Conformal theories

At criticality there is a scale invariance. Under general conditions it generalizes to a conformal symmetry.

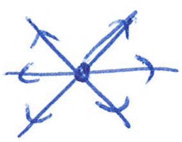
Lorentz invariance : $x'^\mu = \Lambda^\mu{}_\nu x^\nu \rightarrow ds'^2 = ds^2$

preserves distances

Conformal invariance : $x'^\mu = x'^\mu(x^\nu) \rightarrow ds'^2 = \Omega(x)\, ds^2$

conformal factor.

Local scale invariance.


preserves angles but not distances (except when $\Omega = 1$)

$\boxed{d \geq 3}$

$$dx'^2 = \eta_{\mu\nu} \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} dx^\alpha dx^\beta = \Omega(x)\, \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

$x'^\mu = x^\mu + \epsilon^\mu$: infinitesimal transf. $\epsilon^\mu(x)$

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu{}_\alpha + \partial_\alpha \epsilon^\mu$$

$$dx'^2 = \eta_{\alpha\beta}\, dx^\alpha dx^\beta + \eta_{\mu\nu} \partial_\alpha \epsilon^\mu dx^\alpha dx^\nu + \eta_{\mu\nu} \partial_\beta \epsilon^\nu dx^\mu dx^\beta + O(\epsilon^2)$$

$$\eta_{\mu\nu}\left(\partial_\alpha \epsilon^\mu dx^\alpha dx^\nu + \partial_\beta \epsilon^\nu dx^\mu dx^\beta\right) = \delta\Omega\, dx^\mu dx^\nu \eta_{\mu\nu}$$

$$\eta_{\alpha\nu}\, \partial_\mu \epsilon^\alpha dx^\mu dx^\nu + \eta_{\alpha\mu}\, \partial_\nu \epsilon^\alpha dx^\mu dx^\nu = \delta\Omega\, dx^\mu dx^\nu \eta_{\mu\nu}$$

$$\eta_{\alpha\nu}\partial_\mu \epsilon^\alpha + \eta_{\alpha\mu}\, \partial_\nu \epsilon^\alpha = \delta\Omega\, \eta_{\mu\nu}$$

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \delta\Omega\, \eta_{\mu\nu}}$$

$\eta^{\mu\nu} \otimes \qquad 2\partial_\mu \epsilon^\mu = d\,\delta\Omega \quad \Rightarrow \delta\Omega = \frac{2}{d}\partial_\mu \epsilon^\mu$

$$\boxed{\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}\, \partial_\alpha \epsilon^\alpha \eta_{\mu\nu}}$$

$\partial^\mu \rightarrow \qquad \partial^2 \epsilon_\nu + \partial_\nu \partial_\mu \epsilon^\mu = \frac{2}{d} \partial_\nu \partial_\alpha \epsilon^\alpha$

$\partial^\nu \rightarrow \qquad \cancel{2}\partial^2(\partial\epsilon) = \frac{\cancel{2}}{d}\partial^2(\partial\epsilon)$

$$\boxed{\partial^2(\partial\epsilon) = 0}$$

$\boxed{\partial\epsilon \text{ solves Laplace equation.}}$

If we do not allow singularities (except at ∞) then the only solution is $\partial\epsilon = \alpha + \beta_\mu x^\mu$ linear. In electrostatic terms, with no sources we can only have constant $\vec{E}$, namely linear Φ.

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \underbrace{\left(A + B_\rho x^\rho \right)}_{(\partial \epsilon)} \eta_{\mu\nu}$$

$$\partial^\mu \to \quad \partial^2 \epsilon_\nu + \partial_\nu (\partial \epsilon) = \frac{2}{d} \partial^\mu (\partial \epsilon) \eta_{\mu\nu} = \frac{2}{d} \partial^\nu (\partial \epsilon)$$

$$\partial^2 \epsilon_\nu = \frac{2-d}{d} \partial_\nu (\partial \epsilon) = \frac{2-d}{d} B_\nu$$

(If $d=2 \to \partial^2 \epsilon_\nu = 0$)

$$\partial^2 \epsilon_\nu = \text{constant} \Rightarrow \quad \epsilon_\nu = A_\nu + B_{\nu\rho} x^\rho + C_{\nu\underline{\rho\delta}} x^\rho x^\delta$$

symmetric.

$$\partial_\mu \epsilon_\nu = B_{\nu\mu} + 2 C_{\nu\mu\delta} x^\delta$$

$$B_{\nu\mu} + B_{\mu\nu} + 2 \left(C_{\nu\mu\delta} + C_{\mu\nu\delta} \right) x^\delta = \frac{2}{d} \left(A + B_\rho x^\rho \right) \eta_{\mu\nu}$$

$$B_{\nu\mu} + B_{\mu\nu} = \frac{2}{d} A \eta_{\mu\nu}$$

$$B_{\mu\nu} = \frac{1}{d} A \eta_{\mu\nu} + B_{[\mu\nu]}$$

antisymmetric

$$\left.\begin{array}{lll} C_{\nu\mu\delta} + \cancel{C_{\mu\nu\delta}} = \frac{1}{d} B_\delta \eta_{\mu\nu} & (+) \\ C_{\nu\delta\mu} + \cancel{C_{\delta\nu\mu}} = \frac{1}{d} B_\mu \eta_{\delta\nu} & (+) \\ \cancel{C_{\delta\mu\nu}} + \cancel{C_{\mu\delta\nu}} = \frac{1}{d} B_\nu \eta_{\mu\delta} & (-) \end{array}\right\} \to 2 C_{\nu\mu\delta} =$$

$$= \frac{1}{d} \left(B_\delta \eta_{\mu\nu} + B_\mu \eta_{\delta\nu} - B_\nu \eta_{\mu\delta} \right)$$

Most general:

$$\epsilon_\mu = \underbrace{a_\mu}_{\text{transl.}} + \frac{2}{d} A \overbrace{x_\mu}^{\text{dilatation}} + \overbrace{B_{[\mu\nu]} x^\nu}^{\text{rotation}} +$$

$$+ \frac{1}{2d} \underbrace{\left(B_\delta \eta_{\mu\nu} + B_\nu \eta_{\delta\mu} - B_\mu \eta_{\nu\delta} \right) x^\nu x^\delta}_{\text{special conf. transf.}}$$

$$\epsilon_\mu = a_\mu + \alpha x_\mu + b_{[\mu\nu]} x^\nu +$$

$$+ \left(2(\tilde{a}.x)\, x_\mu - \tilde{a}_\mu \{x^2\} \right)$$

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global:

$x^\mu \to x^\mu + a^\mu$

$x^\mu \to \lambda x^\mu$

$x^\mu \to A^\mu{}_\nu x^\nu$

$$x^\mu \underset{\text{inv.}}{\to} \frac{x^\mu}{x^2} \underset{\text{transl.}}{\to} \frac{x^\mu}{x^2} + a^\mu \underset{\text{inv.}}{\to} \frac{x^\mu/x^2 + a^\mu}{\left(x^\mu/x^2 + a^\mu\right)^2} = \frac{x^\mu + x^2 a^\mu}{\frac{x^2}{x^4}\left(x^\mu + a^\mu x^2\right)^2} =$$

$$= \frac{x^2 (x^\mu + x^2 a^\mu)}{(x^2 + 2ax\, x^2 + a^2 x^4)} = \frac{x^\mu + a^\mu x^2}{(1 + 2ax + x^2 a^2)}$$

inversion preserves angles.

$$\delta x^\mu = a^\mu x^2 - 2ax\, x^\mu \qquad \text{(first order in } a^\mu\text{)}$$

