Optical theorem and Froissart bound

Consider first a finite dimensional unitary matrix S.
Define T such that

$$\underset{\uparrow \text{ S-matrix}}{S} = 1 + i \underset{\uparrow \text{ scattering.}}{T}$$

$$S^{+} S = 1 \Rightarrow (1 - i T^{+})(1 + i T) = 1$$

$$i T - i T^{+} + T^{+} T = 0$$

$$-i (T - T^{+}) = T^{+} T$$

consider some state $|\psi_{in}\rangle$

$$-i \langle \psi_{in} | T - T^{+} | \psi_{in} \rangle = \langle \psi_{in} | T^{+} \overbrace{\sum_{|\psi_f\rangle} |\psi_f\rangle \langle \psi_f |}^{\text{identity}} T | \psi_{in} \rangle$$

then

$$-i \Big(\underbrace{\langle \psi_{in} | T | \psi_{in} \rangle - \langle \psi_{in} | T | \psi_{in} \rangle^{*}}_{2 i \, \mathrm{Im} \langle \psi_{in} | T | \psi_{in} \rangle} \Big) = \sum_{|\psi_f\rangle} |\langle \psi_f | T | \psi_{in} \rangle|^2$$

we get

$$\underbrace{\sum_{|\psi_f\rangle} |\langle \psi_f | T | \psi_{in} \rangle|^2}_{\text{total cross section}} = 2 \underbrace{\mathrm{Im} \langle \psi_{in} | T | \psi_{in} \rangle}_{\text{imaginary part of forward amplitude}} .$$

Now we do the same for an ∞ dim. matrix.

$$S = 1 + i\,T$$

$$\underbrace{\langle \vec{q}_j{}' \sigma_j{}' |}_{\langle\beta|} T \,|\, \underbrace{\vec{q}_j\, \sigma_j \rangle}_{|\alpha\rangle} = (2\pi)^4\, \delta^{(4)}\Big(\sum_{j=1}^{n} q_j' - \sum_{j=1}^{m} q_j\Big)\, \mathcal{M}_{\beta\alpha}$$

$$\sum_n \prod_{j=1}^{n} \int \frac{d^3 q_j}{(2\pi)^3} \frac{1}{2\mathcal{E}_j} |\{q_j\}\rangle\langle\{q_j\}| = \mathbb{1}$$ (we ignore polarizations now).

$$T^\dagger T = -i\,(T - T^\dagger)$$

$$\langle p_1' p_2' | T^\dagger \sum_n \prod_{j=1}^{n} \int \frac{d^3 q_j}{(2\pi)^3} \frac{1}{2\mathcal{E}_j} |\{q_j\}\rangle\langle\{q_j\}| T |p_1 p_2\rangle =$$

$$= -i\,\big(\langle p_1' p_2' | T | p_1 p_2\rangle - \langle p_1' p_2' | T^\dagger | p_1 p_2\rangle\big)$$

$$\sum_n \prod_{j=1}^{n} \int \frac{d^3 q_j}{(2\pi)^3} \frac{1}{(2\mathcal{E}_j)} (2\pi)^{\cancel{8}\,4}\, \delta^{(4)}\Big(p_1' + p_2' - \sum q_j\Big)\, \delta^{(4)}\Big(p_1 + p_2 - \sum q_j\Big) \,.$$

$$\cdot\ \mathcal{M}^*_{\{q_j\}|p_1' p_2'\rangle}\ \mathcal{M}_{\{q_j\}|p_1 p_2\rangle} = -i\, \cancel{(2\pi)^4}\, \delta^{(4)}(p_1' + p_2' - p_1 - p_2) \,.$$

$$\cdot\Big(\mathcal{M}_{|p_1' p_2'\rangle |p_1 p_2\rangle} - \mathcal{M}^*_{|p_1 p_2\rangle |p_1' p_2'\rangle}\Big)$$

$$\delta^{(4)}(p_1'+p_2'-\sum q_i)\,\delta^{(4)}(p_1+p_2-\sum q_i) =$$

$$= \delta^{(4)}(p_1+p_2-\sum q_i)\,\underbrace{\delta^{(4)}(p_1'+p_2'-p_1-p_2)}_{\text{we can eliminate it.}}$$

then put $p_1' = p_1;\ p_2' = p_2$

$$2\,\mathrm{Im}\,\mathcal{M}_{\alpha\alpha} = \sum_n \prod_{j=1}^{n} \int \frac{d^3 q_j}{(2\pi)^3}\frac{1}{2E_j}(2\pi)^4\,\delta^{(4)}(p_1+p_2-\sum q_i)\,.$$

$$\cdot\,|\mathcal{M}_{\{q_j\}|p_1 p_2\rangle}|^2$$

$|\alpha\rangle = |p_1\, p_2\rangle$

using normalization from the book. (4.79)

$$d\sigma = \frac{1}{2E_A\,2E_B\,|v_A-v_B|}\prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\,|\mathcal{M}_{\{p_f\}|p_1 p_2\rangle}|^2\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-\sum p_f)$$

$$\sigma = \frac{1}{2E_A\,2E_B\,|v_A-v_B|}\int \prod_f \frac{d^3 p_f}{(2\pi)^3}\frac{1}{2E_f}\,|\mathcal{M}_{\beta\alpha}|^2\,(2\pi)^4\,\delta^{(4)}(p_A+p_B-\sum p_f)$$

