$\beta = -2\varepsilon\lambda_B + \frac{N+8}{48\pi^2}\lambda_B^2 \;\cancel{= -2\varepsilon\lambda_B + \frac{(N+8)}{48\pi^2}g^2}$$

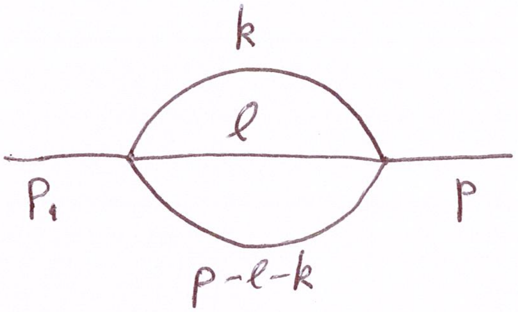
$$= -2\varepsilon\,16\pi^2 g + \frac{(N+8)}{\underset{3}{\cancel{48\pi^2}}}(16\pi^2 g)(\overset{1}{\cancel{16\pi^2}}g) = 16\pi^2\left(-2\varepsilon g + \frac{N+8}{3}g^2\right)$$

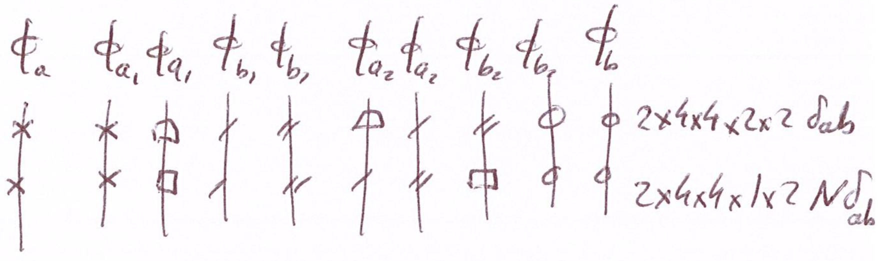
$$\beta = 0 \begin{cases} \lambda_B = 0 \\ \lambda_B^* = \dfrac{2\varepsilon\times 48\pi^2}{N+8} \end{cases}$$

$$;\; g^* = \frac{\lambda}{16\pi^2} = \frac{\cancel{2}\times\overset{6}{\cancel{48}}\,\varepsilon}{\underset{8}{\cancel{16}}\,(N+8)} = \frac{6\varepsilon}{N+8}$$

$$\boxed{\frac{\beta}{16\pi^2} = -2\varepsilon g + \frac{N+8}{3}g^2}$$

$$S = \int d^dx \; \frac{1}{2}(\partial\phi_a)^2 + \frac{\lambda}{4!}\Lambda^{2\varepsilon}(\phi_a^2)^2$$





$2\times4\times4\times2\times2\;\delta_{ab}$

$2\times4\times4\times1\times2\;N\delta_{ab}$

$$\frac{\Lambda^{4\varepsilon}}{\cancel{2}}\left(-\frac{\lambda}{4!}\right)^2 2\times4\times4\times\cancel{2}\,(N+2)\,\delta_{ab}\int\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\;\frac{1}{k^2}\;\frac{1}{l^2}\;\frac{1}{(p-l-k)^2}$$

$$\frac{\Lambda^{4\varepsilon}\lambda^2}{1\textcircled{23}\cancel{4}\;1\cancel{2}34}\,2\times4\times4\,(N+2)\,\delta_{ab}\int\frac{d^dk}{(2\pi)^d}\frac{d^dl}{(2\pi)^d}\int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3\; e^{-\alpha_1k^2-\alpha_2l^2-\alpha_3(p-l-k)^2}$$

$$\boxed{-(\alpha_1+\alpha_3)k^2-(\alpha_2+\alpha_3)l^2}-\alpha_3p^2$$
$$+2\alpha_3pl+2\alpha_3\cancel{p}k-2\alpha_3lk$$

