Axial anomaly

$$\mathcal{L} = \bar\psi (i\not{D})\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$; massless (charged) fermion.

$$D_\mu = \partial_\mu + ieA_\mu$$

$$1 = \frac{1+\gamma_5}{2} + \frac{1-\gamma_5}{2} = P_R + P_L$$

$$\mathcal{L} = \bar\psi (i\not{D}) P_R \psi + \bar\psi (i\not{D}) P_L \psi - \frac{1}{4} F_{\mu} F^{\mu\nu}$$

$$\psi_L = P_L\psi \qquad \bar\psi_L = \psi^\dagger P_L^\dagger \gamma_0 = \psi^\dagger P_L \gamma_0 = \bar\psi P_R$$

$$\gamma^\mu P_L = P_R \gamma^\mu$$

$$\mathcal{L} = \bar\psi \underbrace{(i\not{D}) P_R}_{P_L (i\not{D})} P_R \psi + \bar\psi \underbrace{(i\not{D}) P_L}_{P_R (i\not{D}} P_L \psi - \frac{1}{4} F_{\mu} F^{\mu\nu}$$

$$\mathcal{L} = \bar\psi_R (i\not{D})\psi_R + \bar\psi_L i\not{D} \psi_L - \frac{1}{4} F_{\mu} F^{\mu\nu}$$

It is invariant under $\begin{cases} \psi_R \to e^{i\alpha}\psi_R \\ \psi_L \to e^{i\beta}\psi_L \end{cases}$ Independent phases.

with mass
$$m^2 \bar\psi\psi = m^2 \bar\psi P_L \psi + m^2 \bar\psi P_R \psi$$
$$= m^2 \bar\psi P_L P_L \psi + m^2 \bar\psi P_R P_R \psi$$
$$= m^2 \bar\psi_R \psi_L + m^2 \bar\psi_L \psi_R$$ ← not separately invariant.

$\boxed{\alpha = \beta \text{ only}}$

We define the two currents

$$j^\mu = \bar\psi \gamma^\mu \psi \quad : \text{electromagnetic current.}$$

$$j_5^\mu = \bar\psi \gamma^\mu \gamma^5 \psi \quad : \text{axial current.}$$

$$j^\mu = \bar\psi \gamma^\mu (P_L + P_R)\psi = \bar\psi_L \gamma^\mu \psi_L + \bar\psi_R \gamma^\mu \psi_R$$

$$j_5^\mu = \bar\psi \gamma^\mu \gamma^5 (P_L + P_R)\psi = \bar\psi \gamma^\mu \gamma^5 P_L P_L \psi + \bar\psi \gamma^\mu \gamma^5 P_R P_R \psi$$

$$\gamma^5 P_L = \gamma^5 \left(\frac{1-\gamma_5}{2}\right) = \frac{\gamma^5 - 1}{2} = -P_L$$

$$\gamma^5 P_R = \gamma^5 \left(\frac{1+\gamma_5}{2}\right) = \frac{1+\gamma_5}{2} = P_2$$

$$= -\bar\psi P_R \gamma^\mu P_L \psi + \bar\psi P_L \gamma^\mu P_R \psi$$

$$j_5^\mu = -\bar\psi_L \gamma^\mu \psi_L + \bar\psi_R \gamma^\mu \psi_R$$

e.o.m. $\quad \not\partial \psi_L = 0 \qquad \not\partial \psi_R = 0$

$$\not\partial \psi_L = -ie\not A \psi_L \qquad \not\partial \psi_R = -ie\not A \psi_R$$

$$\partial_\mu \psi_L \gamma^\mu = -ieA_\mu \gamma^\mu \psi_L$$

$$\partial_\mu \psi_L^+ \gamma^0 \gamma^\mu \cancel{\gamma^0} = i\psi_L^+ e A_\mu \gamma^0 \gamma^\mu \cancel{\gamma^0} \Rightarrow \partial_\mu \bar\psi_L \gamma^\mu = ieA_\mu \bar\psi_L \gamma^\mu$$

Naively

$\partial_\mu j^\mu = ?$

$$\partial_\mu(\bar\psi_L \gamma^\mu \psi_L) = \partial_\mu \bar\psi_L \gamma^\mu \psi_L + \bar\psi_L \gamma^\mu \partial_\mu \psi_L$$

$$= ie A_\mu \bar\psi_L \gamma^\mu \psi_L - ie \bar\psi_L \not{A} \psi_L = 0.$$

Same for ψ_R.

$\Rightarrow \partial_\mu j^\mu = 0$ and $\partial_\mu j_5^\mu = 0$

However it does not work if one regularize.