$\beta \to |\{p_f\}\rangle$, $\alpha \to |p_1 p_2\rangle$

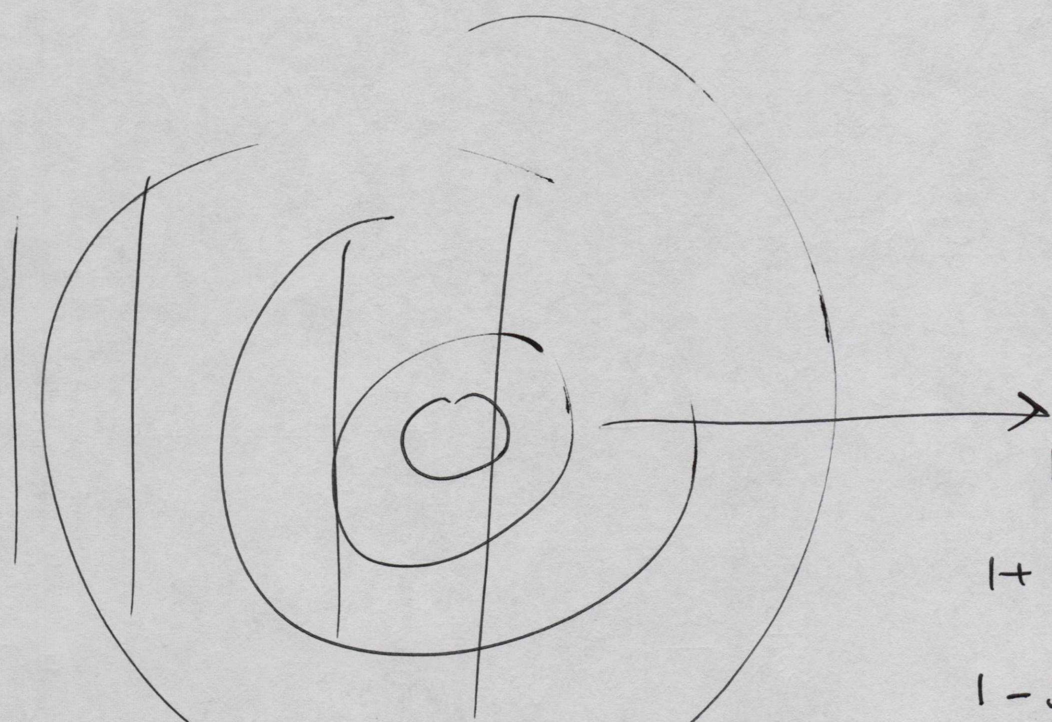
$$\Rightarrow\ 2\,\mathrm{Im}\,\mathcal{M}_{\alpha\alpha} = 2E_A\,2E_B\,|v_A-v_B|\,\sigma_{\text{total}}$$

all processes.

$$|v_A - v_B| = \left| \frac{\vec{p}_1}{\varepsilon_1} - \frac{\vec{p}_2}{\varepsilon_2} \right| = |\vec{p}_1| \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) \qquad (\vec{p}_2 = -\vec{p}_1)$$

$$\operatorname{Im} \mathcal{M}_{\alpha\alpha} = 2\, \varepsilon_1 \varepsilon_2\, |\vec{p}_1| \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) \sigma_T$$

$$\boxed{\operatorname{Im} \mathcal{M}_{\alpha\alpha} = 2 E_{CM}\, |\vec{p}_1|\, \sigma_T}$$ optical theorem



$1 + if$

$1 + i(\operatorname{Re} f + i \operatorname{Im} f)$

$1 - \operatorname{Im} f + i \operatorname{Re} f$

Froissart bound

Naive idea

massive theory $\rightarrow$ potential $V \simeq g \dfrac{e^{-\mu r}}{r}$

when $g e^{-\mu r_0} \sim 1 \Rightarrow \mu r_0 = \ln|g|$ $\quad r_0 = \frac{1}{\mu} \ln|g|$

if $r \gg r_0$ then no potential $\rightarrow$ no scattering

if $r < r_0$ potential is large. $\rightarrow$ scattering

$$\sigma \approx \pi r_0^2 = \frac{\pi}{\mu^2} (\ln|g|)^2$$

μ: mass.

this a constant. But in QFT we can assume $|g|$ can grow as a power of energy or s!

$$g \sim s^\alpha \qquad \ln|g| \sim \alpha \ln s$$

$$\sigma \approx \frac{\pi \alpha^2}{\mu^2} (\ln s)^2$$

naive bound for cross section

ϕ^4 theory optical theorem in perturbation theory and analiticity.

$$\operatorname{Im} \mathcal{M}_{\alpha\alpha} = 2 E_{cm} |\vec{p}_1| \sigma_T$$

check in pert. theory.