$$\frac{N+2}{18}\lambda^2\Lambda^{4\varepsilon}\delta_{ab}\int_0^\infty d\alpha_1d\alpha_2d\alpha_3\frac{\pi^{\cancel{2d/2}}}{(\alpha_1+\alpha_3)^{d/2}(\alpha_2+\cancel{\alpha_3})^{d/2}}\;\cancel{\exp}($$

$$\int\frac{d^dk}{(2\pi)^d}\;\frac{\pi^{d/2}}{(2\pi)^d}\;\frac{1}{(\alpha_2+\alpha_3)^{d/2}}\;e^{-(\alpha_1+\alpha_3)k^2-\alpha_3p^2+2\alpha_3pk+\frac{\cancel{4}\alpha_3^2(p-k)^2}{\cancel{4}(\alpha_2+\alpha_3)}}$$

$$\frac{\alpha_3^2}{(\alpha_2+\alpha_3)}(p^2+k^2-2pk)-(\alpha_1+\alpha_3)k^2-\alpha_3p^2+2\alpha_3p.k$$

$$\frac{\cancel{\alpha_3^2}-\alpha_2\alpha_3-\cancel{\alpha_3^2}}{\alpha_2+\alpha_3}p^2+\frac{\cancel{\alpha_3^2}-\alpha_2\alpha_1-\alpha_2\alpha_3-\alpha_1\alpha_3-\cancel{\alpha_3^2}}{\alpha_2+\alpha_3}k^2+2pk\frac{-\cancel{\alpha_3^2}+\alpha_2\alpha_3+\cancel{\alpha_3^2}}{\alpha_2+\alpha_3}$$

$$\int\frac{d^dk}{(2\pi)^d}\;\frac{1}{(4\pi)^{d/2}}\;\frac{1}{(\alpha_2+\alpha_3)^{d/2}}\;e^{-\frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}p^2-\frac{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}{\alpha_2+\alpha_3}k^2+2pk\frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}}$$

$$\frac{N+2}{18}\lambda^2\Lambda^{4\varepsilon}\delta_{ab}\int d\alpha_1 d\alpha_2 d\alpha_3\,\frac{1}{(4\pi)^d}\,\frac{1}{\cancel{(\alpha_2+\alpha_3)^{d/2}}}\,\frac{\cancel{(\alpha_2+\alpha_3)^{d/2}}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{d/2}}$$

$$e^{-\frac{\alpha_2\alpha_3}{\alpha_2+\alpha_3}p^2}\; e^{\frac{\cancel{4}p^2}{\cancel{4}}\frac{\alpha_2^2\alpha_3^2}{(\alpha_2+\alpha_3)^{\cancel{2}}}\frac{\cancel{\alpha_2+\alpha_3}}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}}$$

$$\frac{N+2}{18}\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^d}\delta_{ab}\int_0^\infty\frac{d\alpha_1 d\alpha_2 d\alpha_3}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^{d/2}}\; e^{-p^2\frac{\alpha_1\alpha_2^2\alpha_3+\alpha_1\alpha_2\alpha_3^2}{(\alpha_2+\alpha_3)(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)}}$$

$\beta_1 = 1/\alpha_1 \quad \beta_2 = 1/\alpha_2 \quad \beta_3 = 1/\alpha_3 \qquad \frac{1}{\beta_1\beta_2}+\frac{1}{\beta_1\beta_3}+\frac{1}{\beta_2\beta_3}=\frac{\beta_3+\beta_2+\beta_1}{\beta_1\beta_2\beta_3}$

$d\alpha_1 = -\frac{d\beta_1}{\beta_1^2}$ ..

