663, Homework III, (3 problems)

Problem 1

In 3 dimensions and using that $K_j = P_j^\dagger$ compute the unitarity constraints for scalar fields and vector fields following from positivity of the matrix elements

$$\langle\Delta, j\,m|C_k^* K_k\; C_l P_l|\Delta, j\,m\rangle \;\geq\; 0 \tag{0.1}$$
$$\langle\Delta, j\,m|(C^{kp})^* K_p K_k\; C^{lq} P_l P_q|\Delta, j\,m\rangle \;\geq\; 0 \tag{0.2}$$

for arbitrary constants C and $j = 0, 1$. For simplicity, in the case of $j = 1$ just consider the first inequality. If you want, for $j = 1$, consider also the condition

$$C_{km_2}^* C_{lm_1}\langle\Delta, j\,m_2|K_k\; P_l|\Delta, j\,m_1\rangle \geq 0 \tag{0.3}$$

for a better bound.
Reference: see *e.g.* hep-th/9712074 by S. Minwalla.

Problem 2

Consider the O.P.E. of two scalar fields ϕ around the middle point between them, that is

$$\phi(\vec{x})\phi(-\vec{x}) = \frac{1}{|2x|^{2\Delta}} + \sum_{\Phi} C_{\phi\phi\Phi}|2x|^a\;(1 + \alpha x^2\partial_y^2 + \ldots)\Phi(y)\big|_{y=0} + \ldots \tag{0.4}$$

That is, determine the exponent a and the coefficient α by matching with the appropriate 3-point function.

Problem 3

Using the lecture notes, provide more detailed calculations showing that, if the lowest two primary operators are scalars ϕ and Φ and also $C_{\phi\phi\phi} = 0$ then $\Delta_\Phi^2 \leq (2\Delta - 1)(\Delta - 1)$ (under the simplifying assumption that $\Delta_\Phi \gg \Delta$).