In ϕ^4 theory we only have particles of mass m.

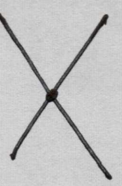
if $E_{cm} < 4m$ (and $E_{cm} > 2m$ as it should) then we can only have 2 particles in the final state (3 is not possible because of $\phi \leftrightarrow -\phi$ symmetry)

In that range of energy σ_T is for 2 particles only.

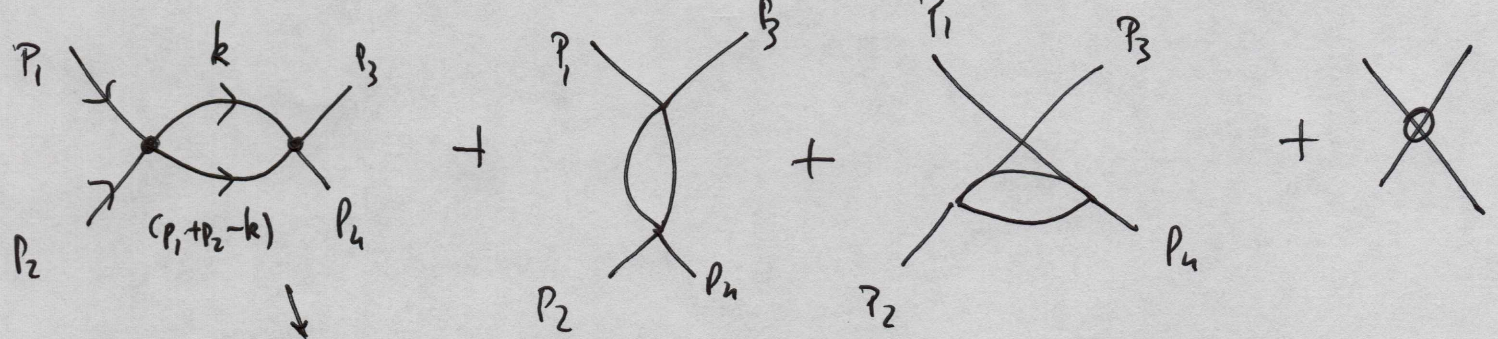
then

$$\operatorname{Im} \mathcal{M}_{\alpha\alpha} = 2 \underbrace{E_{cm}}_{\sqrt{s}} \cancel{|\vec{p}_1|} \int d\Omega \, \frac{1}{64\pi^2} \frac{|\vec{p}_3|}{\cancel{|\vec{p}_1|}} \frac{1}{s} |\mathcal{M}_{\beta\alpha}|^2$$

$$= \frac{|\vec{p}_3|}{32\pi^2 \sqrt{s}} \int d\Omega \, |\mathcal{M}_{\beta\alpha}|^2$$



$-i\lambda$



$$\frac{(-i\lambda)^2}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(\underbrace{p_1+p_2}_{p}-k)^2-m^2+i\varepsilon}=$$

$$=+\frac{\lambda^2}{2}\int_0^1 d\alpha\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2+\alpha p^2-2\alpha pk-m^2+i\varepsilon)^2}$$

$$(k-\alpha p)^2+\alpha(1-\alpha)p^2-m^2+i\varepsilon$$

$$=\frac{\lambda^2}{2}\int_0^1 d\alpha\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2+\alpha(1-\alpha)s-m^2+i\varepsilon)^2}$$

$$=\frac{\lambda^2}{2}\int_0^1 d\alpha\,\frac{i}{(4\pi)^{d/2}}\,\frac{\Gamma(2-d/2)}{\Gamma(2)}\,\frac{1}{\Delta^{2-d/2}}\;; \qquad \Delta=m^2+i\varepsilon-\alpha(1-\alpha)s$$

$$d=4-\varepsilon$$

$$=\frac{i\lambda^2}{2}\,\frac{\Gamma(\varepsilon/2)}{(4\pi)^{2-\varepsilon/2}}\int_0^1 d\alpha\;\Delta^{-\varepsilon/2}$$

$$= \frac{i\lambda^2}{32\pi^2}\,\Gamma(\epsilon/2)\,(4\pi)^{\epsilon/2}\int_0^1 d\alpha\;\Delta^{-\epsilon/2}$$

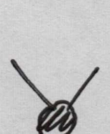
$$\frac{\epsilon}{2}\Gamma(\epsilon/2) = \Gamma(1+\epsilon/2) = 1 + \frac{\epsilon}{2}\Gamma'(1) + \dots$$