Point splitting regularization of the current (product of op. at the same point)

$$j_5^\mu = \bar\psi(\vec{x}_\mu + \tfrac{\vec{\epsilon}_\mu}{2}) \gamma^\mu \gamma^5 \underbrace{e^{-ie\int_{\vec{x}_\mu - \vec{\epsilon}/2}^{\vec{x}_\mu + \vec{\epsilon}/2} dy^\mu A_\mu}}_{\text{for gauge invariance.}} \psi(x_\mu - \tfrac{\epsilon_\mu}{2})$$

$$\partial_\mu j_5^\mu = \partial_\mu \bar\psi(x_\mu + \tfrac{\epsilon_\mu}{2}) \gamma^\mu \gamma^5 e^{-ie\int_{x_\mu-\epsilon_\mu/2}^{x_\mu+\epsilon_\mu/2} dy^\mu A_\mu} \psi(x_\mu - \epsilon_\mu/2)$$

$$+ \bar\psi(x_\mu + \epsilon_\mu/2) \gamma^\mu \gamma^5 e^{-ie\int_{x_\mu-\epsilon_\mu/2}^{x_\mu+\epsilon_\mu/2} dy^\mu A_\mu} \partial_\mu \psi(x_\mu - \epsilon_\mu/2)$$

$$+ \bar\psi(x_\mu + \epsilon_\mu/2) \gamma^\mu \gamma^5 (-ie\, \epsilon^\nu \partial_\mu A_\nu) \psi(x_\mu - \epsilon_\mu/2)$$

$\underline{\mathcal{O}(\varepsilon)}$ only

$$= ie\,\bar\psi(x_\mu + \tfrac{\varepsilon_\mu}{2})\,\cancel{A}(x+\tfrac{\varepsilon}{2})\,\underset{\bar 2}{\gamma^5}\,\psi(x-\varepsilon/2)$$

$$+ie\,\bar\psi(x_\mu + \varepsilon_\mu/2)\,\gamma^5\, ie\,\cancel{A}(x-\epsilon/2)\,\psi(x-\epsilon/2)$$

because of γ_5

$$-ie\,\bar\psi(x_\mu + \epsilon_\mu/2)\,\gamma^\mu\gamma^5\,\epsilon^\nu\partial_\mu\Lambda_\nu)\,\psi(x-\epsilon/2)$$

$$= ie\,\bar\psi(x_\mu + \epsilon_\mu/2)\Big[\cancel{\cancel{A}(x)} + \frac{\epsilon}{2}^\nu\partial\cancel{A} - \cancel{\cancel{A}(x)} + \frac{\epsilon^\nu}{2}\partial_\nu\cancel{A} -$$

$$-\gamma^\mu\epsilon^\nu\partial_\mu\Lambda_\nu\Big]\gamma^5\,\psi(x-\epsilon/2)$$

$$\tfrac{1}{2}\,\epsilon^\nu\partial_\nu\Lambda_\mu\gamma^\mu - \epsilon^\nu\gamma^\mu\partial_\mu\Lambda_\nu = \epsilon^\nu\gamma^\mu(\partial_\nu\Lambda_\mu - \partial_\nu\Lambda_\mu)$$

$$= \epsilon^\nu\gamma^\mu F_{\nu\mu}$$

$$= ie\,\bar\psi(x_\mu + \epsilon_\mu/2)\,\epsilon^\nu\gamma^\mu F_{\nu\mu}(x)\gamma^5\psi(x_\mu - \epsilon_\mu/2)$$

It seems that $\epsilon\to 0$ gives indeed 0.

However $\bar\psi(x+\epsilon/2)\,\psi(x-\epsilon/2)$ is singular.

consider

$$\langle 0|\bar\psi_b(y)\,\psi_a(z)|0\rangle = -i(\cancel{\partial}_z)_{ab}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik(y-z)}$$

$$\langle 0|\bar\psi_b(y)(\gamma^\mu_{ba}\gamma^5_{ba}\psi_a(z)|0\rangle = -i\,\partial_\nu\underbrace{\mathrm{Tr}(\gamma^\nu\gamma^\mu\gamma^5)}_{0}\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik(y-z)}$$

$$=0$$

? free field theory.

Suppose there is a background field $\underline{A_\mu(x)}$.

$$\mathcal{L} = \bar{\psi}\,(i\partial_\mu - eA_\mu)\gamma^\mu\psi$$

If spatially separated (ϵ^μ space-like) then we can compute $\hat{T}\{\bar{\psi}\,\psi\}$

inverse op.

$$(i\partial^x_\mu - eA_\mu(x))\,\gamma^\mu\,\underbrace{\langle 0|\hat{T}\,\psi(x)\,\bar{\psi}(y)|0\rangle_A}\, = i\,\delta^{(4)}(x-y)$$

$$G_F(x-y) = \Delta_F^{(0)}(x-y) + \tilde{\Delta}_F^{(1)}(x-y) + \ldots$$

$$\cancel{i\partial_\mu\gamma^\mu\Delta_F^{(0)}} + i\partial_\mu\gamma^\mu\Delta^{(1)} - eA_\mu\gamma^\mu\Delta_F^{(0)} = \cancel{i\,\delta^{(4)}(x-y)}$$

$$i\partial_\mu\gamma^\mu\Delta^{(1)} = eA_\mu\gamma^\mu\Delta_F^{(0)}$$

$$\not{\partial}\Delta^{(1)} = -ie\not{A}\,\Delta_F^{(0)}$$

$$\not{\partial}_x\,\underbrace{\Delta^{(1)}(x-y)} = -ie\not{A}(x)\,\Delta_F^{(0)}(x-y)$$

$$\not{\partial}_x\int_z \Delta^{(0)}(x-z)\,\tilde{\Delta}^{(1)}(z-y)$$

$$\int_z \delta(x-z)\,\tilde{\Delta}^{(1)}(z-y) = \tilde{\Delta}^{(1)}(x-y) \quad\Rightarrow\quad \tilde{\Delta}^{(1)}(x-y) = -ie\not{A}(x)\,\Delta_F^{(0)}(x-y)$$

