partial waves and unitarity

For fixed total E and $\vec{P}$ the space of states is compact and can be analyzed in partial waves $\to$ discrete

Example: 2 particles : fixed $\vec{E}$ and $\vec{P}=0$

$$(\varepsilon_1, \vec{p}_1) \oplus (\varepsilon_2, \vec{p}_2) \qquad \vec{p}_2 = -\vec{p}_1$$

$$\varepsilon_1^2 = \vec{p}_1^{\,2} + m_1^2$$

$$\varepsilon_2^2 = \vec{p}_2^{\,2} + m_2^2 = \vec{p}_1^{\,2} + m_2^2$$

$$\varepsilon_T = \varepsilon_1 + \varepsilon_2 = \sqrt{\vec{p}_1^{\,2} + m_1^2} + \sqrt{\vec{p}_1^{\,2} + m_2^2} \Rightarrow |\vec{p}_1| \text{ fixed}$$

then for fixed ε_T we only have (θ, φ) angles of $\vec{p}_1$.

We can use spherical harmonics. $Y_{\ell m}(\theta,\varphi)$

For a generic state we can write:

$$|E, \vec{P}, \nu\rangle$$

E: total energy; $\vec{P}$: total momentum (CM); ν: discrete labels

Normalize

$$\langle E', \vec{P}', \nu' | E, \vec{P}, \nu\rangle = \delta(E'-E)\,\delta^{(3)}(\vec{P}'-\vec{P})\,\delta_{\nu\nu'}$$

A scattering event preserves $E, \vec{P}$ then

$$\langle E', \vec{P}', \nu' | S | E, \vec{P}, \nu\rangle = \delta(E'-E)\,\delta(\vec{P}'-\vec{P})\; S_{\nu'\nu}(E,P)$$

S: S-matrix; $S_{\nu'\nu}(E,P)$: unitary matrix

It is convenient to define the M matrix.

$$S_{fi} = \mathbb{1}_{fi} - 2\pi i\, \delta^{(4)}(P_f - P_i)\, M_{fi}$$

↑ Weinberg's book

Peskin & Schroeder book

$$\left(1 + i(2\pi)^4 \delta^{(4)}(p_f - p_i)\,\mathcal{M}\right)$$

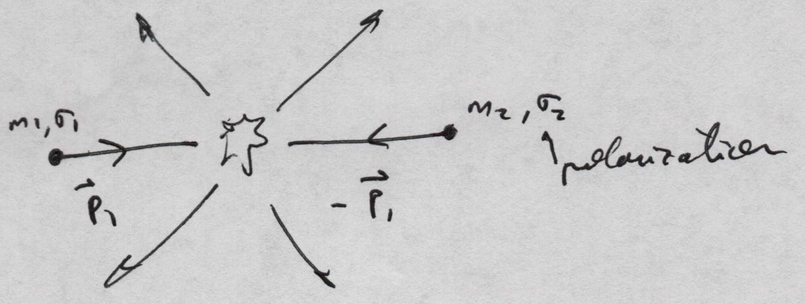
then

$$\langle E', \vec{p}\,', \nu' | S | E, \vec{p}, \nu \rangle = \delta^{(4)}(p' - p)\, \delta_{\nu'\nu} - 2\pi i\, \delta^{(4)}(P_f - P_i)\, M_{\nu'\nu}$$

$$\Rightarrow \quad S_{\nu'\nu} = \delta_{\nu'\nu} - 2\pi i\, M_{\nu'\nu} \qquad \text{(matrix equation)}$$

↑ also called $T_{\nu'\nu}$

Consider two particle states in the center of mass frame.



$|s_1 \sigma_1, s_2 \sigma_2\rangle \to |s_1, s_2, \underbrace{s_T}_{\text{total spin}}, \sigma\rangle$

$|s_1, s_2, s_T, \sigma, l m\rangle \to |s_1, s_2, s_T, l, j, m_j\rangle$ j_z.

↑ total ang. momentum

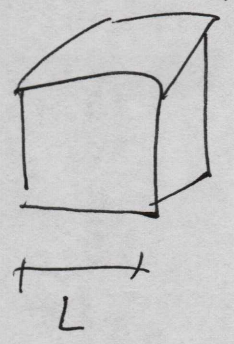
} use Clebsch-Gordan coefficients.