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} + \underbrace{\Gamma'(1)}_{-\gamma} + \dots$$

$$= \frac{i\lambda^2}{32\pi^2}\,\frac{2}{\epsilon}\left(1 - \frac{\epsilon}{2}\gamma + \frac{\epsilon}{2}\ln 4\pi - \frac{\epsilon}{2}\int_0^1 d\alpha \ln\Delta + \dots\right)$$

$$= \frac{i\lambda^2}{16\pi^2\epsilon} + \frac{i\lambda^2}{32\pi^2}\left(-\gamma + \ln 4\pi - \int_0^1 d\alpha \ln\left(m^2 - \alpha(1-\alpha)s - i\epsilon\right)\right)$$

subtraction



$$= -i\lambda + \cancel{\frac{3i\lambda^2}{16\pi^2\epsilon}} + \frac{3i\lambda^2}{32\pi^2}\left(-\gamma + \ln 4\pi\right) -$$

$$- \frac{i\lambda^2}{32\pi^2}\left(\int_0^1 d\alpha\left[\ln\left(m^2 - \alpha(1-\alpha)s - i\epsilon\right) + \ln\left(m^2 - \alpha(1-\alpha)t - i\epsilon\right) + \ln\left(m^2 - \alpha(1-\alpha)u - i\epsilon\right)\right]\right)$$

Notation from the book ↗ this is $i\mathcal{M}$

$$\mathcal{M} = -\lambda + \frac{3\lambda^2}{32\pi^2}\left(-\gamma + \ln 4\pi\right) - \frac{\lambda^2}{32\pi^2}\left(\int_0^1 d\alpha\left[\ln\left(m^2 - \alpha(1-\alpha)s - i\epsilon\right) + \ln\left(m^2 - \alpha(1-\alpha)t - i\epsilon\right) + \ln\left(m^2 - \alpha(1-\alpha)u - i\epsilon\right)\right]\right)$$

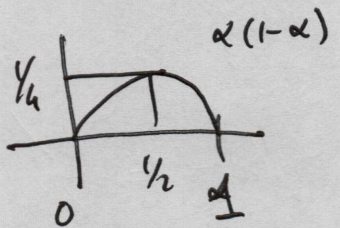
$\mathrm{Im}\,\mathcal{M} \neq 0$ only if argument of ln is negative.

For physical values $t \leq 0$ $u \leq 0$ $\Rightarrow$ argument positive.

Only imag. part from

$$\mathrm{Im}\,\mathcal{M} = -\frac{\lambda^2}{32\pi^2}\,\mathrm{Im}\int_0^1 d\alpha \; \ln\left(m^2 - \alpha(1-\alpha)s - i\varepsilon\right)$$

It should be evaluated at $\theta = 0$ for optical theorem but it does not depend on θ. (or $t=0$)

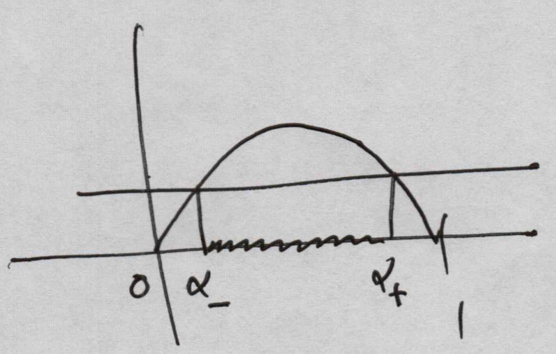


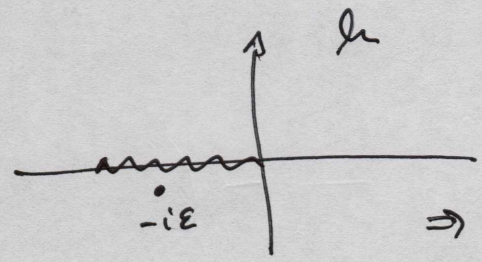
if $m^2 - \frac{1}{4}s < 0 \Rightarrow$ im. part

$\boxed{s > 4m^2}$ physical region

$m^2 - \alpha(1-\alpha)s < 0 \quad \Rightarrow \quad \alpha(1-\alpha) > m^2/s$

$$\alpha^2 - \alpha + m^2/s = 0 \qquad \alpha = \frac{1 \pm \sqrt{1 - 4m^2/s}}{2} = \frac{\sqrt{s} \pm \sqrt{s - 4m^2}}{2\sqrt{s}}$$





$\Rightarrow \mathrm{Im}\,\ln = -\pi$

$$\mathrm{Im}\,\mathcal{M} = \frac{\lambda^2}{32\pi}(\alpha_+ - \alpha_-) = \frac{\lambda^2}{32\pi}\frac{\sqrt{s - 4m^2}}{\sqrt{s}}$$

identical particles

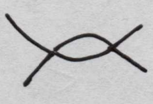
$$\frac{1}{2}\frac{|\vec{p}_3|}{32\pi^2\sqrt{s}}\int d\Omega\;\lambda^2 = \frac{1}{2}\frac{4\pi\,|\vec{p}_3|}{32\pi^2\sqrt{s}}\,\lambda^2 = \frac{1}{16\pi}\,\frac{1}{2}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\lambda^2 =$$

$$\left(\text{Here } |\vec{p}_3| = |\vec{p}_1| = \sqrt{\varepsilon^2 - m^2} = \sqrt{\frac{s}{4}-m^2} = \frac{1}{2}\sqrt{s-4m^2}\right)$$

$$= \frac{1}{32\pi}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}\,\lambda^2$$ agrees!