$$\Delta_F^{(1)}(x-y) = -ie\int_z \Delta_F^{(0)}(x-z)\,\not{A}(z)\,\Delta_F^{(0)}(z-y)$$

$$\Delta_F \simeq \Delta_F^{(0)}(x-y) - ie \int_t \Delta_F^{(0)}(x-z)\, \not{A}(z)\, \Delta_F^{(0)}(z-y) + \cdots$$

$$\langle 0|\, \overset{x}{\bar{\psi}}(x+\tfrac{\epsilon}{2})\, \gamma^\mu \gamma^5\, \overset{y}{\psi}(x-\epsilon/2)|0\rangle =$$

(-) sign

$$= ie \int_z \mathrm{Tr}\big(\gamma^\mu \gamma^5\, \underset{i\not\partial \Delta(x-z)}{\Delta_F^{(0)}(x-z)}\, \not{A}(z)\, \underset{i\not\partial \Delta(z-y)}{\Delta_F^{(0)}(z-y)}\big) + \cdots$$

$$= ie \int_z \mathrm{Tr}(\gamma^\mu \gamma^5 \gamma^\alpha \gamma^\beta \gamma^\rho)\, \partial_\alpha^x \Delta(x-z)\, A_\beta(z)\, \partial_\rho^z \Delta(z-y)$$

$$= 4iie\, \epsilon^{\mu\alpha\beta\rho} \int_z \partial_\alpha^x \Delta(x-z)\, A_\beta(z)\, \partial_\rho^z \Delta(z-y)$$

$$= -4e\, \epsilon^{\mu\alpha\beta\rho} \int d^d z \int \frac{d^d k_1}{(2\pi)^d}\, \partial_\alpha^x \frac{e^{ik_1(x-z)}}{k_1^2+i\epsilon} \int \frac{d^d k_2}{(2\pi)^d}\, \frac{\partial_\rho^z e^{ik_2(z-y)}}{k_2^2+i\epsilon}\, A_\beta(z)$$

$$= -4e\, \epsilon^{\mu\alpha\beta\rho} \int d^d z \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d}\, \frac{ik_{1\alpha}\, ik_{2\rho}\, e^{ik_2(z-y)+ik_1(x-z)}}{(k_1^2+i\epsilon)(k_2^2+i\epsilon)}\, A_\beta(z)$$

$$= -4e\, \epsilon^{\mu\alpha\beta\rho} \int d^d z \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d}\, \frac{ik_{1\alpha}\, ik_{2\rho}}{k_1^2+i\epsilon\;\, k_2^2+i\epsilon}\, e^{ik_1(x+\epsilon/2) - ik_2(x-\epsilon/2)}\, e^{i(k_2-k_1)z} A_\beta(z)$$

$$= 4e\, \epsilon^{\mu\alpha\beta\rho} \int d^d z \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \frac{\overbrace{(-ik_{1\alpha})}^{\partial/\partial z_\alpha} ik_{2\rho}}{(k_1^2+i\epsilon)(k_2^2+i\epsilon)} \times$$

$$\times e^{i(k_2-k_1)(z-x)} e^{i(k_1+k_2)\frac{\epsilon}{2}} A_\beta(z)$$

$\partial/\partial z$ by parts

$$= -4e\, \epsilon^{\mu\alpha\beta\rho} \underbrace{\int d^d z}_{(2\pi)^d \delta(k_1-k_2)} \int \frac{d^d k_1}{(2\pi)^d} \int \frac{d^d k_2}{(2\pi)^d} \frac{ik_{2\rho}}{k_1^2 k_2^2} e^{i(k_2-k_1)(z-x)} \times$$

$$\times e^{i(k_1+k_2)\frac{\epsilon}{2}} \underbrace{\partial_\alpha A_\beta}_{\underbrace{\frac{1}{2}F_{\alpha\beta}}_{\text{assume } x \text{ constant}}}$$

$$= -2e\, \epsilon^{\mu\alpha\beta\rho} \int \frac{d^d k}{(2\pi)^d} \frac{ik_\rho}{(k^2+i\epsilon)^2} e^{ik\cdot\epsilon} F_{\alpha\beta}$$

$$\frac{\partial}{\partial \epsilon_\rho} \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\epsilon}}{(k^2+i\epsilon)^2} = -2\pi^2 i \frac{\epsilon_\rho}{\epsilon^2}$$

$$\int_0^\infty d\alpha\, \alpha\, e^{i(k^2+i\epsilon)\alpha} = \int_0^\infty d\alpha\, \alpha\, e^{-(\epsilon-ik^2)\alpha} = \frac{1}{(\epsilon-ik^2)^2} = \frac{-1}{(k^2+i\epsilon)^2}$$

$$-\frac{\partial}{\partial \epsilon_\rho} \int_0^\infty d\alpha\, \alpha \int \frac{d^d k}{(2\pi)^d} e^{-\epsilon\alpha} e^{ik^2\alpha + ik\epsilon}$$