Scalo factor:

$$dx'^\mu = \frac{dx^\mu + 2a^\mu (xdx)}{h} - \frac{1}{h^2}\left(x^\mu + x^2 a^\mu\right)\left(2(adx) + 2a^2(xdx)\right)$$

$$h = 1 + 2ax + x^2 a^2$$

$$ds'^2 = dx'^\mu dx'_\mu = \frac{1}{h^2}\left(dx^2 + 4(adx)(xdx) + 4a^2(xdx)^2\right) +$$

$$+\frac{1}{h^4}\underbrace{\left(x^2 + 2x^2(ax) + x^4a^2\right)}_{\to x^2 h} 4\left((adx) + a^2(xdx)\right)^2 -$$

$$-\frac{2}{h^3} 2\left((adx) + a^2(xdx)\right)\left(xdx + x^2(adx) + 2(ax)(xdx) + 2a^2(xdx)x^2\right)$$

$$= \frac{1}{h^2}\Big(dx^2 + 4(adx)(xdx) + 4a^2(xdx)^2 + \frac{4x^2\cancel{h}}{h}\left((adx)^2 + 2a^2(adx)(xdx)\right.$$

$$\left. + a^4(xdx)^2\right) - \frac{4}{h}\left((adx) + a^2(xdx)\right)\Big(\left(1 + 2(ax) + 2a^2x^2\right)(xdx) +$$

$$+ x^2(adx)\Big)\Big)$$

$$= \frac{1}{h^2}\Big(dx^2 + (adx)(xdx)\Big[4 + \frac{8x^2a^2}{h} - \frac{4}{h}\left(1 + 2(ax) + 2a^2x^2\right) -$$

$$- \frac{4}{h}a^2x^2\Big] + (xdx)^2\Big[4a^2 + \frac{4a^4x^2}{h} - \frac{4a^2}{h}\left(1 + 2ax + 2a^2x^2\right)\Big] +$$

$$+ (a\,dx)^2 \left[\cancel{\frac{4x^2}{h}} - \cancel{\frac{4x^2}{h}} \right] \Big)$$

$$= \frac{1}{h^2} \Big(dx^2 + (a\,dx)(x\,dx)\frac{4}{h} \left[\cancel{1} \cancel{+2ax} \cancel{+a^2x^2} \cancel{+2x^2a^2} \cancel{-1} \cancel{-2ax} \cancel{-2a^2x^2} \cancel{-a^2x^2} \right]$$

$$+ (x\,dx)^2 \frac{4}{h} a^2 \left[\cancel{1} \cancel{+2ax} \cancel{+a^2x^2} \cancel{+a^2x^2} \cancel{-1} \cancel{-2ax} \cancel{-2a^2x^2} \right] \Big)$$

$$= \frac{dx^2}{h^2}$$

$$\boxed{dx'^2 = \frac{dx^2}{(1+2ax+x^2a^2)^2}}$$

For translations & rotations $dx'^2 = dx^2$

For dilatations $x'^\mu = \lambda x^\mu$

$$dx'^2 = \lambda^2 dx^2$$

Conf. transf. on fields :

conf. dim

$$\langle \phi_1(r_1) \; \ldots \; \phi_n(r_n) \rangle = \prod_{i=1}^{n} \left| \frac{D(r_i')}{D(r_i)} \right|^{\frac{\Delta_i}{d}} \langle \phi_1(r_1') \ldots \phi_n(r_n') \rangle$$

$$\frac{D(r')}{D(r)} = \frac{1}{(1 + 2(ax) + a^2 x^2)^d}$$

$$\phi(x) \to \phi'(x') = \left| \frac{\partial x'}{\partial x} \right|^{-\Delta/d} \phi(x)$$

$$\langle \phi_1'(x_1') \ldots \phi_n'(x_n') \rangle = \prod_{i=1}^{n} \left| \frac{\partial x'}{\partial x} \right|^{-\Delta_i/d} \langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle$$

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \qquad dx'^2 = \left| \frac{\partial x'}{\partial x} \right|^2 dx^2$$

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(\quad)^d} \qquad S_{\alpha\mu} = \frac{\partial x'^\mu}{\partial x^\alpha} \; ; \quad S S^t = \frac{1}{(\;)^2} \mathbb{1} ; \; (\det S)^2 = \frac{1}{(\;)^{2d}}$$

$$\langle \phi_1'(x_1') \ldots \phi_n'(x_n') \rangle = \prod_{i=1}^{n} (1 + 2ax + a^2 x^2)^{\Delta_i} \langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle$$

$$\langle \phi_1'(x_1') \ldots \phi_n'(x_n') \rangle = \lambda^{-\sum \Delta_i} \langle \phi_1(x_1) \ldots \phi_n(x_n) \rangle$$

Energy momentum tensor generates translations.

$$Q_{(a)} = a^\mu P_\mu = \int d^{d-1}x \; a^\mu T_{\mu 0}$$

$x^\mu \to x^\mu + a^\mu$

Current

$$\boxed{j_{a\nu} = a^\mu T_{\mu\nu}} \quad \to \partial^\nu j_{a\nu} = 0 \qquad \partial^\mu T_{\mu\nu} = 0$$

For a dilatation we can define a current.

$$j_{D\nu} = x^\mu T_{\mu\nu} \qquad ; \; a^\mu = x^\mu$$

$$\partial^\nu j_{D\nu} = \underbrace{\partial^\nu T_{\mu\nu}}_{0} x^\mu + T_{\mu\mu} = 0$$

$j_{D\nu}$ is conserved if $T_{\mu\mu} = 0$. (T is traceless).

For special conf. transf.:

$$j_{SCT\,\nu} = \delta x^\mu T_{\mu\nu} = (-2(ax)x^\mu + a^\mu x^2) T_{\mu\nu}$$

$$\partial^\nu j_\nu = T_{\mu\nu}\left(-2(ax)g^{\mu\nu} - 2a^\nu x^\mu + a^\mu 2x^\nu\right)$$

$$= -2(ax)\underbrace{T_{\mu\mu}}_{\text{traceless.}} + 2\underbrace{T_{\mu\nu}}_{\text{symmetric}}(a^\mu x^\nu - a^\nu x^\mu) = 0$$

if $T_{\mu\nu}$ is symmetric and traceless then we have new conserved currents:

$$j_{D\nu} = x^\mu T_{\mu\nu}$$

$$j_{SCT\,\nu} = (a^\mu x^2 - 2(ax)\, x^\mu)\, T_{\mu\nu} \qquad (\text{arbitrary } a^\mu).$$

and the theory is conformally invariant.

$$\langle \phi_1(x'_1)\, \phi_2(x'_2)\rangle = F(|x'_1 - x'_2|)$$

transl. & rot. invariant.

$$x'^{\mu}_1 - x'^{\mu}_2 = \frac{x_1^\mu + a^\mu x_1^2}{(1+2a x_1 + x_1^2 a^2)} - \frac{x_2^\mu + a^\mu x_2^2}{(1+2a x_2 + x_2^2 a^2)}$$

$$|x'_1 - x'_2|^2 = \frac{1}{h_1^2}\left(x_1^2 + a^2 x_1^4 + 2(a x_1) x_1^2\right) + \frac{1}{h_2^2}\left(x_2^2 + a^2 x_2^4 + 2(a x_2) x_2^2\right)$$

$$- \frac{2}{h_1 h_2}\left((x_1 x_2) + (a x_1) x_2^2 + (a x_2) x_1^2 + a^2 x_1^2 x_2^2\right)$$

$$= \frac{x_1^2 \cancel{h_1}}{h_1^{\cancel{2}}} + \frac{x_2^2 \cancel{h_2}}{h_2^{\cancel{2}}} - \frac{2(x_1 x_2)}{h_1 h_2} - \frac{2}{h_1 h_2}\left((a x_1) x_2^2 + (a x_2) x_1^2 + a^2 x_1^2 x_2^2\right)$$

$$= \frac{1}{h_1 h_2}\Big[x_1^2\left(1 + \cancel{2(a x_2)} + \cancel{a^2 x_2^2}\right) + x_2^2\left(1 + \cancel{2(a x_1)} + \cancel{a^2 x_1^2}\right) -$$

$$-2 x_1 x_2 - \cancel{2(a x_1) x_2^2} - \cancel{2(a x_2) x_1^2} - \cancel{2a^2 x_1^2 x_2^2}\Big]$$

$$= \frac{|x_1 - x_2|^2}{h_1 h_2}$$

rot + transl. + scale inv.