Alternative calculation

$$\operatorname{Im}\mathcal{M} = \operatorname{Im}\left(+i\,\frac{(-i\lambda)^2}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2-m^2+i\varepsilon}\,\frac{i}{(p-k)^2-m^2+i\varepsilon}\right)$$

Im part comes from  as we saw.

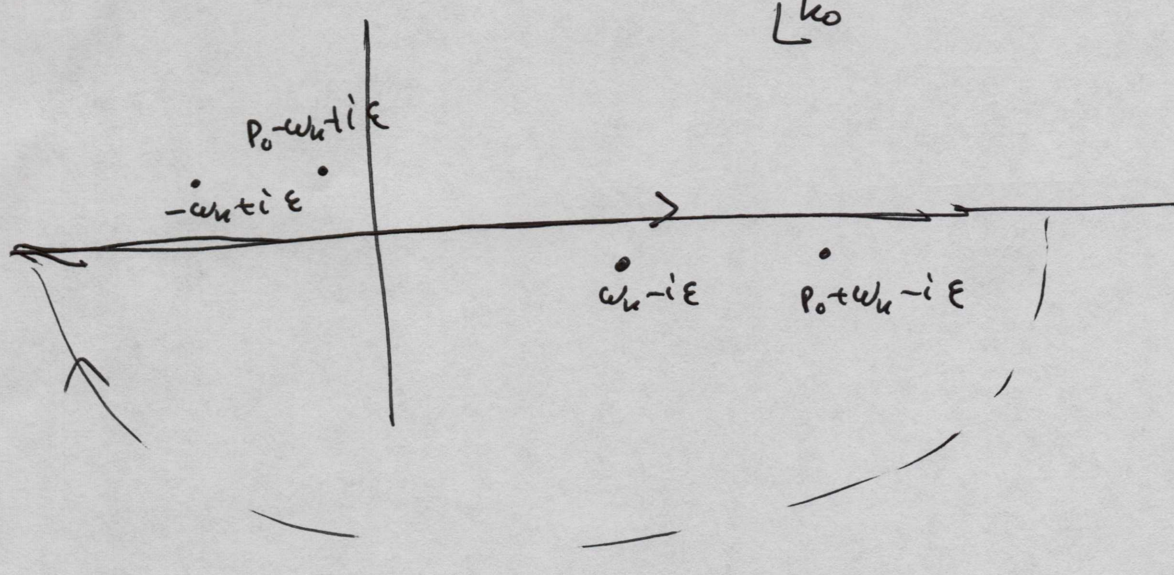
$$= -\frac{\lambda^2}{2}\operatorname{Im} i\int\frac{dk_0}{2\pi}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{k_0^2-\omega_k^2+i\varepsilon}\,\frac{1}{(p_0-k_0)^2-\omega_k^2+i\varepsilon}$$

poles $k_0^2 = \omega_k^2 - i\varepsilon$ $\quad k_0 = \pm\omega_k + i\delta$ $\quad k_0^2 = \omega_k^2 \pm 2i\delta\,\omega_k$

$\qquad = \pm\omega_k \mp i\varepsilon$ $\qquad\qquad\qquad\qquad \downarrow \mp$

$(k_0-p_0)^2 = \omega_k^2 - i\varepsilon$ $\quad k_0 = p_0 \pm \omega_k \mp i\varepsilon$

$$\frac{1}{k_0-\omega_k+i\varepsilon}\;\frac{1}{k_0+\omega_k-i\varepsilon}\;\frac{1}{k_0-p_0-\omega_k+i\varepsilon}\;\frac{1}{k_0-p_0+\omega_k\ldots}$$



$$= -\frac{\lambda^2}{2}\, y_m \underbrace{\left(-\frac{\cancel{2\pi} i}{\cancel{2\pi}}\cdot i\right)}_{1} \int \frac{d^3k}{(2\pi)^3}\left[\frac{1}{2\omega_k}\,\frac{1}{(-p_0)}\,\frac{1}{2\omega_k-p_0-i\epsilon} + \underbrace{\frac{1}{p_0}\,\frac{1}{p_0+2\omega_k}\,\frac{1}{2\omega_k}}_{\text{real.}}\right]$$

$$= \frac{\lambda^2}{2p_0}\, y_m \int \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\,\underbrace{\frac{1}{2\omega_k-p_0-i\epsilon}}_{P.P.\frac{1}{2\omega_k-p_0}+i\pi\,\delta(2\omega_k-p_0)}$$

$$= \frac{\pi\lambda^2}{2p_0}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\,\delta(2\omega_k-p_0) = \begin{cases} 0 \quad \text{if} \quad p_0<2m \\ \frac{\pi\lambda^2}{2p_0^2}\,4\pi\int\frac{k^2dk}{(2\pi)^3}\,\delta(2\omega_k-p_0)\,; \quad p_0>2m \end{cases}$$