<u>Notice</u>

$\langle \vec{p}\,' | \vec{p} \rangle = \delta^{(3)}(\vec{p}\,' - \vec{p})$ here vs. $\langle \vec{p}\,' | \vec{p} \rangle = 2E_p (2\pi)^3\, \delta(\vec{p}\,' - \vec{p})$ book.

Rates and cross sections

To compute probabilities we discretize momentum to get all labels discrete. In order to do that we put the system in a box w/ periodic b.c.



$$V = L^3$$

$$\vec{p} = \frac{2\pi}{L}(n_1, n_2, n_3) \qquad n_j \in \mathbb{Z}$$

$$\delta^{(3)}(p-p') = \frac{1}{(2\pi)^3}\int_V d^3x\, e^{i(\vec{p}-\vec{p}\,')\cdot\vec{x}} = \frac{V}{(2\pi)^3}\underbrace{\delta_{\vec{p},\vec{p}\,'}}_{}$$

discrete δ (Kronecker)

An initial state will be a free particles state.

$$|\psi^{(\alpha)}_{in}\rangle = |\varepsilon_1, p_1, \sigma_1 ; --- \varepsilon_{N_\alpha}, p_{N_\alpha}, \sigma_{N_\alpha}\rangle$$

$$\langle\psi^{(\alpha')}_{in}|\psi^{(\alpha)}_{in}\rangle = \prod_{j=1}^{N_\alpha}\delta^{(3)}(p_j - p'_j)\quad \delta_{\sigma_1\sigma'_1} -- \delta_{\sigma_{N_\alpha}\sigma'_{N_\alpha}}$$

$$= \left(\frac{V}{(2\pi)^3}\right)^{N_\alpha}\prod_j \delta_{\vec{p}_j\,',\vec{p}_j}\quad \delta_{\sigma_1\sigma'_1} -- \delta_{\sigma_{N_\alpha}\sigma'_{N_\alpha}}$$

we normalize the states.

$$|\alpha_{in}\rangle = \left(\frac{(2\pi)^3}{V}\right)^{\frac{N_\alpha}{2}}|\psi^{(\alpha)}_{in}\rangle$$

Consider a final state $|\beta\rangle \neq |\alpha\rangle$ then we get

$$S^{Box}_{\alpha\to\beta} = \left(\frac{(2\pi)^3}{V}\right)^{\frac{N_\alpha + N_\beta}{2}} S_{\alpha\to\beta} \quad ; \quad P(\alpha\to\beta) = |S^{Box}_{\alpha\to\beta}|^2 = |S^{box}_{\beta\alpha}|^2$$

$$S_{\alpha\to\beta} = S_{\beta\alpha}$$

To compute a transition probability we turn the interaction on for a time T.

$$\delta(E_\alpha - E_\beta) = \frac{1}{2\pi}\int_{-T/2}^{T/2} e^{i(E_\alpha - E_\beta)t}\, dt \quad \Rightarrow \quad \delta(0) = \frac{T}{2\pi}$$

In a small range of final momenta dp_j the probability is constant and the number of states is $d^3 n_j = \frac{V}{(2\pi)^3} d^3 p_j$

$$\prod_{j=1}^{N_\beta} d^3 n_j = \left(\frac{V}{(2\pi)^3}\right)^{N_\beta} \prod_{j=1}^{N_\beta} d^3 p_j$$

$$dP(\alpha\to\beta) = P(\alpha\to\beta)\prod_{j=1}^{N_\beta} d^3 n_j = \left(\frac{(2\pi)^3}{V}\right)^{\frac{N_\alpha+N_\beta}{2}} \left(\frac{V}{(2\pi)^3}\right)^{N_\beta} \cdot |S_{\beta\alpha}|^2 \cdot \prod_{j=1}^{N_\beta} d^3 p_j$$

$$dP(\alpha\to\beta) = \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha} |S_{\beta\alpha}|^2 \prod_{j=1}^{N_\beta} d^3 p_j$$