$$\frac{N+2}{18}\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^d}\delta_{ab}\int_0^\infty\frac{d\beta_1 d\beta_2 d\beta_3\,(\beta_1\beta_2\beta_3)^{d/2}}{\beta_1^2\beta_2^2\beta_3^2\,(\beta_1+\beta_2+\beta_3)^{d/2}}\; e^{-p^2\frac{\cancel{1/\beta_2+1/\beta_3}}{\cancel{\beta_1\beta_2\beta_3}\cancel{\left(\frac{1}{\beta_2}+\frac{1}{\beta_3}\right)}\frac{(\beta_1+\beta_2+\beta_3)}{\cancel{\beta_1\beta_2\beta_3}}}}$$

$$\frac{N+2}{18}\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^d}\delta_{ab}\int_0^\infty\int_0^\infty d\rho\,\frac{\delta(\rho-\beta_1-\beta_2-\beta_3)}{\rho^{d/2}}\,(\beta_1\beta_2\beta_3)^{\frac{d}{2}-2}\,e^{-\frac{p^2}{\rho}}$$

$\beta_i \to \beta_i\rho$ - - -

$$\frac{N+2}{18}\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^d}\delta_{ab}\int_0^\infty d\beta_i\int_0^\infty\frac{\delta(1-\beta_1-\beta_2-\beta_3)}{\rho}\,\rho^{3-d/2}\,\rho^{\frac{3d}{2}-6}\,(\beta_1\beta_2\beta_3)^{\frac{d}{2}-2}\,e^{-p^2/\rho}$$

③

$$\frac{N+2}{18}\,\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^d}\,\delta_{ab}\int_0^1 d\beta_i\,\delta(1-\Sigma\beta_i)(\beta_1\beta_2\beta_3)^{\frac{d}{2}-2}\int_0^\infty d\rho\,\rho^{d-4}\,e^{-p^2/\rho}$$

$$v=1/\rho \qquad \int_0^\infty \frac{dv}{v^2}\,v^{4-d}\,e^{-p^2v}$$

$$\int_0^\infty dv\,v^{2-d}\,e^{-p^2v} =$$

$$=(p^2)^{-1-2+d}\qquad\Gamma(3-d)$$

$$\frac{N+2}{18}\,\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^d}\,\Gamma(3-d)(p^2)^{d-3}\int_0 d\beta_i\,\delta(1-\Sigma\beta_i)\,(\beta_1\beta_2\beta_3)^{\frac{d}{2}-2}$$

$$\int_0^1 d\beta_1\int_0^{1-\beta_1} d\beta_2\,(\beta_1\,\beta_2\,(1-\beta_1-\beta_2))^{\frac{d}{2}-2} = \int_0^1 d\beta_1\int_0^1 d\tilde\beta_2\,(1-\beta_1)\beta_1^{\frac{d}{2}-2}(1-\beta_1)^{\frac{d}{2}-2}\tilde\beta_2^{\frac{d}{2}-2}$$

$$\beta_2=(1-\beta_1)\tilde\beta_2 \qquad (1-\beta_1)^{\frac{d}{2}-2}(1-\hat\beta_2)^{\frac{d}{2}-2}$$

$$=\int_0^1 d\beta_1\int_0^1 d\beta_2\,(1-\beta_1)^{d-3}\,\beta_1^{\frac{d}{2}-2}\,\beta_2^{\frac{d}{2}-2}\,(1-\beta_2)^{\frac{d}{2}-2}$$

$$=B\left(d-2,\tfrac{d}{2}-1\right)\,B\left(\tfrac{d}{2}-1,\tfrac{d}{2}-1\right)$$

$$\boxed{\frac{N+2}{18}\,\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^d}\,\Gamma(3-d)\,(p^2)^{d-3}\,B\left(d-2,\tfrac{d}{2}-1\right)\,B\left(\tfrac{d}{2}-1,\tfrac{d}{2}-1\right)}$$

$d=4-2\varepsilon \qquad 3-d=-1+2\varepsilon \qquad d-3=1-2\varepsilon$

$$\frac{N+2}{18}\,\frac{\lambda^2\Lambda^{4\varepsilon}}{(4\pi)^{4-2\varepsilon}}\,\Gamma(-1+2\varepsilon)\,p^{2-4\varepsilon}\,B(2-2\varepsilon,1-2\varepsilon)\,B(1-2\varepsilon,1-2\varepsilon)$$