$\omega_k^2 = k^2+m^2 \qquad \omega_k\, d\omega_k = k\, dk$

$$= \frac{4\pi}{2\times 8\pi^3}\,\frac{\lambda^2}{p_0^2}\,\frac{1}{2}\int_m^\infty \omega_k\, d\omega_k\, k = \frac{\lambda^2}{8\pi}\,\frac{k\,\omega_k}{p_0^2} = \frac{\lambda^2}{16\pi}\,\frac{\sqrt{\overset{s/4}{\epsilon^2}-m^2}}{p_0} = \frac{\lambda^2}{32\pi}\,\frac{\sqrt{s-4m^2}}{\sqrt{s}}$$

$\omega_k = p_0/2$

$p_0>2m$

$$\operatorname{Im} \mathcal{M} = \frac{\lambda^2}{\cancel{32}\pi} \frac{\sqrt{s-4m^2}}{\sqrt{s}} \quad \checkmark$$

<u>Now</u>

$$-\frac{\lambda^2}{2} \operatorname{Im} i \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{1}{k_0^2 - \omega_k^2 + i\varepsilon} \frac{1}{(p_0-k_0)^2 - \omega_k^2 + i\varepsilon}$$

$$\delta(k_0^2 - \omega_k^2) = \sum_i \frac{\delta(k_0 - k_{0i})}{|2k_{0i}|} = \frac{\delta(k_0 - \omega_k)}{2\omega_k} + \frac{\delta(k_0 + \omega_k)}{2\omega_k}$$

$$-\frac{\lambda^2}{2} \operatorname{Im} i \frac{1}{2\pi} \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{2\omega_k} \frac{1}{(p_0 - \omega_k)^2 - \omega_k^2 + i\varepsilon} + \frac{1}{2\omega_k} \frac{1}{(p_0+\omega_k)^2 - \omega_k^2 + i\varepsilon} \right.$$

$$p_0^2 - 2p_0\omega_k + i\varepsilon \qquad p_0^2 + 2p_0\omega_k$$

$$p_0(p_0 - 2\omega_k + i\varepsilon) \qquad p_0(p_0 + 2\omega_k)$$

$$\frac{1}{k_0^2 - \omega_k^2 + i\varepsilon} \to -i2\pi\delta(k_0^2 - \omega_k^2)$$

$$-\frac{\lambda^2}{2} \operatorname{Im} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} \frac{1}{(p_0 - \omega_k)^2 - \omega_k^2 + i\varepsilon}$$

$$-2\pi i\, \delta((p_0 - \omega_k)^2 - \omega_k^2) = -2\pi i\, \delta(p_0(p_0 - 2\omega_k))$$

$$= -\frac{2\pi i}{p_0} \delta(p_0 - 2\omega_k)$$

$$\frac{\lambda^2}{2}\,\frac{2\pi}{p_0}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_k}\,\delta(p_0-2\omega_k) = 2\,\mathrm{Im}\,\mathcal{M}$$

replacing $\dfrac{1}{k^2-m^2+i\varepsilon} \to -2\pi i\,\delta(k^2-m^2)$

gives the discontinuity $(=2\,\mathrm{Im}\,\mathcal{M})$

particles on-shell !

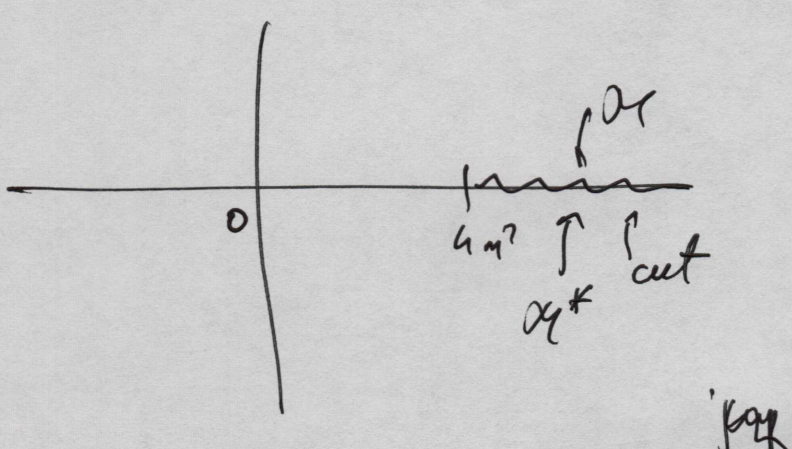
On the real axis for $s<4m^2$ $\mathcal{M}(s)$ is real

and then

$$\mathcal{M}(s) = \left(\mathcal{M}(s^*)\right)^*$$

this can be analytically continued.

It is called real analyticity.



$\mathcal{M}(s+i\varepsilon) = \mathcal{M}^*(s-i\varepsilon)$

$\mathrm{Im}\,\mathcal{M}$ has a jump

$\mathrm{Re}\,\mathcal{M}$ is continuous

Froissart bound

$2 \to 2$ scalars, elastic scattering.

identical particles.