probability of final state to be in the interval determined by

Since $\beta \neq \alpha$

$$S_{\beta\alpha} = -2\pi i \, \delta^{(4)}(\underset{\beta}{P_f} - \underset{\alpha}{P_i}) \, \underset{\beta\alpha}{M_{fi}}$$

$$|S_{\beta\alpha}|^2 = 4\pi^2 \, \delta(\varepsilon_\beta - \varepsilon_\alpha) \underbrace{\delta_T(0)}_{\frac{T}{2\pi}} \, \delta(\vec{P}_\beta - \vec{P}_\alpha) \underbrace{\delta_V(0)}_{\frac{V}{(2\pi)^3}} |\mathcal{M}_{fi}|^2$$

$$dP(\alpha\to\beta) = 4\pi^2 \, \delta(\varepsilon_\beta - \varepsilon_\alpha) \, \delta(\vec{P}_\beta - \vec{P}_\alpha) \, \frac{T}{2\pi} \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha - 1} \cancel{\ \ } |\mathcal{M}_{\beta\alpha}|^2 \cdot \prod_{j=1}^{N_\beta} d^3 p_j$$

Transition rate:

$$\frac{dP(\alpha\to\beta)}{T} = d\Gamma(\alpha\to\beta) = 2\pi \, \delta^{(4)}(P_\beta - P_\alpha) \left(\frac{(2\pi)^3}{V}\right)^{N_\alpha - 1} |\mathcal{M}_{\beta\alpha}|^2 \times \prod_{j=1}^{N_\beta} d^3 p_j$$

•) Decay rates $\underline{N_\alpha = 1}$

$$d\Gamma(\alpha\to\beta) = 2\pi \, |\mathcal{M}_{\beta\alpha}|^2 \, \delta^{(4)}(P_\beta - P_\alpha) \prod_{j=1}^{N_\beta} d^3 p_j$$

••) Cross section $\underline{N_\alpha = 2}$

$$d\Gamma(\alpha\to\beta) = \frac{(2\pi)^4}{V} \, \delta^{(4)}(P_\beta - P_\alpha) \, |\mathcal{M}_{\beta\alpha}|^2 \prod_{j=1}^{N_\beta} d^3 p_j$$

density $1/V$ $\quad \underbrace{\frac{u_\alpha}{V}}_{\text{flux}}$ ← relative velocity

cross section.

$$d\sigma_{\alpha\to\beta} = \frac{d\Gamma(\alpha\to\beta)}{\text{flux}} = \frac{(2\pi)^4}{u_\alpha} |\mathcal{M}_{\beta\alpha}|^2 \delta^{(4)}(p_\beta - p_\alpha) \prod_{j=1}^{N_\beta} d^3 p_j$$

In the center of mass: $\quad (\varepsilon_1, \vec{p}_1) \quad (\varepsilon_2, -\vec{p}_1)$

$$u_\alpha = \left| \frac{\vec{p}_1}{\varepsilon_1} - \frac{\vec{p}_2}{\varepsilon_2} \right| \qquad \vec{p}_1 = \frac{m v_1}{\sqrt{1-v_1^2}} \qquad \varepsilon = \frac{m}{\sqrt{1-v_1^2}}$$

$$\vec{p}_1/\varepsilon_1 = \vec{v}_1 \quad \checkmark$$

<u>Case of $2\to 2$ scattering</u> (CM frame)

$$(\varepsilon_1, \vec{p}_1) \oplus (\varepsilon_2, -\vec{p}_1) \to (\varepsilon_3, \vec{p}_3) \oplus (\varepsilon_4, -\vec{p}_3) \quad ; \quad \varepsilon_j^2 + \vec{p}_j^{\,2} = m_j^2$$

$$d\sigma_{\alpha\to\beta} = \frac{(2\pi)^4}{u_\alpha} |\mathcal{M}_{\beta\alpha}|^2 \delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\, \delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\, d^3p_3\, d^3p_4$$

we integrate over $\vec{p}_4$ using $\delta^{(3)}$. then ε_4 is fixed $\varepsilon_4 = \sqrt{\underset{p_4^2}{p_3^2} + m_4^2}$