$$\Gamma(-1+2\varepsilon)=\frac{\Gamma(2\varepsilon)}{-1+2\varepsilon}=\frac{\Gamma(1+2\varepsilon)}{(-1+2\varepsilon)\,2\varepsilon}=-\frac{1}{2\varepsilon}\frac{(1+2\varepsilon\gamma_1)}{1-2\varepsilon}=-\frac{1}{2\varepsilon}\left(1-2\varepsilon\gamma_1+2\varepsilon\right)$$

$$\frac{N+2}{18}\frac{\lambda^2\,\Lambda^{4\varepsilon}}{(4\pi)^4}\left(-\frac{1}{2\varepsilon}\right)p^2\,p^{-4\varepsilon}\;B(2,1)\;B(1,1)\;+\text{finite constant}$$

$$\boxed{\frac{\Gamma(2)\Gamma(1)}{\Gamma(3)}}_{2}\qquad\boxed{\frac{\Gamma(1)\Gamma(1)}{\Gamma(2)}}_{1}$$

$$=-\frac{1}{\varepsilon}\frac{N+2}{4\times 18}\frac{\lambda^2}{(16\pi^2)^2}\,p^2\left(1+4\varepsilon\ln\Lambda-4\varepsilon\ln p\right)\qquad\boxed{g=\frac{\lambda}{16\pi^2}}$$

$$=-\frac{1}{\varepsilon}\frac{N+2}{4\times 18}g^2p^2-\frac{1}{4\times\cancel{\varepsilon}}\frac{N+2}{\underset{9}{\cancel{18}}}g^2p^2\,\overset{2}{4\cancel{\varepsilon}}\ln\frac{\Lambda}{p}$$

$$=-\frac{1}{\varepsilon}\frac{N+2}{4\times 18}g^2p^2-\frac{2}{4\times 9}(N+2)\,g^2p^2\ln\frac{\Lambda}{p}+\underset{\text{const}}{\text{finite}}+\ldots$$

$$p^2+m^2-\Sigma(p^2)$$

$$\cancel{\partial p^{(2)}}\qquad\Gamma^{(2)}=p^2+\frac{\lambda}{6}\Delta(0)+\frac{1}{\varepsilon}\frac{N+2}{4\times 18}g^2p^2+\frac{2}{9\times 4}(N+2)g^2p^2\ln\frac{\Lambda}{p}+\ldots$$

$$-\frac{p^2}{2}\ln p^2$$

$$\left.\frac{\partial\Gamma^{(2)}}{\partial p^2}\right|_{p^2=0}=1+\frac{1}{\varepsilon}\frac{N+2}{18\times 4}g^2+\frac{2}{9\times 4}(N+2)\,g^2\ln\frac{\Lambda}{p}+\frac{2}{9}(N+2)g^2\cancel{p^2}\frac{1}{\cancel{p^2}}$$

$$\left.\frac{\partial\Gamma_k^{(2)}}{\partial p^2}\right|_{p^2=\mu^2}\cancel{\varepsilon 0}=Z_\phi\left.\left(\quad\right)\right|_{p^2=\mu^2}=1$$

$$Z_\phi = 1 + \frac{1}{\epsilon}\frac{N+2}{4\times18}g^2 + \frac{2}{9\times4}(N+2)g^2\ln\frac{\Lambda}{\mu} + \frac{2}{9\times4}(N+2)g^2$$

$$\ln Z_\phi = -\frac{1}{\epsilon}\frac{N+2}{4\times18}g^2 + \frac{2}{9\times4}(N+2)g^2\ln\frac{\Lambda}{\mu} + \frac{2}{9\times4}(N+2)g^2$$

$$\frac{\partial\ln Z_\phi}{\partial\Lambda} = \frac{2}{9\times4}\frac{(N+2)}{\Lambda}g^2$$

$$\eta = -\Lambda\frac{\partial\ln Z_\phi}{\partial\Lambda} = +\frac{2}{9\times4}(N+2)g^2$$