$$\langle \phi'_1(x'_1)\, \phi'_2(x'_2)\rangle = \frac{C}{|x'_1 - x'_2|^{\Delta_1+\Delta_2}} = h_1^{\Delta_1} h_2^{\Delta_2} \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

$$\frac{(h_1 h_2)^{\frac{\Delta_1+\Delta_2}{2}}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = h_1^{\Delta_1} h_2^{\Delta_2} \frac{1}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \Rightarrow \boxed{\Delta_1 = \Delta_2}$$

$$\langle \phi_1(x_1)\, \phi_2(x_2) \rangle = \frac{C}{|x_1 - x_2|^{2\Delta}}$$

$$\boxed{\Delta_1 = \Delta_2}$$

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle = \frac{C}{|x_1 - x_2|^a\, |x_1 - x_3|^b\, |x_2 - x_3|^c}$$

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3$$

$$h_1^{\Delta_1}\, h_2^{\Delta_2}\, h_3^{\Delta_3} = (h_1 h_2)^{a/2}\, (h_1 h_3)^{b/2}\, (h_2 h_3)^{c/2}$$

$$a + b = 2\Delta_1 \quad ; \quad a + c = 2\Delta_2 \quad ; \quad b + c = 2\Delta_3$$

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \quad \checkmark$$

$$2a + b + c = 2\Delta_1 + 2\Delta_2 \quad \Rightarrow \quad a = 2\Delta_1 + 2\Delta_2 - \Delta_1 - \Delta_2 - \Delta_3$$

$$\boxed{a = \Delta_1 + \Delta_2 - \Delta_3}$$

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3) \rangle = \frac{C}{|x_1 - x_2|^{\Delta_1 + \Delta_2 - \Delta_3}\, |x_1 - x_3|^{\Delta_1 - \Delta_2 + \Delta_3}\, |x_2 - x_3|^{-\Delta_1 + \Delta_2 + \Delta_3}}$$

$$\langle \phi_1(x_1)\, \phi_2(x_2)\, \phi_3(x_3)\, \phi_4(x_4) \rangle = F(u,v) \prod_{i<j} x_{ij}^{-a_{ij}}$$

$$u = \frac{|x_1-x_2|^2|x_3-x_4|^2}{|x_1-x_3|^2|x_2-x_4|^2} \quad \xleftarrow{2\leftrightarrow 4} ; \quad v = \frac{|x_1-x_4|^2|x_2-x_3|^2}{|x_1-x_3|^2|x_2-x_4|^2}$$

invariant (cross ratios).

$$h_1^{\Delta_1} h_2^{\Delta_2} h_3^{\Delta_3} h_4^{\Delta_4} = \prod_{i<j} h_i^{a_{ij}/2}\, h_j^{a_{ij}/2}$$

$$\left.\begin{aligned}
\Delta_1 &= a_{12}/2 + a_{13}/2 + a_{14}/2 \\
\Delta_2 &= a_{23}/2 + a_{24}/2 + a_{12}/2 \\
\Delta_3 &= a_{13}/2 + a_{23}/2 + a_{34}/2 \\
\Delta_4 &= a_{14}/2 + a_{24}/2 + a_{34}/2
\end{aligned}\right\}$$

4 eqns.
6 variables.

$$\sum \Delta_i = a_{12} + a_{13} + a_{23} + a_{14} + a_{24} + a_{34} = \Delta$$

$$\Delta_1 + \Delta_2 = \frac{\Delta}{2} - \frac{a_{34}}{2} + a_{12}/2$$

$$a_{12} - a_{34} = 2\Delta_1 + 2\Delta_2 - \Delta$$

$$\Delta_3 + \Delta_4 = \frac{\Delta}{2} - \frac{a_{12}}{2} + \frac{a_{34}}{2}$$

Solution $\quad a_{ij} = \alpha\Delta + \Delta_i + \Delta_j$

$$\Delta_1 = \frac{3\alpha}{2}\Delta + \frac{3\Delta_1}{2} + \frac{\Delta_2+\Delta_3+\Delta_4}{2}$$

$$= \frac{3\alpha}{2}\Delta + \Delta_1 + \frac{\Delta}{2} \qquad \alpha = -1/3$$

$$\boxed{a_{ij} = -\frac{\Delta}{3} + \Delta_i + \Delta_j}$$

$$\boxed{\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\,\phi_4(x_4)\rangle = \frac{F(u,v)}{\prod_{i<j} x_{ij}^{\Delta_i+\Delta_j-\Delta/3}}}$$

Same op $\qquad 2\Delta - \frac{4\Delta}{3} = \frac{2\Delta}{3}$

$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle = \frac{F(u,v)}{\prod_{i<j} x_{ij}^{2\Delta/3}}$$

$$= \frac{F(u,v)}{x_{12}^{2\Delta/3}x_{13}^{2\Delta/3}x_{23}^{2\Delta/3}x_{24}^{2\Delta/3}x_{34}^{2\Delta/3}x_{14}^{2\Delta/3}} = \frac{1}{x_{14}^{2\Delta}\,x_{23}^{2\Delta}}\;\frac{x_{14}^{4\Delta/3}\,x_{23}^{4\Delta/3}\,F(U,V)}{x_{12}^{2\Delta/3}\,\underbrace{x_{13}^{2\Delta/3}}\,\underbrace{x_{24}^{2\Delta/3}}\,x_{34}^{2\Delta/3}}$$

$2\leftrightarrow 4$

$$F(U,V) = F(V,u) \qquad = \frac{1}{x_{14}^{2\Delta}\,x_{23}^{2\Delta}}\,V^{2\Delta/3}\left(\frac{V}{u}\right)^{2\Delta/3} F(U,V)$$

$$f(u,v) = \frac{V^{2\Delta/3}}{U^{2\Delta/3}}\,F(U,V) \quad ; \quad f(v,u) = \frac{U^{2\Delta/3}}{V^{2\Delta/3}}\,F(U,V) = \frac{U^{2\Delta}}{V^{2\Delta}}\,f(u,v)$$

Special conf transf.

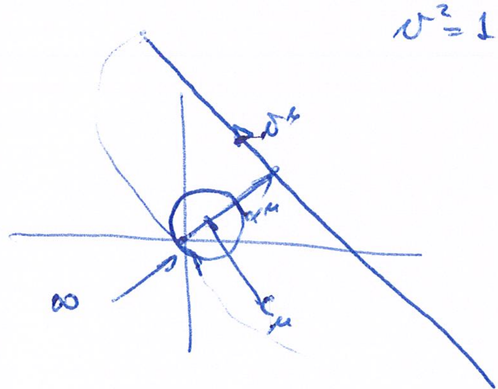
$$x'_\mu = \frac{x_\mu + x^2 a_\mu}{1+2ax+a^2x^2} \quad ; \qquad \text{inv} \to \text{transl.} \to \text{inv.}$$

inversion:

$\boxed{\text{straight line}}$ $\quad x_\mu = \alpha_\mu + \sigma v_\mu \qquad \sigma = -\infty \ldots \infty$

$v^2 = 1 \qquad \alpha v = 0$

$$x'_\mu = \frac{\alpha_\mu + \sigma v_\mu}{\alpha^2 + \sigma^2}$$



$\sigma = \infty \to x'_\mu = 0$

$\sigma = 0 \to x'_\mu = \alpha_\mu / \alpha^2$

In fact it goes to a circle. It is in a plane (α_μ, v_μ)