$F(s,t,u)$

Assumptions ($m^2 = 1$ units)

For $s > 4$ (physical region) we get

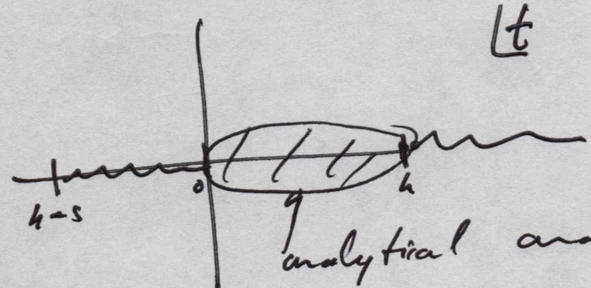
$$t = -2k^2(1-c\theta) \quad \text{scattering angle}$$

$$k^2 = \frac{s-4}{4}$$

$\Rightarrow$ $t < 0$ physical region

$$\boxed{4 - s < t < 0}$$

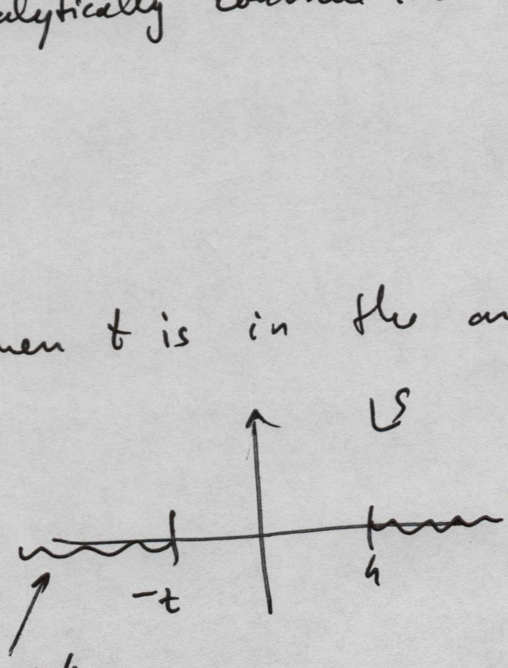
analytically continue in t



analytical and bounded $|F(s,t)| \le C s^N$, $s \to \infty$, N indep. of s.

when t is in the analyticity domain

$\Rightarrow$



more generally

cuts for $u > 4$ $s > 4$ $t > 4$

Partial waves

$$F(s,t) = \frac{1}{2}\frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\,P_l(c\theta)\,f_l(s)$$

Converges also in $|t| < 4$

$$S_l = 1 + 2i f_l \qquad |S_l|^2 = (1+2if)(1-2if^*) =$$

$$= 1 - 4\,\mathrm{Im} f + 4|f|^2 \leq 1$$

$$\Rightarrow \quad \mathrm{Im} f_l \geq |f_l|^2 \geq 0 \quad \text{(physical region).}$$

◦) Take $s > 4$ and $0 < t < 4$

$$\mathrm{Im}\,F(s,t) = \frac{1}{2}\frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\,P_l(c\theta)\,\mathrm{Im} f_l(s)$$

$$C s^N > |F(s,t)| > \mathrm{Im}\,F(s,t) = \frac{1}{2}\frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\,P_l(c\theta)\,\mathrm{Im} f_l(s)$$

Notice $1 - c\theta = -\dfrac{t}{2k^2}$ $\qquad c\theta = 1 + \dfrac{t}{2k^2} > 1 \Rightarrow P_l(c\theta) > 0$

($t > 0$ here).

Also $\mathrm{Im} f_l > 0$

$$\Rightarrow \quad C s^N > \frac{1}{2}\frac{\sqrt{s}}{k}(2l+1)\,P_l(c\theta_0)\,\mathrm{Im} f_l(s) \quad \left(\begin{array}{c}\text{one term is}\\ \text{smaller than the}\\ \text{sum}\end{array}\right)$$

$$\mathrm{Im}\, f_\ell(s) < \frac{2k}{\sqrt{s}} \, \frac{C s^N}{(2\ell+1)} \, \frac{1}{P_\ell(\cos\theta_0)}$$

we can take $\cos\theta_0 = 1 + \frac{4}{2k^2} = 1 + \frac{2}{k^2} = 1 + \frac{2}{\frac{s-4}{4}} = 1 + \frac{8}{s-4} = \frac{s+4}{s-4}$

$$\mathrm{Im}\, f_\ell(s) < \frac{\tilde{C}\, s^N}{(2\ell+1)} \, \frac{1}{P_\ell\left(\frac{s+4}{s-4}\right)} \qquad \left(\frac{k}{\sqrt{s}} \underset{s\to\infty}{\to} 1\right)$$

But then

$$|f_\ell| \le \left(\mathrm{Im}\, f_\ell\right)^{1/2} < \frac{\sqrt{\tilde{C}}\; s^{N/2}}{\sqrt{2\ell+1}} \, \frac{1}{\sqrt{P_\ell\left(\frac{s+4}{s-4}\right)}}$$