" " " $|\vec{p}_3|$ using δ. $(d^3p_3 \to p_3^2 dp_3\, d\Omega)$

Also $u_\alpha = \left| \frac{\vec{p}_1}{\varepsilon_1} - \frac{\vec{p}_2}{\varepsilon_2} \right| = |\vec{p}_1| \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} \right) = p_1 \frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2} = p_1 \frac{E}{\varepsilon_1\varepsilon_2}$ ← total energy

then

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{(2\pi)^4}{|\vec{p}_1|} \frac{\varepsilon_1\varepsilon_2}{E} |\mathcal{M}_{\beta\alpha}|^2 |\vec{p}_3| \varepsilon_3 \frac{\varepsilon_4}{E} = (2\pi)^4 \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}{E^2} |\mathcal{M}_{\beta\alpha}|^2$$

we used $\int_{\varepsilon_3} \delta(\varepsilon_3+\varepsilon_4-\varepsilon_1-\varepsilon_2) = \frac{1}{1+\partial\varepsilon_4/\partial\varepsilon_3} = \frac{1}{1+\varepsilon_3/\varepsilon_4} = \varepsilon_4/E$

One can define $f = -4\pi^2 \sqrt{\frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4}{E^2}}\, \mathcal{M}_{\beta\alpha}$

Such that

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = |f|^2$$

Normalization of partial waves states.

In momentum

$$\langle \vec{p}_1{}' \sigma_1' \ \vec{p}_2{}' \sigma_2' \ n' \,|\, \vec{p}_1 \sigma_1 \ \vec{p}_2 \sigma_2 \ n \rangle = \delta^{(3)}(\vec{p}_1{}' - \vec{p}_1)\, \delta^{(3)}(\vec{p}_2{}' - \vec{p}_2)\, \delta_{\sigma_1\sigma_1'} \delta_{\sigma_2\sigma_2'} \delta_{nn'}$$

In ang. momentum: (CM)

$$\langle E' \ \vec{p}\,' \ j' \sigma' \ell' s' n' \,|\, E \ 0 \ j \sigma \ \ell s n \rangle = \delta^{(3)}(\vec{p}\,')\, \delta(E'-E)\, \delta_{jj'}\, \delta_{\sigma\sigma'} \delta_{\ell\ell'} \delta_{s's} \delta_{nn'}$$

We take:

$$\langle \vec{p}_1 \sigma_1 \ \vec{p}_2 \sigma_2 \ n' \,|\, E \ \vec{p} \ j \sigma \ \ell s n \rangle = \sqrt{\frac{E}{|\vec{p}_1|\, \varepsilon_1 \varepsilon_2}}\ \delta^{(3)}(\vec{p} - \vec{p}_1 - \vec{p}_2)\, \delta_{nn'}\,.$$

$$\cdot\ \delta(E - \varepsilon_1 - \varepsilon_2) \sum_{m\mu} \underbrace{\langle s_1 \sigma_1 \ s_2 \sigma_2 \,|\, s \mu \rangle}_{\text{compose spin to total spin}} \ \underbrace{\langle \ell m \ s \mu \,|\, j \sigma \rangle}_{\text{ang. mom. + spin} \equiv \text{total ang. momentum } j} \ \underbrace{Y_{\ell m}(\hat{p}_1)}_{\text{ang. momentum wave-function.}}$$

We check:

$$\langle E'\vec{p}\,'j'\sigma'l's'n' \mid E\,0\,j\sigma\, lsn\rangle =$$

$$= \int d^3p_1\, d^3p_2 \sum_{\substack{\sigma_1\sigma_2\\ n''}} \langle E'\vec{p}\,'j'\sigma'l's'n' \mid \vec{p}_1\sigma_1\vec{p}_2\sigma_2 n''\rangle\langle \vec{p}_1\sigma_1\vec{p}_2\sigma_2 n'' \mid E\,0\,j\sigma\, lsn\rangle$$