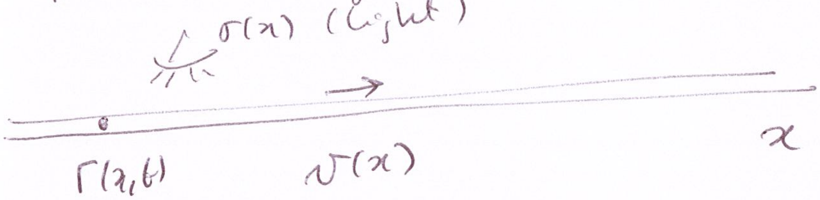
$$\eta^* = +\frac{2}{\cancel{9}}\frac{(N+2)}{4}\frac{\overset{4}{\cancel{36}}\epsilon^2}{(N+8)^2} = +\frac{8(N+2)}{4(N+8)^2}\epsilon^2 = \frac{2(N+2)}{(N+8)^2}\epsilon^2$$

$$\underset{\overset{\uparrow}{\epsilon=1/2}}{\eta} = \frac{N+2}{2(N+8)^2} =$$

$N=1$	0.0185	0.03~~6~~
$N=2$	0.02	0.038
$N=3$	0.021	0.038

$$\Lambda \frac{\partial\Gamma}{\partial\Lambda} + \beta(g)\frac{\partial\Gamma}{\partial g} = \frac{n}{2}\gamma(g)\Gamma$$

Bacterial analogue.



$$\Gamma(t+\Delta t, x+\Delta x) = \Gamma(t,x) + \sigma(x)\Delta t\,\Gamma(t,x)$$

$$\partial_t\Gamma + v\,\partial_x\Gamma = \sigma\Gamma$$

Solution: $\Gamma(x,t)$, given $\Gamma(x,t_0)$

Find $(x_0,t_0) \xrightarrow{t_0\to t} (x,t)$

$x(t) \,/\, x(t_0)=x_0$ $\qquad \frac{dx}{dt}=v(x)$

$$\int_{x_0}^{x}\frac{d\tilde{x}}{v(\tilde{x})} = t-t_0$$

$$\Gamma(x,t) = e^{\int_{t_0}^{t}\sigma(x(\tilde{t}))d\tilde{t}}\,\Gamma(\overset{x_0(x,t)}{x_0},t_0)$$

$$\partial_t\Gamma = \sigma(x)\Gamma + e^{\int}\frac{\partial\Gamma}{\partial x_0}\partial_t x_0$$

$$\partial_x\Gamma = e^{\int}\frac{\partial\Gamma}{\partial x_0}\frac{\partial x_0}{\partial x}$$

We need:

$$\partial_t x_0 + v(x)\,\frac{\partial x_0}{\partial x} = 0$$

$$\int_{x_0}^{x} \frac{d\tilde{x}}{v(\tilde{x})} = t - t_0$$

$$\frac{dx}{v(x)} - \frac{dx_0}{v(x_0)} = dt$$

$$\frac{dx}{v(x)} - \frac{1}{v(x_0)}\left(\frac{\partial x_0}{\partial x}\right)dx - \frac{1}{v(x_0)}\frac{\partial x_0}{\partial t}\,dt = dt$$

$$\frac{\partial x_0}{\partial x} = \frac{v(x_0)}{v(x)} \qquad \frac{\partial x_0}{\partial t} = -v(x_0)$$

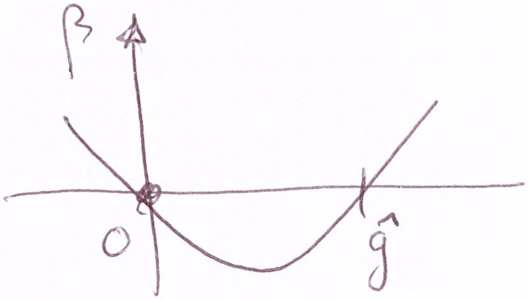
$$\partial_t x_0 + v(x)\frac{\partial x_0}{\partial x} = -v(x_0) + v\cancel{(x)}\,\frac{v(x_0)}{\cancel{v(x)}} = \underline{0 \checkmark}.$$