Take $\quad c_\mu = \dfrac{\alpha_\mu}{2\alpha^2} \qquad \|c_\mu\| = \dfrac{\alpha}{2\alpha^2} = \dfrac{1}{2\alpha} = R.$

$$x'_\mu - c_\mu = \frac{\alpha_\mu + \sigma v_\mu}{\alpha^2+\sigma^2} - \frac{\alpha_\mu}{2\alpha^2} = \frac{2\alpha^2\alpha_\mu + 2\alpha^2\sigma v_\mu - \alpha^2\alpha_\mu - \sigma^2\alpha_\mu}{2\alpha^2(\alpha^2+\sigma^2)}$$

$$x'_\mu - c_\mu = \frac{(\alpha^2-\sigma^2)\alpha_\mu + 2\alpha^2\sigma v_\mu}{2\alpha^2(\alpha^2+\sigma^2)}$$

$$\|x' - c_\mu\|^2 = \frac{(\alpha^2-\sigma^2)^2\alpha^2 + 4\alpha^4\sigma^2}{4\alpha^4(\alpha^2+\sigma^2)^2} = \frac{\alpha^2\cancel{(\alpha^2+\sigma^2)^2}}{4\alpha^4\cancel{(\alpha^2+\sigma^2)^2}} = \frac{1}{4\alpha^2}$$

$$\boxed{R = \frac{1}{2\alpha}} \quad \checkmark$$

$$x'_\mu = \frac{x_\mu + x^2 a_\mu}{1+2ax+a^2x^2}$$

$$x_\mu = \alpha_\mu + \sigma v_\mu \qquad \alpha v = 0 \qquad v^2 = 1$$

$$x'_\mu = \frac{\alpha_\mu + \sigma v_\mu + (\alpha^2+\sigma^2) a_\mu}{1+2(a\alpha) + 2\sigma(av) + a^2\alpha^2 + a^2\sigma^2}$$

$$x'_\mu - \frac{a_\mu}{a^2} = \frac{\alpha_\mu + \sigma v_\mu - \frac{1+2(a\alpha)+2\sigma(av)}{a^2} a_\mu}{1+2(a\alpha)+2\sigma(av)+a^2\alpha^2+a^2\sigma^2}$$

$$\tilde{x}_\mu = \left(x'_\mu - \frac{a_\mu}{a^2}\right) = \frac{\sigma\left(v_\mu - \frac{2(av)}{a^2} a_\mu\right) + \alpha_\mu - \frac{1+2(a\alpha)}{a^2} a_\mu}{1+2(a\alpha)+a^2\alpha^2+2\sigma(av)+a^2\sigma^2}$$

$$\boxed{t_\mu = v_\mu - \frac{2(av)}{a^2} a_\mu}$$ tangent at origin $(\sigma \to \infty)$

$\tilde{x}_\mu = 0 \quad \sigma \to \infty$

$$t^2 = v^2 + \left(\frac{2(av)}{a^2}\right)^2 a^2 - \frac{4(av)}{a^2} av = 1 + \frac{4(av)^2}{a^2} - \frac{4(av)^2}{a^2} = 1$$

$$ta = va - 2(av) = -(av)$$

$$\tilde{x}_\mu = \frac{\sigma t_\mu + \rho_\mu}{1+2(a\alpha)+a^2\alpha^2+2\sigma(av)+a^2\sigma^2} \quad ; \; \rho_\mu = \alpha_\mu - \frac{1+2(a\alpha)}{a^2} a_\mu$$

$\tilde{x}_\mu$ is in a plane, should be circle through origin

Denominator contains

$$a^2\left(\sigma^2 + \frac{2(av)}{a^2}\sigma\right) = a^2\underbrace{\left(\sigma + \frac{av}{a^2}\right)}_{\eta}{}^2 - \frac{(av)^2}{a^2}$$

$$\sigma = \eta - \frac{av}{a^2}$$

$$\tilde{x}_\mu = \frac{\eta t_\mu + \rho_\mu - \frac{av}{a^2}t_\mu}{1+2(a\alpha)+a^2\alpha^2 - \frac{(av)^2}{a^2} + a^2\eta^2}$$

Define $\quad n_\mu = \rho_\mu - \frac{av}{a^2}t_\mu$

$$t_\mu n_\mu = \rho t - \frac{av}{a^2}$$

$$\rho t = \alpha t - \frac{1+2(a\alpha)}{a^2}\, a t = -\cancel{\frac{2(av)}{a^2}}\, a\alpha + \frac{1+\cancel{2(a\alpha)}}{a^2}(av)$$

$$= \frac{av}{a^2} \quad , \Rightarrow \boxed{t.n = 0}$$

$$n^2 = \alpha^2 - 2\frac{(1+2(a\alpha))}{a^2}(\alpha a) + \frac{(1+2(a\alpha))^2}{a^2} + \frac{(av)^2}{a^4} - \frac{2(av)^2}{a^4}$$

$$= \alpha^2 - \frac{2(\alpha a)}{a^2} - \cancel{\frac{4(\alpha a)^2}{a^2}} + \frac{1}{a^2} + \cancel{\frac{4(a\alpha)^2}{a^2}} + \frac{4(a\alpha)}{a^2} \pm \frac{(av)^2}{a^4}$$

$$n^2 = \frac{1}{a^2}\left(\alpha^2 a^2 + 2(\alpha a) + 1\right) - \frac{(av)^2}{a^4}$$

$$\boxed{\tilde{x}_\mu = \frac{\eta t_\mu + n_\mu}{a^2(n^2+\eta^2)}} = \frac{\cancel{n}\,(\xi t_\mu + n_\mu/n)}{n^{\cancel{2}} a^2\,(1+\xi^2)}$$

$$\xi = \eta/n$$

$n_\mu/n = \hat{n}_\mu$ unit vector

$$\tilde{x}_\mu = \frac{1}{na^2}\,\frac{\xi t_\mu + \hat{n}_\mu}{1+\xi^2}$$

$$x'_\mu = \frac{a_\mu}{a^2} + \frac{1}{na^2}\,\frac{\xi t_\mu + \hat{n}_\mu}{1+\xi^2}$$

$$\frac{\xi t_\mu + \hat{n}_\mu}{1+\xi^2} - r\hat{n}_\mu = \frac{\xi t_\mu + (1-r-r\xi^2)\hat{n}_\mu}{1+\xi^2}$$

$$\xi^2 + (1-r)^2 - 2r\xi^2(1-r) + r^2\xi^4 \overset{r=1/2}{=} \xi^2 + \frac{1}{4} - \frac{1}{2}\xi^2 + \frac{1}{4}\xi^4 =$$

$$= \frac{1}{4}\left(1 + \xi^4 + 2\xi^2\right) = \frac{(1+\xi^2)^2}{4}$$

$$\frac{\xi t_\mu + \hat{n}_\mu}{1+\xi^2} = \frac{1}{2}\hat{n}_\mu + \frac{\xi t_\mu + \frac{1}{2}(1-\xi^2)\hat{n}_\mu}{1+\xi^2}$$

$$\frac{2\xi}{1+\xi^2} = c\phi \qquad 1 - c^2\phi = 1 - \frac{4\xi^2}{(1+\xi^2)^2} = \frac{(1-\xi^2)^2}{(1+\xi^2)^2} = s^2\phi$$

$$\frac{\xi t_\mu + \hat{n}_\mu}{1+\xi^2} = \frac{1}{2}\hat{n}_\mu + \frac{1}{2}\left(c\phi\, t_\mu + s\phi\, \hat{n}_\mu\right)$$

Finally:

$$\underbrace{x'_\mu}_{\text{circle}} = \underbrace{\boxed{\frac{a_\mu}{a^2} + \frac{1}{2na^2}\hat{n}_\mu}}_{\text{Center}} + \underbrace{\frac{1}{2na^2}}_{\text{radius.}}\left(c\phi\, t_\mu + s\phi\, \hat{n}_\mu\right)$$

$$t_\mu = v_\mu - \frac{2(av)}{a^2} a_\mu$$

$$n_\mu = x_\mu - \frac{1+2(ax)}{a^2} a_\mu - \frac{av}{a^2} t_\mu$$

$$\hat{n}_\mu = \frac{n_\mu}{n} \quad ; \; n = \| n_\mu \|$$

$$c\phi = \frac{2\xi}{1+\xi^2} = \frac{2\eta/n}{1+\eta^2/n^2} = \frac{2\eta n}{n^2+\eta^2} \quad ; \; \eta = \sigma + \frac{av}{a^2}$$

Conf. transf. as Lorentz transf. in $R^{d+1,1}$

Consider $R^{d+1,1}$ and light-cone

$$Y_1^2 + \cdots + Y_d^2 + Y_{d+1}^2 - Y_0^2 = 0$$

$$Y_\pm = Y_0 \pm Y_{d+1}$$

$$Y_\mu^2 - Y_+ Y_- = 0 \Rightarrow Y_+ = Y_\mu^2 / Y_-$$

space of "light-rays"

$$\boxed{x_\mu = Y_\mu / Y_-}$$

Lorentz transf. action on x_μ.