(4d) $$\int dk_0\, e^{ik_0^2\alpha + ik_0\epsilon_0} = \sqrt{\frac{\pi}{-i\alpha}}\, e^{+\frac{\epsilon_0^2}{+4i\alpha}} = e^{\frac{i\pi}{4}} \sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{i\epsilon_0^2}{4\alpha}}$$

$$\int dk_j\, e^{-ik_j^2\alpha - ik_j\epsilon_j} = \sqrt{\frac{\pi}{i\alpha}}\, e^{\frac{-\epsilon_j^2}{4i\alpha}} = e^{-\frac{i\pi}{4}}\sqrt{\frac{\pi}{\alpha}}\, e^{\frac{i\epsilon_j^2}{4\alpha}} \quad 8$$

$$\int \frac{d^4k}{(2\pi)^4}\, e^{-\epsilon\alpha}\, e^{ik^2\alpha + ik\epsilon} = \frac{e^{\frac{i\pi}{4}-\frac{3i\pi}{4}}}{e^{-2i\pi/4}}\,\frac{\pi^2}{\alpha^2}\,\frac{e^{-\frac{i\epsilon^2}{4\alpha}}}{(2\pi)^4}$$

$$= -i\,\frac{\pi^2}{\alpha^2}\,\frac{e^{-\frac{i\epsilon^2}{4\alpha}}}{(2\pi)^4}$$

$$(-)\frac{\partial}{\partial\epsilon_\rho}\int_0^\infty d\alpha\, \cancel{\alpha}\, e^{-\epsilon\alpha}\, \frac{(-i)\pi^2}{\alpha^{\cancel{2}}}\, e^{-\frac{i\epsilon^2}{4\alpha}}$$

$$\cancel{i}\,\pi^2 \int_0^\infty \frac{d\alpha}{\alpha}\, e^{-\epsilon\alpha}\left(\cancel{-2i}\,\frac{\epsilon_\rho}{4\alpha}\right) e^{-\frac{i\epsilon^2}{4\alpha}}$$

$$\frac{\pi^2\epsilon_\rho}{2}\int_0^\infty \frac{d\alpha}{\alpha^2}\, e^{-\frac{i\epsilon^2}{4\alpha} - \epsilon\alpha} = \frac{\pi^2\epsilon_\rho}{2}\int_0^\infty d\mu\, e^{-\frac{i\epsilon^2}{4}\mu}\, \cancel{e^{-\epsilon/\mu}}$$

$\downarrow ?$

$\mu = 1/\alpha$

$$= \frac{\pi^2\epsilon_\rho}{2}\,\frac{1}{(i\epsilon^2/4)} = -i\,\frac{2\pi^2}{16\pi^4}\,\frac{\epsilon_\rho}{\epsilon^2}$$

$$= +2e\, \epsilon^{\mu\alpha\beta\rho}\, F_{\alpha\beta}\left(+\frac{2\pi^2}{16\pi^4}\, i\, \frac{\epsilon_\rho}{\epsilon^2}\right) = \frac{ie}{4\pi^2}\, \epsilon^{\mu\alpha\beta\rho} F_{\alpha\beta}\, \frac{\epsilon_\rho}{\epsilon^2}$$

$\frac{1}{4}\delta_\rho^\nu$ averag.

$$\Rightarrow\; ie\, \epsilon^\nu F_{\nu\mu}\, \frac{ie}{4\pi^2}\, \epsilon^{\mu\alpha\beta\rho} F_{\alpha\beta}\, \frac{\epsilon_\rho}{\epsilon^2} = -\frac{e^2}{4\pi^2}\, \epsilon^{\mu\alpha\beta\rho} F_{\nu\mu} F_{\alpha\beta}\left(\frac{\epsilon^\nu\epsilon_\rho}{\epsilon^2}\right)$$

$$= -\frac{e^2}{16\pi^2} \epsilon^{\mu\alpha\beta\rho} F_{\rho\mu} F_{\alpha\beta}$$

$$= \frac{e^2}{16\pi^2} \epsilon^{\alpha\beta\rho\mu} F_{\alpha\beta} F_{\rho\mu}$$

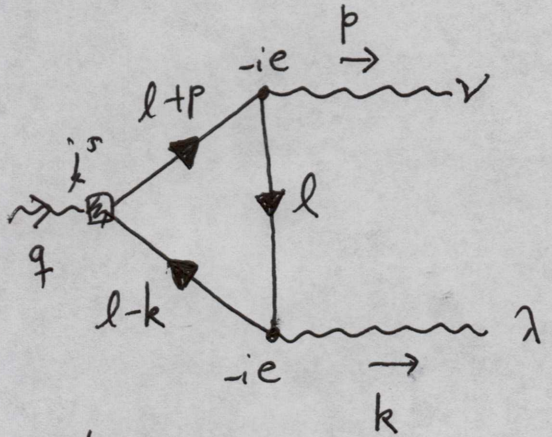
$$\boxed{\partial_\mu j^\mu_5 = \frac{e^2}{16\pi^2} \epsilon^{\alpha\beta\rho\mu} F_{\alpha\beta} F_{\rho\mu}}$$