Define $g_0(g, \Lambda, \Lambda_0)$

$$\boxed{\int_{g_0}^{g} \frac{d\tilde{g}}{\beta(\tilde{g})} = \ln\frac{\Lambda}{\Lambda_0}} \qquad \Gamma(g, \Lambda p_i) = e^{\frac{n}{2}\int_{\Lambda_0}^{\Lambda}\gamma(g(\Lambda, g_0))\frac{d\Lambda}{\Lambda}}\;\Gamma(g_0, \Lambda_0, p_i)$$

$$\beta = 16\pi^2 \, (-2\varepsilon)\,(g)\,\left(1 - \frac{N+8}{6\varepsilon} g\right)$$

$$= -32\pi^2 \varepsilon \, g\,(1 - g/\hat{g}) = \frac{32\pi^2\varepsilon}{\hat{g}}\, g\,(g-\hat{g})$$



$$\frac{\hat{g}}{32\pi^2\varepsilon} \int_{g_0}^{g} \frac{d\tilde{g}}{g(g-\hat{g})} = \ln \frac{\Lambda}{\Lambda_0}$$

$$-\frac{1}{g} + \frac{1}{g-\hat{g}} = -\frac{g-\hat{g}-g}{g(g-\hat{g})} = \frac{\hat{g}}{g(g-\hat{g})}$$

$$-\ln g + \ln(g-\hat{g}) \Big|_{g_0}^{g} = 32\pi^2\varepsilon \ln \frac{\Lambda}{\Lambda_0}$$

$$\ln(1-\hat{g}/g)\Big|_{g_0}^{g} = 32\pi^2\varepsilon \ln \frac{\Lambda}{\Lambda_0}$$

$$\ln \frac{1-\hat{g}/g}{1-\hat{g}/g_0} = 32\pi^2\varepsilon \ln \Lambda/\Lambda_0$$

$$1-\hat{g}/g = (1-\hat{g}/g_0)\left(\frac{\Lambda}{\Lambda_0}\right)^{32n^2\varepsilon}$$

$$1-\hat{g}/g_0 = \left(\frac{\Lambda_0}{\Lambda}\right)^{32n^2\varepsilon}(1-\hat{g}/g)$$

$$\hat{g}/g_0 = 1-\left(\frac{\Lambda_0}{\Lambda}\right)^{32n^2\varepsilon}(1-\hat{g}/g)$$

$$g_0 = \frac{\hat{g}}{1+\frac{\hat{g}-g}{g}\left(\frac{\Lambda_0}{\Lambda}\right)^{32n^2\varepsilon}}$$

$$\boxed{\Lambda\to\infty \qquad g_0\to\hat{g}} \qquad\qquad \boxed{\varepsilon>0}$$

$$\Lambda\to\Lambda_0 \qquad g_0\to g$$

$$\Lambda\to 0 \qquad g_0\to 0$$

$$\Gamma(g,\Lambda\to\infty,p_0) = e^{\frac{n}{2}\int_{\Lambda_0}^{\Lambda}\eta(g(\lambda,g_0))\frac{d\lambda}{\lambda}}\,\Gamma(\hat{g},\Lambda_0,p_0)$$

⑤

$$g(\Lambda, g_0) = \frac{\hat{g}}{1 + \frac{\hat{g} - g_0}{g_0} \left(\frac{\Lambda}{\Lambda_0} \right)^{32\pi^2 \varepsilon}}$$

if

$g_0 \not\simeq \hat{g}$

$$g(\Lambda, g_0) \simeq \hat{g}_{\infty} \qquad \eta(\hat{g}) = \hat{\eta}$$

$$\Gamma(g, \Lambda \to \infty, p_i) = e^{\frac{n}{2} \hat{\eta} \ln \frac{\Lambda}{\Lambda_0}} \Gamma(\hat{g}, \Lambda_0, p_i)$$

$$= \left(\frac{\Lambda}{\Lambda_0} \right)^{\frac{n}{2} \hat{\eta}} \Gamma(\hat{g}, \Lambda_0, p_i)$$

$\uparrow$

at critical point.