Rotations of Y_μ → rotations of x_μ ✓

Boosts in Y_0, Y_{d+1} →

$$\left.\begin{aligned} Y_+ &\to \lambda Y_+ \\ Y_- &\to \frac{1}{\lambda} Y_- \end{aligned}\right\} \text{preserve internal } Y_\mu^2 - Y_+ Y_-$$

$$x_\mu \to \lambda \frac{Y_\mu}{Y_-} = \lambda x_\mu \quad \text{scale transf.}$$

"Boosts" in Y_μ, Y_+ or Y_μ, Y_-

$$Y'_\mu = Y_\mu + a_\mu Y_-$$

$$Y'_- = Y_-$$

$$Y'_+ = Y_+ + 2(aY) + a^2 Y_-$$

$$Y'^2_\mu = Y_\mu^2 + 2(aY)Y_- + a^2 Y_-^2$$

$$-Y'_+ Y'_- = -Y_+ Y_- - 2(aY)Y_- - a^2 Y_-^2$$

$$Y'^2_\mu - Y'_+ Y'_- = Y_\mu^2 - Y_+ Y_- \checkmark$$

$$x'_\mu = \frac{Y_\mu + a_\mu Y_-}{Y_-} = \frac{Y_\mu}{Y_-} + a_\mu = x_\mu + a_\mu$$ translation!

$$Y'_\mu = Y_\mu + a_\mu Y_+$$

$$Y'_+ = Y_+$$

$$Y'_- = Y_- + 2(aY) + a^2 Y_+$$

$$Y_+ = Y^2/Y_- = Y_- x^2$$

$$x'_\mu = \frac{Y_\mu + a_\mu Y_+}{Y_- + 2(aY) + a^2 Y_+} = \frac{Y_- x_\mu + Y_- x^2 a_\mu}{Y_- + 2(ax)Y_- + a^2 Y_- x^2} = \frac{x_\mu + a_\mu x^2}{1 + 2(ax) + a^2 x^2}$$ SCT !!

Conformal algebra.

$$[P_\mu, \mathcal{O}(x)] = -i\,\partial_\mu \mathcal{O}(x)$$

$$[D, \mathcal{O}(x)] = -i\,(\Delta + x^\mu \partial_\mu)\mathcal{O}(x)$$

$$[M_{\mu\nu}, \mathcal{O}(x)] = -i\,(\Sigma_{\mu\nu} + x_\mu \partial_\nu - x_\nu \partial_\mu)\mathcal{O}(x)$$

$$[K_\mu, \mathcal{O}(x)] = -i\,(2x_\mu \Delta + 2x^\lambda \Sigma_{\lambda\mu} + 2x_\mu (x^\rho \partial_\rho) - x^2 \partial_\mu)\mathcal{O}(x)$$

---

at $\underline{x=0}$ $\quad [D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0)$, $\quad [P_\mu, \mathcal{O}(0)] = -i\partial_\mu \mathcal{O}(0)$

$[M_{\mu\nu}, \mathcal{O}(0)] = -i\Sigma_{\mu\nu}\mathcal{O}(0)$

$[K_\mu, \mathcal{O}(0)] = 0 \longleftarrow$ Definition of primary operators.

Conf. Algebra

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i\,(\delta_{\mu\rho} M_{\nu\sigma} + \delta_{\nu\sigma} M_{\mu\rho} - \delta_{\mu\sigma} M_{\nu\rho} - \delta_{\nu\rho} M_{\mu\sigma})$$

$$[M_{\mu\nu}, P_\rho] = i\,(\delta_{\nu\rho} P_\mu - \delta_{\mu\rho} P_\nu)$$

$$[D, P_\mu] = -i P_\mu$$

$$[D, K_\mu] = i K_\mu$$

$$[P_\mu, K_\nu] = 2i\,(\delta_{\mu\nu} D - M_{\mu\nu})$$

$\boxed{\text{O.P.E}}$ operator product expansion. (1)

$$\phi_1(x_1)\,\phi_2(x_2) \underset{x_1\to x_2}{=} \sum_{\mathcal{O}} \lambda_{\mathcal{O}}\, C_{\mathcal{O}}(x_{12}, \partial_y)\,\mathcal{O}(y)\Big|_{y=x_2}$$

But

$$\langle\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle\rangle = \frac{C_{123}}{X_{12}^{\Delta_1+\Delta_2-\Delta_3}\, X_{13}^{\Delta_1+\Delta_3-\Delta_2}\, X_{23}^{\Delta_2+\Delta_3-\Delta_1}}$$

Example $\Delta_1=\Delta_2=\Delta$ $\quad \phi_3 \to \Delta_\Phi;\ \Phi=\phi_3$

$$\langle\langle \phi_0(x_1)\,\phi(x_2)\,\Phi(x_3)\rangle\rangle = \frac{C_{\phi\phi\Phi}}{X_{12}^{2\Delta-\Delta\phi}\, X_{13}^{\Delta\phi}\, X_{23}^{\Delta\phi}}$$

$$\frac{1}{X_{13}^{\Delta\phi}} \underset{x_1\to x_2}{=} \frac{1}{\left[(X_{23}+X_{12})^2\right]^{\Delta\phi/2}} = \frac{1}{X_{23}^{\Delta\phi}\left(1+2\frac{X_{12}\cdot X_{23}}{X_{23}^2}+\frac{X_{12}^2}{X_{23}^2}\right)^{\Delta\phi/2}}$$

$$\frac{1}{(1+\varepsilon)^\alpha} = 1-\alpha\varepsilon+\frac{1}{2}\alpha(\alpha+1)\varepsilon^2 + \ldots$$

$$\frac{1}{X_{13}^{\Delta\phi}} = \frac{1}{X_{23}^{\Delta\phi}}\left(1-\Delta\phi\frac{X_{12}\cdot X_{23}}{X_{23}^2} - \frac{\Delta\phi}{2}\frac{X_{12}^2}{X_{23}^2} + \frac{1}{\cancel{2}}\frac{\Delta\phi}{\cancel{2}}\left(\frac{\Delta\phi}{2}+1\right)\cancel{4}\frac{(X_{12}X_{23})^2}{X_{23}^4}\right.$$

So, when $x_1 \to x_2$

$$\langle\langle \phi(x_1)\phi(x_2)\Phi(x_3)\rangle\rangle = \frac{C_{\phi\phi\Phi}}{x_{12}^{2\Delta-\Delta_\phi}\, x_{23}^{2\Delta_\phi}}\left(1 - \Delta_\phi \frac{x_{12}\cdot x_{23}}{x_{23}^2} - \frac{\Delta_\phi}{2}\frac{x_{12}^2}{x_{23}^2} + \right.$$

$$\left. + \frac{1}{2}\Delta_\phi(\Delta_\phi+2)\frac{(x_{12}\cdot x_{23})^2}{x_{23}^4} + \cdots \right) \quad ; \left(x_{12} \to 0\right)$$

Now:

normalization of $\lambda_{\mathcal{O}}$

$$C_{\mathcal{O}}(x_{12}, \partial_y) = x_{12}^a\left(1 + \alpha\, x_{12}^\mu \partial_y^\mu + \beta\, x_{12}^\mu x_{12}^\nu \partial_\mu^y + \gamma\, x_{12}^2 \partial_y^2 + \dots\right)$$

$$\langle\langle \phi(x_1)\phi(x_2)\Phi(x_3)\rangle\rangle = \sum_{\mathcal{O}} \lambda_{\mathcal{O}}\, C_{\mathcal{O}}(x_{12}, \partial_y) \underbrace{\langle\langle \mathcal{O}(y)\Phi(x_3)\rangle\rangle}_{\substack{\mathcal{O}=\Phi \\ \underbrace{\qquad}_{\frac{1}{|y-x_3|^{2\Delta_\phi}}}}}\Bigg|_{y=x_2}$$

$$= \lambda_\Phi\, x_{12}^a\left(1 + \alpha x_{12}^\mu \partial_y^\mu + \beta x_{12}^\mu x_{12}^\nu \partial_\mu^y + \gamma x_{12}^2 \partial_y^2 + \dots\right)\frac{1}{|y-x_3|^{2\Delta_\phi}}\Bigg|_{y=x_2}$$

$$= \underbrace{\lambda_\Phi \frac{x_{12}^a}{x_{23}^{2\Delta_\phi}}} + \cdots$$

we need $\quad C_{\phi\phi\Phi} = \lambda_\Phi \qquad a = -2\Delta + \Delta_\phi$