$$= \int d^3p_1\, d^3p_2 \sum_{\substack{\sigma_1\sigma_2\\ n''}} \sqrt{\frac{E'}{|\vec{p}_1|\varepsilon_1\varepsilon_2}}\sqrt{\frac{E}{|\vec{p}_1|\varepsilon_1\varepsilon_2}}\ \delta^{(3)}(\vec{p}\,'-\vec{p}_1-\vec{p}_2)\,\delta_{n'n''}\,\delta^{(3)}(\vec{p}_1+\vec{p}_2)\,\delta_{n''n}$$

$$\delta(E'-\varepsilon_1-\varepsilon_2)\,\delta(E-\varepsilon_1-\varepsilon_2)\sum_{m'\mu'}\langle s'\mu'\mid s_1\sigma_1 s_2\sigma_2\rangle\langle j'\sigma'\mid l'm's'\mu'\rangle\, Y^*_{l'm'}(\hat{p}_1)$$

$$\sum_{m\mu}\langle s_1\sigma_1 s_2\sigma_2\mid s\mu\rangle\langle lms\mu\mid j\sigma\rangle\, Y_{lm}(\hat{p}_1)$$

$$\sum_{\sigma_1\sigma_2}\langle s'\mu'\mid s_1\sigma_1 s_2\sigma_2\rangle\langle s_1\sigma_1 s_2\sigma_2\mid s\mu\rangle = \delta_{ss'}\,\delta_{\mu\mu'}$$

$\int d^3p_2 \to$ done using $\delta^{(3)}(\vec{p}_1+\vec{p}_2)$

$$\int d^3p_1 = \int p_1^2\,dp_1\int d\Omega_1 = \int p_1\varepsilon_1\,d\varepsilon_1\int d\Omega_1$$

$$\int d\Omega_1\, Y^*_{l'm'}(\hat{p}_1)\, Y_{lm}(\hat{p}_1) = \delta_{ll'}\,\delta_{mm'}$$

$$= \int p_1\varepsilon_1\,d\varepsilon_1\ \delta_{nn'}\ \frac{E}{\sqrt{|\vec{p}_1|^2\varepsilon_1^2\varepsilon_2^2}}\ \delta^{(3)}(\vec{p}\,'-\vec{p}_1-\vec{p}_2)\,\delta(E'-E)\,\delta(E-\varepsilon_1-\varepsilon_2)\ \cdot$$

$$\sum_{m\mu}\langle j'\sigma'\mid lms\mu\rangle\langle lms\mu\mid j\sigma\rangle \to \delta_{jj'}\,\delta_{\sigma\sigma'}$$

$$= \delta_{nn'}\, \delta(E'-E)\, \delta^{(3)}(\vec{p}\,')\, \delta_{jj'}\, \delta_{\sigma\sigma'}\, E \int_0^\infty \frac{\cancel{p_1}\, \cancel{\epsilon_1}\, d\epsilon_1}{\cancel{p_1}\, \cancel{\epsilon_1}\, \epsilon_2}\, \delta(E-\epsilon_1-\epsilon_2)$$

$$\left(\int_0^\infty \frac{d\epsilon_1}{\epsilon_2}\, \delta(E-\epsilon_1-\epsilon_2(\epsilon_1)) = \frac{1}{\epsilon_2}\, \frac{1}{1+\partial\epsilon_2/\partial\epsilon_1} = \frac{1}{\epsilon_2}\, \frac{1}{1+\epsilon_1/\epsilon_2} = \frac{1}{E} \right.$$

$$\epsilon_2 = \sqrt{p_2^2+m_2^2} = \sqrt{p_1^2+m_2^2} \qquad \frac{\partial\epsilon_2}{\partial\epsilon_1} = \frac{\cancel{2}\, p_1\, \partial p_1/\partial\epsilon_1}{\cancel{2}\sqrt{p_1^2+m_2^2}} = \frac{\cancel{p_1}}{\epsilon_2}\, \frac{\epsilon_1}{\cancel{p_1}}$$

$$\left. \partial p_1/\partial\epsilon_1 = \epsilon_1/p_1 \right)$$

$$= \delta_{nn'}\, \delta(E'-E)\, \delta^{(3)}(\vec{p}\,')\, \delta_{jj'}\, \delta_{\sigma\sigma'} \quad \checkmark$$

correct normalization.