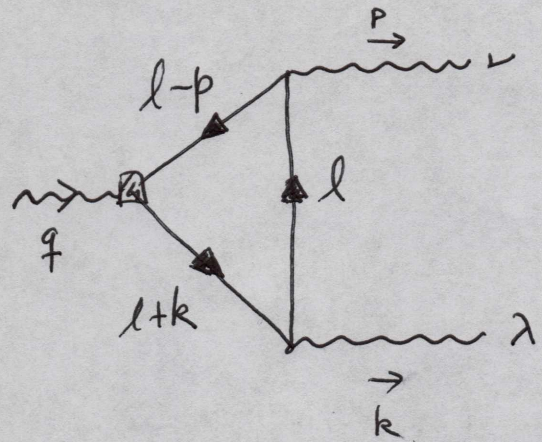
Equivalent calculation : triangle anomaly

$$\int d^4x\; e^{-iqx} \langle p,k|\, \partial_\mu j^{\mu 5}\,|0\rangle = i q_\mu \int d^4x\; e^{-iqx} \langle p,k|\, j^{\mu 5}\,|0\rangle$$

$$= (2\pi)^4\, \delta^{(4)}(p+k-q)\; \epsilon^*_\nu(p)\; \epsilon^*_\lambda(k)\; \underbrace{i q_\mu\, \mathcal{M}^{\mu\nu\lambda}(p,k)}_{=0\,?}$$

photon polarizations





fermion loop ↓

$$\cancel{(-)}(\cancel{+}ie)^2\, i q_\mu \int \frac{d^dl}{(2\pi)^d}\, \mathrm{Tr}\left\{ \frac{i(\not l+\not p)\gamma^\mu\gamma^5\, i(\not l-\not k)\gamma^\lambda\, i\not l\gamma^\nu}{(l+p)^2 (l-k)^2 l^2} + \frac{i(\not l-\not p)\gamma^\nu\, i\not l\gamma^\lambda\, i(\not l+\not k)\gamma^\mu\gamma^5}{(l-p)^2 (l+k)^2 l^2} \right.$$

$$\not p+\not k = \not p+\not l+\not k-\not l \qquad \not p-\not l+\not k+\not l = \not p+\not k$$

$$\left. ie^2 \int \frac{d^dl}{(2\pi)^d}\, \mathrm{Tr}\left\{ \frac{(\not l+\not p)\not q\gamma^5(\not l-\not k)\gamma^\lambda\not l\gamma^\nu}{(l+p)^2(l-k)^2 l^2} + \frac{(\not l-\not p)\gamma^\nu\not l\gamma^\lambda(\not l+\not k)\not q\gamma^5}{(l-p)^2(l+k)^2 l^2} \right\} \right\}$$

$$\not a\,\not a = a^2$$

$$e^2 \int \frac{d^dl}{(2\pi)^d}\, \mathrm{Tr}\left\{ \frac{\gamma^5(\not l-\not k)\gamma^\lambda\not l\gamma^\nu}{(l-k)^2 l^2} + \frac{(\not l+\not p)\gamma^5\gamma^\lambda\not l\gamma^\nu}{(l+p)^2 l^2} + \right.$$

$$\left. + \frac{(\not l-\not p)\gamma^\nu\not l\gamma^\lambda\gamma^5}{(l-p)^2 l^2} + \frac{\gamma^\nu\not l\gamma^\lambda(\not l+\not k)\gamma^5}{(l+k)^2 l^2} \right\}$$

If we can shift $l \to l+k$ and $l \to l+p$ then

$$e^2 \int \frac{d^d l}{(2\pi)^d} \,\mathrm{Tr}\left\{ \frac{\gamma^\nu \gamma^5 \not{l} \gamma^\lambda (\not{l}+\not{k})}{l^2 (l+k)^2} + \frac{\gamma^5 \gamma^\nu \not{l} \gamma^\lambda (\not{l}+\not{k})}{l^2 (l+k)^2} + \right.$$

$$\left. + \frac{\gamma^5 \gamma^\lambda \not{l} \gamma^\nu (\not{l}+\not{p})}{l^2 (l+p)^2} + \frac{\gamma^\lambda \gamma^5 \not{l} \gamma^\nu (\not{l}+\not{p})}{l^2 (l+p)^2} \right\}$$

Since $\{\gamma^\nu, \gamma^5\} = 0 = \{\gamma^5, \gamma^\lambda\}$ then it seems to cancel.

However we cannot shift in a divergent integral:

example:

$$\Delta(a) = \int_{-\infty}^{\infty} dx \,\big(\underbrace{f(x+a) - f(x)}_{\cancel{f(x)} + a f'(x) + \frac{1}{2} a^2 f''(x) + \cdots - \cancel{f(x)}} \big) = a\big(f(\infty) - f(-\infty)\big) + \frac{1}{2} a^2 \big(f'(\infty) - f'(-\infty)\big)$$

If the integral converges then $f(\pm\infty) = 0$ ... etc.

then $\Delta(a) = 0$

but for a divergent integral where e.g. $f(\pm\infty) \neq 0$ then

$$\Delta(a) = \underbrace{a\big(f(\infty) - f(-\infty)\big)}_{\text{surface term.}}$$

(and more terms if $f'(\pm\infty) \neq 0$ etc.)