$\lambda_{\mathcal{O}}$ is given by 3 point function coefficient. ③

Now, let's get α, β, γ

$$\frac{1}{[(y-x_3)^2]^{\Delta\phi}} = \frac{1}{((x_{23} + \underbrace{(y-x_2)}_{\xi})^2)^{\Delta\phi}} = \frac{1}{x_{23}^{2\Delta\phi}} \frac{1}{\left(1 + \frac{2x_{23}\xi}{x_{23}^2} + \frac{\xi^2}{x_{23}^2}\right)^{\Delta\phi}}$$

$$= \frac{1}{x_{23}^{2\Delta\phi}}\left(1 - 2\Delta\phi \frac{x_{23}\cdot\xi}{x_{23}^2} - \Delta\phi \frac{\xi^2}{x_{23}^2} + \frac{1}{2}\Delta\phi(\Delta\phi+1)\, 4\, \frac{(x_{23}\cdot\xi)^2}{x_{23}^4} + \dots\right)$$

$$\partial_\xi^\mu \Big|_{\xi=0} = -\frac{2\Delta\phi}{x_{23}^{2\Delta\phi}} \frac{x_{23}^\mu}{x_{23}^2}$$

$$\partial_\xi^\mu \partial_\xi^\nu \Big|_{\xi=0} = \left(-\frac{2\Delta\phi\, \eta^{\mu\nu}}{x_{23}^2} + 4\Delta\phi(\Delta\phi+1)\frac{x_{23}^\mu x_{23}^\nu}{x_{23}^4}\right)\frac{1}{x_{23}^{2\Delta\phi}}$$

$$C_{\mathcal{O}}(x_{12}, \partial_y)\frac{1}{(y-x_3)^{2\Delta\phi}}\Bigg|_{y=x_2} = \frac{x_{12}^{-2\Delta+\Delta\phi}}{x_{23}^{2\Delta\phi}}\Bigg[1 - \alpha\frac{2\Delta\phi}{x_{23}^2} x_{23}\cdot x_{12} +$$

$$+\beta\left(-\frac{2\Delta\phi\, x_{12}^2}{x_{23}^2} + 4\Delta\phi(\Delta\phi+1)\frac{(x_{12}\cdot x_{23})^2}{x_{23}^4}\right) +$$

$$+\gamma\, x_{12}^2\left(-\frac{2\Delta\phi\, \overset{\swarrow \text{dimension}}{d}}{x_{23}^2} + 4\Delta\phi\frac{(\Delta\phi+1)}{x_{23}^2}\right) + \dots\Bigg]$$

Then:

$$\frac{C_{\phi\phi\Phi}}{x_{12}^{2\Delta-\Delta_\phi}\, x_{23}^{2\Delta_\phi}}\left(1-\Delta_\phi\frac{x_{12}\cdot x_{23}}{x_{23}^2}-\frac{\Delta_\phi}{2}\frac{x_{12}^2}{x_{23}^2}+\frac{1}{2}\Delta_\phi(\Delta_\phi+2)\frac{(x_{12}\cdot x_{23})^2}{x_{23}^4}+\dots\right)$$

$$=\frac{\lambda_\Phi}{x_{12}^{-a}\, x_{23}^{2\Delta_\phi}}\left[1-2\alpha\Delta_\phi\frac{x_{12}\cdot x_{23}}{x_{23}^2}+4\beta\Delta_\phi(\Delta_\phi+1)\frac{(x_{12}\cdot x_{23})^2}{x_{23}^4}+\right.$$

$$\left.+\frac{x_{12}^2}{x_{23}^2}\Big(-2\beta\Delta_\phi+\gamma\big(-2d\Delta_\phi+4\Delta_\phi(\Delta_\phi+1)\big)\Big)+\dots\right]$$

As we said: $\lambda_\Phi = C_{\phi\phi\Phi}$ $\qquad a = -2\Delta+\Delta_\Phi$

$$\alpha=\frac{1}{2}\qquad \beta=\frac{1}{8}\frac{\cancel{\Delta_\phi}(\Delta_\phi+2)}{\cancel{\Delta_\phi}(\Delta_\phi+1)}$$

$$-\frac{\cancel{\Delta_\phi}}{2}=-2\beta\cancel{\Delta_\phi}+\cancel{\Delta_\phi}\,\gamma\,(-2d+4\Delta_\phi+4)$$

$$2\gamma(2-d+2\Delta_\phi)=\frac{1}{4}\frac{\Delta_\phi(\Delta_\phi+2)}{\Delta_\phi(\Delta_\phi+1)}-\frac{1}{2}=\frac{\Delta_\phi(\Delta_\phi+2)-2\Delta_\phi(\Delta_\phi+1)}{4\Delta_\phi(\Delta_\phi+1)}$$

$$=\frac{\Delta_\phi(\Delta_\phi\cancel{+2}-2\Delta_\phi\cancel{-2})}{4\Delta_\phi(\Delta_\phi+1)}=-\frac{\Delta_\phi^{\cancel{2}}}{4\cancel{\Delta_\phi}(\Delta_\phi+1)}$$

$$\gamma_2 = -\frac{1}{8}\,\frac{\Delta_\phi}{(\Delta_\phi+1)(2\Delta_\phi+2-d)}$$

$$\boxed{\mathcal{C}_\Phi(x_{12},\partial_y) = \frac{1}{x_{12}^{2\Delta-\Delta_\phi}}\left(1+\frac{1}{2}x_{12}^\mu\partial_y^\mu + \frac{1}{8}\frac{\Delta_\phi+2}{\Delta_\phi+1}x_{12}^\mu x_{12}^\nu \partial_{\mu\nu}^y - \frac{1}{16}\frac{\Delta_\phi}{(\Delta_\phi+1)(\Delta_\phi+1-d/2)}x_{12}^2\partial_y^2 + \mathcal{O}(x_{12}^3)\cdots\right)}$$

3 point function.

$$\phi(x_1)\phi(x_2) = \sum_{\Phi} C_{\phi\phi\Phi}\,\mathcal{C}_\Phi(x_{12},\partial_y)\,\Phi(y)\Big|_{y=x_2} + \sum_{\text{spin}} +$$

scalar ops.

$$+\frac{1}{x_{12}^{2\Delta}}$$

Also

$$\langle\langle\phi(x_1)\phi(x_2)\rangle\rangle = \frac{1}{x_{12}^{2\Delta}} \Rightarrow$$ identity operator.

$$C_{\phi\phi\mathbb{1}} = 1 \qquad \Delta_{\mathbb{1}} = 0$$

Simple 4-point function

$$\langle\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle\rangle =$$

$$= \sum_{\mathcal{O}_1,\mathcal{O}_2} \lambda_{\mathcal{O}_1}\lambda_{\mathcal{O}_2}\, C_{\mathcal{O}_1}(x_{12},\partial_{y_1})\, C_{\mathcal{O}_2}(x_{34},\partial_{y_2}) \langle\langle \mathcal{O}_1(y_1)\,\underbrace{\mathcal{O}_2(y_2)\rangle\rangle}_{\boxed{\mathcal{O}_1 = \mathcal{O}_2}} \Big|_{\substack{y_1 = x_2\\ y_2 = x_4}}$$

$$= \sum_{\Phi} C^2_{\phi\phi\Phi} \frac{1}{x_{12}^{2\Delta-\Delta_\phi}\, x_{34}^{2\Delta-\Delta_\phi}} (1+\dots)(1+\dots) \frac{1}{(y_1-y_2)^{2\Delta_\phi}} \Big|_{\substack{y_1 = x_2\\ y_2 = x_4}}$$

Φ ↑ Scalar contribution

$$= \sum_{\Phi} \frac{C^2_{\phi\phi\Phi}}{x_{12}^{2\Delta}\, x_{34}^{2\Delta}} \left(\frac{x_{12}\, x_{34}}{x_{24}^2}\right)^{\Delta_\phi} + \dots$$