<u>Formula</u>

$$P_\ell(z) = \frac{1}{\pi}\int_0^\pi \underbrace{\left(z + \sqrt{z^2-1}\,\cos\varphi\right)}_{\text{smallest it can be is } z-\sqrt{z^2-1} \quad (\varphi=\pi)}^{\ell} d\varphi$$

consider $z>1$

$$P_\ell(z) \ge \frac{1}{\pi}\int_0^\pi d\varphi \left(z-\sqrt{z^2-1}\right)^\ell = \left(z-\sqrt{z^2-1}\right)^\ell$$

$z = \frac{s+4}{s-4}$ $\rightarrow$ $P_\ell\left(\frac{s+4}{s-4}\right) \ge \left(\frac{\sqrt{s}-2}{\sqrt{s}+2}\right)^\ell = e^{\ell \ln\left(\frac{1-2/\sqrt{s}}{1+2/\sqrt{s}}\right)} \simeq e^{-\frac{4\ell}{\sqrt{s}}}$

Better bound. look at $\varphi \sim 0$ contribution

$$P_\ell\left(\frac{s+4}{s-4}\right) \geq c\, e^{4\ell/\sqrt{s}}$$

$$|f_\ell| \leq \frac{\sqrt{\tilde{c}}\; s^{N/2}}{\sqrt{2\ell+1}\,\sqrt{c}}\, e^{-2\ell/\sqrt{s}} = \hat{c}\,\frac{s^{N/2}}{\sqrt{2\ell+1}}\, e^{-2\ell/\sqrt{s}}$$

$s\to\infty$ bound.

ii) Now we look at the physical region. $(s>4 \quad 4-s<t<0)$

$$F(s,t) = \frac{\sqrt{s}}{2k}\sum_{\ell=0}^{\infty}(2\ell+1)\,P_\ell(\cos\theta)\,f_\ell(s)$$

$$|F(s,t)| \leq \frac{\sqrt{s}}{2k}\sum_{\ell=0}^{\infty}(2\ell+1)\,|f_\ell(s)| \qquad (\,|P_\ell(\cos\theta)|\leq 1)$$

$$= \frac{\sqrt{s}}{2k}\sum_{\ell=0}^{L}(2\ell+1)\,|f_\ell(s)| +$$

$$+\frac{\sqrt{s}}{2k}\sum_{\ell=L+1}^{\infty}(2\ell+1)\,|f_\ell(s)|$$

L arbitrary but we take it as function of s to get a good bound for $0 \le l \le L$ we use $|f_l| \le 1$.

for $L+1 \le l \le \infty$ we use previous bound.

$$|F(s,t)| \le \frac{\sqrt{s}}{2k}\sum_{l=0}^{L}(2l+1) + \frac{\sqrt{s}}{2k}\sum_{l=L+1}^{\infty}\frac{(2l+1)}{\sqrt{2l+1}}\,\hat{c}\, s^{N/2} e^{-2l/\sqrt{s}}$$

$$\sum_{l=L+1}^{\infty} \sqrt{2l+1}\, e^{-2l/\sqrt{s}} \rightsquigarrow \int_{L+1}^{\infty} dl\, \sqrt{2l+1}\, e^{-2l/\sqrt{s}} \sim$$

$$\sim \sqrt{2}\int_{L+1}^{\infty} dl\, \sqrt{l}\, e^{-2l/\sqrt{s}}$$

$$y = \frac{2l}{\sqrt{s}} \qquad \sim \frac{\sqrt{2}}{2\sqrt{2}}\int_{2L/\sqrt{s}}^{\infty} s^{3/4}\, dy\, \sqrt{y}\, e^{-y}$$

$$\sim \frac{1}{2} s^{3/4}\, e^{-2L/\sqrt{s}}\, \frac{\sqrt{2L}}{s^{1/4}}$$

$$\sim \frac{1}{\sqrt{2}} s^{1/2}\sqrt{L}\, e^{-2L/\sqrt{s}}$$

$$|F(s,t)| \le \underset{\sim 1}{\frac{\sqrt{s}}{2k}}\ \underbrace{\cancel{2}L^2}_{\text{estimate}} + \frac{\sqrt{s}}{2k}\,\hat{c}\,\frac{s^{N/2}}{\sqrt{2}}\, s^{1/2}\sqrt{L}\, e^{-2L/\sqrt{s}}$$

We want L as large as needed to drop the second term but, not more since we want L^2 as a bound.

$L = \tilde{L} . \sqrt{s}$

$$s^{n/2 + 1/2 + 1/4} \sqrt{\tilde{L}}\, e^{-2\tilde{L}} \Bigg\} \Rightarrow \frac{s^{n/2 + 3/4} \sqrt{\ln s}}{s^n}$$

take $\tilde{L} = n \ln s$ $\qquad \to 0$

for n large. (fixed)

$$|F(s,t)| \leq n\underbrace{s(\ln s)^2}_{\tilde{L}^2}$$

Froissart bound.

Using optical theorem $\underline{t=0}$

$$\sigma_T \sim \frac{\mathrm{Im} F}{s}$$

$$\boxed{\sigma_T \leq C(\ln s)^2}$$