Consider first ~~the~~ ambiguity in the first diagram:

$$-ie^2\int\frac{d^4l}{(2\pi)^4}\,\frac{Tr\{(\not{l}+\not{p})\gamma^\mu\gamma^5(\not{l}-\not{k})\gamma^\lambda\not{l}\gamma^\nu\}}{(l+p)^2(l-k)^2\,l^2}$$

linear divergence $\frac{l^4\,l^3}{l^6}$ → we need:

$$-ie^2\int\frac{d^4l}{(2\pi)^4}\,\frac{Tr\{\not{l}\gamma^\mu\gamma^5\not{l}\gamma^\lambda\not{l}\gamma^\nu\}}{(l+p)^2(l-k)^2\,l^2}$$

$$\simeq -ie^2\int\frac{d^4l}{(2\pi)^4}\,\frac{Tr\{\not{l}\gamma^\lambda\not{l}\gamma^\nu\not{l}\gamma^\mu\gamma^5\}}{l^6}$$

Using: $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha\gamma^\beta\gamma^5) = -4i\,(\eta^{\mu\nu}\epsilon^{\rho\sigma\alpha\beta} - \eta^{\mu\rho}\epsilon^{\nu\sigma\alpha\beta} +$

$+\eta^{\rho\nu}\epsilon^{\mu\sigma\alpha\beta} - \eta^{\alpha\beta}\epsilon^{\sigma\mu\nu\rho} +$

$+\eta^{\sigma\beta}\epsilon^{\alpha\mu\nu\rho} - \eta^{\sigma\alpha}\epsilon^{\beta\mu\nu\rho})$

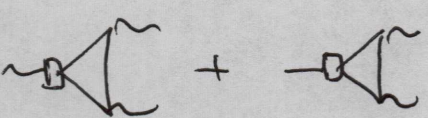
we get

$$\simeq +ie^2(-4i)\int\frac{d^4l}{(2\pi)^4}\,\frac{(+l^2)}{l^{\cancel{6}\,4}}\,\epsilon^{\lambda\nu\alpha\mu}\,l_\alpha \Rightarrow \frac{4e^2}{16\pi^4}\,\underbrace{\frac{\cancel{\Lambda^4}}{\cancel{\Lambda^4}}}_{2\pi^2}\,\underbrace{\int d\Omega\,\hat{l}_\rho a^\rho\,\hat{l}_\alpha}_{\frac{1}{4}\eta_{\rho\alpha}}\;\epsilon^{\lambda\nu\alpha\mu}$$

$$\frac{e^2}{8\pi^2}\,a_\alpha\,\epsilon^{\lambda\nu\alpha\mu}$$

the second diagram is the same with $(k \leftrightarrow p)$
$(\mu \leftrightarrow \nu)$

then the ambiguity is the same:

 $\rightarrow \quad \dfrac{e^2}{8\pi^2}\, a_\alpha\, \epsilon^{\lambda\nu\alpha\mu}$

$$a_\alpha = a_1 k_\alpha + a_2 p_\alpha$$

Now we compute the div of axial anomaly.
the shifts cancel. We get

$$i q_\mu \mathcal{M}^{\mu\nu\lambda} = \frac{ie^2}{8\pi^2}\, a_\alpha\, q_\mu\, \epsilon^{\lambda\nu\alpha\mu}$$

Now we compute the div. of the e.m. current

$$i k_\lambda \mathcal{M}^{\mu\nu\lambda} = e^2 \int \frac{d^4 l}{(2\pi)^4}\, \mathrm{Tr}\left\{ \frac{(\not{l}+\not{p})\gamma^\mu\gamma^5(\not{l}-\not{k})\overset{k-l+l}{\not{k}}\,\not{l}\,\gamma^\nu}{(l+p)^2\,(l-k)^2\,l^2} + \right.$$

$$\left. + \frac{(\not{l}-\not{p})\gamma^\nu\,\not{l}\,\overset{k+l-l}{\not{k}}(\not{l}+\not{k})\gamma^\mu\gamma_5}{(l-p)^2\,(l+k)^2\,l^2} \right\} = e^2 \int \frac{d^4 l}{(2\pi)^4}\, \mathrm{Tr}\left\{ - \frac{(\not{l}+\not{p})\gamma^\mu\gamma^5\,\not{l}\,\gamma^\nu}{(l+p)^2\,l^2} + \right.$$

$l \to l+p$ cancel.

$$\left. + \frac{(\not{l}+\not{p})\gamma^\mu\gamma^5(\not{l}-\not{k})\gamma^\nu}{(l+p)^2\,(l-k)^2} + \frac{(\not{l}-\not{p})\gamma^\nu\,\not{l}\,\gamma^\mu\gamma_5}{(l-p)^2\,l^2} - \frac{(\not{l}-\not{p})\gamma^\nu(\not{l}+\not{k})\gamma^\mu\gamma_5}{(l-p)^2(l+k)^2} \right\}$$

$$= e^2 \int \frac{d^4 l}{(2\pi)^4} \left\{ \frac{Tr\left((\not{l} - \not{k}) \gamma^\nu (\not{l} + \not{p}) \gamma^\mu \gamma^5 \right)}{(l+p)^2 (l-k)^2} - \frac{Tr\left((\not{l} - \not{p}) \gamma^\nu (\not{l} + \not{k}) \gamma^\mu \gamma_5 \right)}{(l-p)^2 (l+k)^2} \right\}$$