But

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \underset{\uparrow}{\simeq} \frac{x_{12}^2 X_{34}^2}{X_{24}^2 X_{24}^2} \qquad v = \frac{X_{14}^2 X_{23}^2}{X_{13}^2 X_{24}^2} \simeq \frac{X_{24}^2 X_{24}^2}{X_{24}^2 X_{24}^2} \simeq 1$$

$x_1 \to x_2 \quad x_3 \to x_4$

$$\langle\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle\rangle = \sum_{\Phi} \frac{C^2_{\phi\phi\Phi}}{X_{12}^{2\Delta} X_{34}^{2\Delta}}\, u^{\Delta_\phi/2} + \dots$$

$$\langle\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle\rangle = \frac{f(u,v)}{x_{12}^{2\Delta}\,x_{34}^{2\Delta}} = \frac{\sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, G_{\mathcal{O}}(u,v)}{x_{12}^{2\Delta}\,x_{34}^{2\Delta}}$$

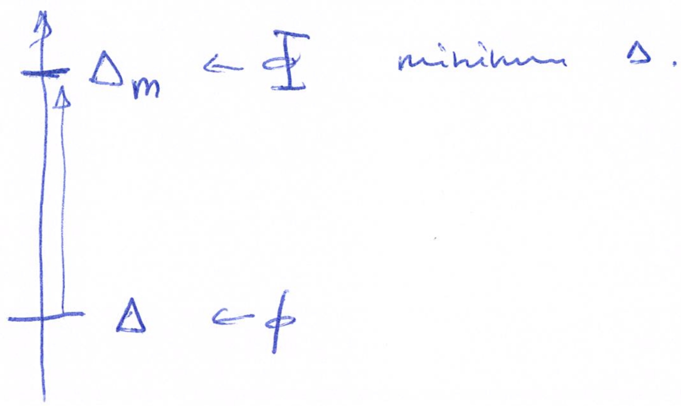
$\boxed{G_{\mathcal{O}}(u,v):\ \text{Conformal blocks}}$

when $u\to 0$ $v\to 1$ $x_1\to x_2$ $x_3\to x_4$

$$f(u,v) \simeq \sum_{\Phi} C^2_{\phi\phi\Phi}\, u^{\Delta_\Phi/2} + 1$$

contribution from identity.

let's say



and that $\langle\phi\phi\phi\rangle = 0$ (e.g. $\phi \mapsto -\phi$ symmetry)

then

$$f(u,v) \underset{\substack{u\to 0\\ v\to 1}}{\simeq} 1 + \sum_{\Phi} C^2_{\phi\phi\Phi}\, u^{\Delta_m/2} + \dots$$

higher orders in u.

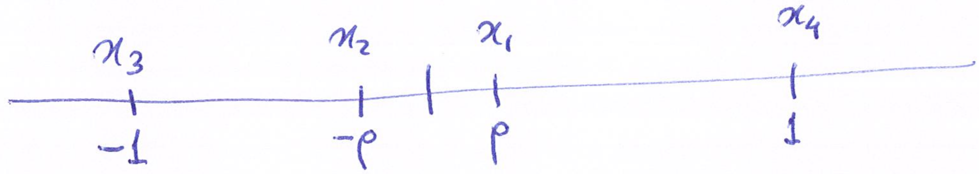
But

$$\boxed{v^{\Delta} f(u,v) = u^{\Delta} f(v,u)} \Rightarrow (v^{\Delta} - u^{\Delta}) + \sum_{\Phi} C^2_{\phi\phi\Phi}\left(v^{\Delta}u^{\Delta_m/2} - u^{\Delta}v^{\Delta_m/2}\right) + \dots = 0$$

More precisely

$$v^{\Delta} - u^{\Delta} + \sum_{\vartheta} c^{2}_{\phi\phi\vartheta} \left(v^{\Delta} G_{\vartheta}(u,v) - u^{\Delta} G_{\vartheta}(v,u) \right) = 0$$

Special configuration — Bootstrap bounds



$\rho \in \mathbb{R}_{<1}$

$$u = \frac{x_{12}^2\, x_{34}^2}{x_{13}^2\, x_{24}^2} = \frac{4\rho^2 \times 4}{(1+\rho)^4} = \frac{16\rho^2}{(1+\rho)^4}$$

$$V = \frac{x_{14}^2\, x_{23}^2}{x_{13}^2\, x_{24}^2} = \frac{(1-\rho)^4}{(1+\rho)^4}$$

$$\sqrt{u} = \frac{4\rho}{(1+\rho)^2} \qquad \sqrt{v} = \frac{(1-\rho)^2}{(1+\rho)^2}$$

$$\sqrt{u} + \sqrt{v} = \frac{(1+\rho)^2}{(1+\rho)^2} = 1$$

Special point $\rho = \rho_0$ / $\underline{u = v}$

$$16\rho^2 = (1-\rho)^4 \rightarrow 4\rho = (1-\rho)^2 = 1 - 2\rho + \rho^2 \rightarrow \rho^2 - 6\rho + 1 = 0$$

$$\rho = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm \frac{1}{2}\underbrace{\sqrt{32}}_{16 \times 2} = 3 \pm 2\sqrt{2} \qquad \text{but } \underline{\rho < 1} \Rightarrow \text{take } -$$

$$\boxed{\rho_0 = 3 - 2\sqrt{2}} \qquad \rho_0^2 = 9 + 8 - 12\sqrt{2} = 17 - 12\sqrt{2}$$

$$(1+\rho_0)^2 = (4 - 2\sqrt{2})^2 = 16 + 8 - 16\sqrt{2} = 24 - 16\sqrt{2} = 8(3 - 2\sqrt{2}) = 8\rho_0$$

$$(1-\rho_0)^2 = (-2 + 2\sqrt{2})^2 = 4(\sqrt{2}-1)^2 = 4(2 + 1 - 2\sqrt{2}) = 4\rho_0 \quad \checkmark$$

$$u_0 = \frac{16\cancel{\rho_0^2}}{64\cancel{\rho_0^2}} = \frac{1}{4} \qquad V_0 = \frac{16\cancel{\rho_0^2}}{64\cancel{\rho_0^2}} = \frac{1}{4} \qquad u_0 = V_0 \ \checkmark$$

$$\sqrt{u_0} + \sqrt{V_0} = 1 \ \checkmark$$

$$\rho_0 \simeq 3 - 2 \times 1.4 \simeq 0.2$$

4-point function: (2)

$$\langle\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle\rangle$$

O.P.E around 0.

$$\phi(x_1)\,\phi(x_2) = \phi(\rho)\,\phi(-\rho) = \frac{1}{(2\rho)^{2\Delta}} + \sum_{\mathcal{O}} \frac{C_{\phi\phi\mathcal{O}}}{(2\rho)^{2\Delta-\Delta_{\mathcal{O}}}}\left(1+\rho^2\alpha\,\partial_x^2+\dots\right)\mathcal{O}\Big|_{x=0}$$

no linear term in ρ.
$\rho \leftrightarrow -\rho$ symmetry.