$l \to l + k - p$ we get surface term

$$= e^2 \int \frac{d^4 l}{(2\pi)^4} \left. \frac{(-4i)\, \epsilon^{\alpha\nu\beta\mu} (l-k)_\alpha (l+p)_\beta}{l^4} \right|_{\text{surface.}}$$

$$- l_\beta p_\alpha$$

$$= -4ie^2 \int \frac{d^4 l}{(2\pi)^4} \frac{\epsilon^{\alpha\nu\beta\mu}}{l^4} \left(-k_\alpha l_\beta + l_\alpha p_\beta \right)$$

$$= +4ie^2\, \epsilon^{\alpha\nu\beta\mu} (k_\alpha + p_\alpha) \int \frac{d^4 l}{(2\pi)^4} \frac{l_\beta}{l^4}$$

$l \to l + k - p.$

$$= 4ie^2\, \epsilon^{\alpha\nu\beta\mu} (k_\alpha + p_\alpha) \frac{\cancel{2\pi^2}}{\cancel{16}\pi^{\cancel{4}\,2}\ 8} (k_\beta - p_\beta) \frac{1}{\cancel{4}}$$

$$= \frac{ie^2}{8\pi^2} \epsilon^{\alpha\nu\beta\mu} (k_\alpha + p_\alpha)(k_\beta - p_\beta)$$

$$= -\frac{ie^2}{4\pi^2} \epsilon^{\alpha\nu\beta\mu} k_\alpha p_\beta$$

In total (with ambiguity)

$$i k_\lambda \mathcal{M}^{\mu\nu\lambda} = -\frac{ie^2}{4\pi^2}\, \epsilon^{\overset{\lambda\nu\alpha\mu}{\alpha\nu\beta\mu}}\, k_{\underset{\lambda}{\alpha}}\, p_{\underset{\alpha}{\beta}} + \frac{e^2}{8\pi^2}\, a_\alpha\, \epsilon^{\lambda\nu\alpha\mu}\, i k_\lambda$$

$a_\alpha = 2 p_\alpha$

$i k_\lambda \mathcal{M}^{\mu\nu\lambda} = 0$ ✓.

But then

$$\boxed{i q_\mu \mathcal{M}^{\mu\nu\lambda} = \frac{ie^2}{4\pi^2}\, p_\alpha\, k_\mu\, \epsilon^{\lambda\nu\alpha\mu}}$$

fixes axial anomaly.

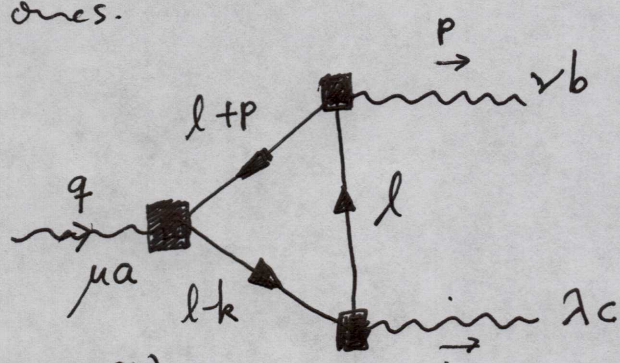
Non-abelian case

The triangle diagram is the same up to non-abelian generator matrices.

Suppose only left fermions see the gauge field.

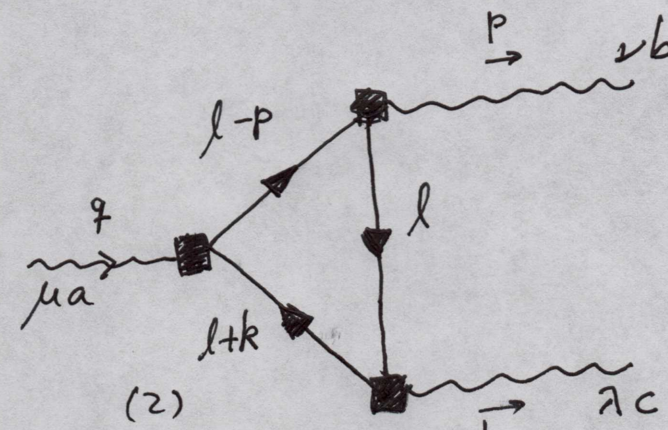
$$j^{\mu a} = \bar{\psi}\,\gamma^\mu \underbrace{P_L}_{\left(\frac{1-\gamma_5}{2}\right)} \psi \qquad \Bigg\| \qquad \Gamma^{\mu\nu\rho}_{abc} = \langle 0 | \hat{T}\{ j^\mu_a(x)\, j^\nu_b(y)\, j^\rho_c(z)\} | 0 \rangle$$

We can put non-Abelian background fields ~~as~~ instead of abelian ones.



(1)

$+$



(2)

$$\cancel{-}\, g^2 \cancel{(-i)^2}\, \mathrm{Tr}(t^a t^b t^c) \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\left\{ \frac{i(\not{l}+\not{p})\,\gamma^\nu P_L\; i\not{l}\,\gamma^\lambda P_L\; i(\not{l}-\not{k})\,\gamma^\mu P_L}{(l+p)^2\,(l-k)^2\,l^2} \right\}$$

$$+\,(p \leftrightarrow k,\ \nu \leftrightarrow \lambda,\ b \leftrightarrow c)$$

$$-\frac{ig^2}{2}\, \mathrm{Tr}(t^a t^b t^c) \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not{l}+\not{p})\gamma^\nu \not{l}\gamma^\lambda (\not{l}-\not{k})\gamma^\mu (1-\gamma_5)\}}{(l+p)^2\,(l-k)^2\,l^2}$$

$$+\,(p \leftrightarrow k,\ \nu \leftrightarrow \lambda,\ b \leftrightarrow c)$$

Another point. We have

$$\mathrm{Tr}(t^a t^b t^c) = \tfrac{1}{2}\mathrm{Tr}(t^a\{t^b, t^c\}) + \tfrac{1}{2}\mathrm{Tr}(t^a[t^b, t^c])$$

$$= \overset{\swarrow \text{definition}}{D^{abc}} + \frac{i}{2} f^{bcd}\,\mathrm{Tr}\,t^a t^d = D^{abc} + \frac{iN}{2} f^{abc}$$

$$\mathrm{Tr}(t^a t^c t^b) = D^{abc} - \frac{iN}{2} f^{abc} \qquad (\mathrm{Tr}(t^a t^b) = N\delta^{ab})$$

the commutator terms correspond to equal time commutators:

$$\Gamma^{\mu\nu\rho}_{abc} = \langle 0| T\{ j^\mu_a(x)\, j^\nu_b(y)\, j^\rho_c(z)\}|0\rangle$$

$$\partial_\mu \Gamma^{\mu\nu\rho}_{abc} \to [j, j]$$. They do not represent anomalies.

We then have for the anomaly (only in γ_5 part also)

$$\frac{ig^2}{2} D^{abc} \left\{ \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not l + \not p)\gamma^\nu \not l \gamma^\lambda (\not l - \not k)\gamma^\mu \gamma_5\}}{(l+p)^2 (l-k)^2 l^2} + \right.$$

$$\left. + \int \frac{d^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\{(\not l + \not k)\gamma^\lambda \not l \gamma^\nu (\not l - \not p)\gamma^\mu \gamma_5\}}{(l+k)^2 (l-p)^2 l^2} \right\}$$

This is the same as before. We cannot have all currents conserved ($ik_\mu \Pi^{\mu\nu\rho}_{abc} = 0 \qquad iq_\mu \Pi^{\mu\nu\rho}_{abc} = 0$)

In this case all the currents are the same $\Rightarrow j^\mu_L$ not conserved unless $D^{abc} = 0$

If one of the currents corresponds to a global current (i.e. not gauged) then we can have $D^{abc} \neq 0$ and put the anomaly in the global current.

<u>Example of standard model</u> (first generation).

(recall that before spontaneous symmetry breaking all fermions are massless).

	$SU(3)$	$SU(2)$	$U(1)_Y$
$l_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$	3	2 $(1/2)$	$-1/6$
u_R^*	$\bar{3}$	1 (0)	$+2/3$
d_R^*	$\bar{3}$	1 (0)	$-1/3$
$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	1	2 $(1/2)$	$1/2$
e_R^r	1	1	-1

$\Big\}$ they are left handed (u_R^*)

examples

$SU(3) - SU(3) - U(1)_Y$

$$\mathrm{Tr}\left(T_Y \underbrace{\{T_a, T_b\}}_{SU(3)}\right)$$

If we label as $|j\rangle$ the different states we have

$$\mathrm{Tr}\left(T_Y \{T_a T_b\}\right) = \sum_{|j\rangle} \langle j| \, T_Y \, \{T_a, T_b\} |j\rangle$$

gives zero unless

$|j\rangle$ is 3 or $\bar{3}$

$\{T_a, T_b\}$ has a part proportional to T_c and then $\underbrace{\delta_{ab} \cdot \mathbb{1}}_{\text{this survives.}}$

We get anomaly $\sim \underset{\uparrow \text{triplets}}{\mathrm{Tr}} (T_Y) = \underbrace{2 \times \left(-\frac{1}{6}\right)}_{-1/3} + \frac{2}{3} - \frac{1}{3}$

$-1/3 + \frac{2}{3} - \frac{1}{3} = 0$ ✓

$SU(2) - SU(2) - U(1)_Y$

Same as $SU(3)$ but we get

$$\mathrm{Tr}\, T_Y = 3 \times \left(-\frac{1}{6}\right) + \frac{1}{2} = 0$$

↑ doublets ↑ 3 colors!

$U(1)_Y - U(1)_Y - U(1)_Y$

$$\mathrm{Tr}(T_Y^3) \stackrel{?}{=} 0$$

$$\mathrm{Tr}(T_Y^3) = 3 \times 2 \times \left(-\frac{1}{6}\right)^3 + 3 \times \left(\frac{2}{3}\right)^3 + 3 \times \left(-\frac{1}{3}\right)^3 +$$

$$+ 2 \times \left(\frac{1}{2}\right)^3 + (-1)^3$$

$$= -\frac{1}{36} + \frac{8}{9} - \frac{1}{9} + \frac{1}{4} - 1$$

$$= -\frac{1}{36} - \frac{36}{36} + \frac{32 - 4 + 9}{36} = \frac{\overbrace{32-4+9-1}^{4}-36}{36}$$

$= 0$ ✓