$$\langle\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle\rangle \simeq \frac{1}{(2\rho)^{2\Delta}}\langle\langle \phi(x_3)\,\phi(x_4)\rangle\rangle +$$

$$+\sum_{\mathcal{O}} \frac{C_{\phi\phi\mathcal{O}}}{(2\rho)^{\Delta}}(2\rho)^{\Delta_{\mathcal{O}}}\Big(\langle\langle \overset{\text{operator}}{\mathcal{O}(0)}\,\phi(x_3)\,\phi(x_4)\rangle\rangle + \overset{\text{order}}{\mathcal{O}(\rho^2)}\dots\Big)$$

$$= \frac{1}{(2\rho)^{2\Delta}}\frac{1}{2^{2\Delta}} + \sum_{\mathcal{O}} \frac{C_{\phi\phi\mathcal{O}}}{(2\rho)^{\Delta}}(2\rho)^{\Delta_{\mathcal{O}}} \frac{C_{\phi\phi\mathcal{O}}}{\underset{1}{|x_3|^{\Delta_{\mathcal{O}}}}\,\underset{1}{|x_4|^{\Delta_{\mathcal{O}}}}\,\underset{2}{x_{34}^{2\Delta-\mathcal{O}}}} + \mathcal{O}$$

$$= \frac{1}{(4\rho)^{2\Delta}}\left(1+\sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}^2\,\frac{\cancel{2^{2\Delta}}\,(2\rho)^{\Delta_{\mathcal{O}}}}{2^{\cancel{2\Delta}-\cancel{\Delta}_{\mathcal{O}}}} + \mathcal{O}(\rho^{\Delta_{\mathcal{O}}+2})\dots\right)$$

$$= \frac{1}{(4\rho)^{2\Delta}}\left(1+\sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}^2\,(4\rho)^{\Delta_{\mathcal{O}}} + \mathcal{O}(\rho^{\Delta_{\mathcal{O}}+2})\dots\right)$$

$$\langle\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\rangle\rangle = \frac{f(u,v)}{x_{12}^{2\Delta}\,x_{34}^{2\Delta}} = \frac{f(u,v)}{(4\rho)^{2\Delta}}$$

$$f(u,v) = 1 + \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}^2 \underbrace{(4\rho)^{\Delta_{\mathcal{O}}}\left(1+\mathcal{O}(\rho^2)\right)}_{\text{conformal block.}}$$

Suppose $C_{\phi\phi\phi} = 0$ (e.g. $\phi \to -\phi$ symmetry).

then $\mathcal{O} \neq \phi$.

Suppose lowest $\Delta_{\mathcal{O}} \gg \Delta$.

Crossing symmetry:

$$u^\Delta f(v,u) = v^\Delta f(u,v)$$

lower $\mathcal{O}(\rho^2)$

$$(u^\Delta - v^\Delta) + \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}^2 \left(u^\Delta (4\tilde\rho)^{\Delta_{\mathcal{O}}} - v^\Delta (4\rho)^{\Delta_{\mathcal{O}}}\right) + \dots = 0$$

$\tilde\rho \to (u \leftrightarrow v)$

Take $\rho = \rho_0 + \tilde\varepsilon$

$\sqrt{u} + \sqrt{v} = 1$

$\sqrt{v} = 1 - \sqrt{u} \Rightarrow v = (1-\sqrt{u})^2$

Define $z = \sqrt{u} \rightarrow u = z^2$

$z_0 = 1/2$ Take $z_0 = \frac{1}{2} + \varepsilon$

$$V = (1-z)^2 = \left(\frac{1}{2} - \varepsilon\right)^2$$

$$U = \left(\frac{1}{2} + \varepsilon\right)^2 \qquad U \leftrightarrow V \qquad \boxed{\varepsilon \leftrightarrow -\varepsilon}$$

$$U^\Delta = \left(\frac{1}{2} + \varepsilon\right)^{2\Delta} = \frac{1}{2^{2\Delta}} (1+2\varepsilon)^{2\Delta}$$

$$x^a; \quad a x^{a-1}, \quad a(a-1) x^{a-2}, \quad a(a-1)(a-2) x^{a-3}$$

$$(1+2\varepsilon)^{2\Delta} = 1 + 2\Delta(2\varepsilon) + \frac{(2\Delta)(2\Delta-1)}{2}(2\varepsilon)^2 + \frac{(2\Delta)(2\Delta-1)(2\Delta-2)}{6}(2\varepsilon)^3 + \dots$$

$$U^\Delta - V^\Delta = 8\Delta\varepsilon + \frac{\cancel{2} \times 4 \times 8}{\cancel{6}\,3} \Delta(2\Delta-1)(\Delta-1)\varepsilon^3 + \dots \equiv 8\Delta\left(\varepsilon + \frac{4}{3}(2\Delta-1)(\Delta-1)\varepsilon^3\right.$$

$$\sqrt{u} = z = \frac{4\rho}{(1+\rho)^2} \qquad \rho^2 + 2\rho + 1 - \frac{4\rho}{z} = 0 \qquad \rho^2 + 2\left(1 - \frac{2}{z}\right)\rho + 1 = 0$$

$$\rho = \frac{-2(1-2/z) \pm \sqrt{4(1-2/z)^2 - 4}}{2} = -1 + 2/z \pm \sqrt{\cancel{1} - \frac{4}{z} + \frac{4}{z^2} - \cancel{1}}$$

$$\rho = -1 + \frac{2}{z} \pm \frac{2}{z}\sqrt{1-z} \qquad \rho = -1 + \frac{2}{z} - \frac{2}{z}\sqrt{1-z}$$

$$\rho(1/2) = -1 + 4 \pm 4\sqrt{\frac{1}{2}} = 3 \pm 2\sqrt{2} \qquad -$$

$$\rho = -1 + \frac{2}{z} - \frac{2}{z}\sqrt{1-z}$$

$$\rho = -1 + \frac{2}{\frac{1}{2}+\varepsilon}\left(1 - \sqrt{\tfrac{1}{2}-\varepsilon}\right)$$

$$\rho = -1 + \frac{4}{1+2\varepsilon}\left(1 - \frac{1}{\sqrt{2}}(1-2\varepsilon)^{1/2}\right)$$

$$= -1 + \frac{4}{1+2\varepsilon} - \frac{4}{\sqrt{2}}\frac{\sqrt{1-2\varepsilon}}{1+2\varepsilon}$$

leading $\Delta_O \gg 1$

$$\rho(\varepsilon)^{\Delta_O} - \rho(-\varepsilon)^{\Delta_O} \simeq 4(3-2\sqrt{2})^{\Delta_O - 1}\Delta_O(3\sqrt{2}-4)\left(\varepsilon + \frac{4}{3}\Delta_O^2\varepsilon^3 + \dots\right)$$

$$= 4\rho_0^{\Delta_O \cancel{-1}}\Delta_O\sqrt{2}\,\cancel{\rho_0}\left(\varepsilon + \frac{4}{3}\Delta_O^2\varepsilon^3 + \dots\right)$$

$$= 4\sqrt{2}\,\rho_0^{\Delta_O}\Delta_O\left(\varepsilon + \frac{4}{3}\Delta_O^2\varepsilon^3 + \dots\right)$$

$$8\Delta\left(\varepsilon + \frac{4}{3}(2\Delta-1)(\Delta-1)\varepsilon^3\right) + \sum_{O} C^2_{\phi\phi O}\frac{4^{\Delta_O}}{4^{\Delta}}4\sqrt{2}\rho_0^{\Delta_O}\Delta_O\left(\varepsilon + \frac{4}{3}\Delta_O^2\varepsilon^3 + \dots\right)$$

$O \leftrightarrow O$ including

$\tilde{C}^2_{\phi\phi O}$

$u^\Delta \sim v^\Delta$ $\quad \Delta \ll \Delta_O$

$$8\Delta + \sum_O \tilde{C}^2_{\phi\phi O} = 0$$

$$8\Delta\,\cancel{\tfrac{4}{3}}(2\Delta-1)(\Delta-1) + \sum_O \cancel{\tfrac{4}{3}}\,\tilde{C}^2_{\phi\phi O}\Delta_O^2 = 0$$

$\Delta_{\mathcal{O}} \geqslant \Delta_{min} \gg \Delta$ (assumption).

$$\sum_{\mathcal{O}} \tilde{C}^2_{\phi\phi\mathcal{O}} \Delta^2_{\mathcal{O}} \geq \sum_{\mathcal{O}} \tilde{C}^2_{\phi\phi\mathcal{O}} \Delta^2_{min} = +8\Delta \Delta^2_{min}$$

$$+\cancel{8\Delta}\, \Delta^2_{min} \lesssim +\cancel{8\Delta}\, (2\Delta-1)(\Delta-1)$$

$$\Delta^2_{min} \lesssim (2\Delta-1)(\Delta-1)$$

$\Delta_{min} \gg 1 \rightarrow \Delta \gg 1$ $\quad \Delta_{min} \lesssim \sqrt{2}\,\Delta$

we cannot have $\boxed{\Delta_{min} \gg \sqrt{2}\Delta}$