For an operator $\mathcal{O}$ that conserves (commutes) the total energy, momentum and ang. momentum:

$$\langle E' \vec{p}\,' j' \sigma' \ell' s' n' | \mathcal{O} | E \overset{\overset{\text{CM}}{\downarrow}}{0} j \sigma \ell s n \rangle =$$

$$= \delta^{(3)}(\vec{p}\,')\, \delta_{jj'}\, \delta_{\sigma\sigma'}\, \delta(E'-E)\; \mathcal{O}_{j\ell' s' n' \ell s n}$$

For $2\to2$ scattering on the CM we have:

$$f = -4\pi^2 \sqrt{\frac{|\vec{p}_3|}{|\vec{p}_1|}\,\frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4}{E^2}}\; \mathcal{M}_{\beta\alpha}$$

$$\mathcal{M}_{\beta\alpha} \Rightarrow \quad \langle\beta|\mathcal{M}|\alpha\rangle = \delta^{(4)}(P_\beta - P_\alpha)\, \mathcal{M}_{\beta\alpha}$$

$$|\alpha\rangle = |\vec{p}_1\sigma_1\vec{p}_2\sigma_2\rangle \qquad |\beta\rangle = |\vec{p}_3\sigma_3\vec{p}_4\sigma_4\rangle$$

$$\langle\beta|\mathcal{M}|\alpha\rangle = \int dE'\, d^3\vec{p}\,' \sum_{j'\sigma'\ell' n' s'\mu'} \int dE\, d^3p \sum_{j\sigma\ell m s\mu} \cdot$$

$$\cdot \langle \vec{p}_3\sigma_3\vec{p}_4\sigma_4 | E'\vec{p}\,' j'\sigma'\ell' s' n'\rangle \langle E'\vec{p}\,' j'\sigma'\ell' s' n' | \mathcal{M} | E\vec{p}\, j\sigma\ell s n\rangle$$

$$\langle E\vec{p}\, j\sigma\ell s n | \vec{p}_1\sigma_1\vec{p}_2\sigma_2\rangle$$

(11)

$$= \int dE\, dE'\, d^3\vec{p}\, d^3\vec{p}\,' \sum_{\substack{j\sigma l s m\mu \\ j'\sigma' l' s' m'\mu'}} \sqrt{\frac{E \cdot E}{|\vec{p}_3|\, \varepsilon_3 \varepsilon_4\, |\vec{p}_1|\, \varepsilon_1 \varepsilon_2}}\; \delta^{(3)}(\vec{p}\,' - \vec{p}_3 - \vec{p}_4)\, \delta(E' - \varepsilon_3 - \varepsilon_4) \,\cdot$$

$$\cdot\; \delta^{(3)}(\vec{p} - \vec{p}_1 - \vec{p}_2)\, \delta(E - \varepsilon_1 - \varepsilon_2) \;\cdot\; \langle s_3 \sigma_3 s_4 \sigma_4 | s' \mu' \rangle \langle l' m' s' \mu' | j' \sigma' \rangle\, Y_{lm}(\hat{p}_3)$$

$$\langle s\mu | s_1 \sigma_1 s_2 \sigma_2 \rangle \langle j\sigma | l m s \mu \rangle\, Y^*_{lm}(\hat{p}_1)\;\; \delta^{(3)}(\vec{p} - \vec{p}\,') \cdot \delta_{jj'}\, \delta_{\sigma\sigma'} \,\cdot$$

$$\cdot\, \delta(E - E')\;\; \mathcal{M}_{j\, l's'n'\, lsn}(E)$$

$$= \sum_{\substack{j\sigma l s m\mu \\ l' s' m'\mu'}}{}' \sqrt{\frac{E^2}{|\vec{p}_1||\vec{p}_3|\, \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4}}\;\; \underbrace{\delta^{(3)}(\vec{p}_3 + \vec{p}_4 - \vec{p}_1 - \vec{p}_2)\, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)}_{\delta^{(4)}(P_\beta - P_\alpha)} \;\times$$

$$\times\; \langle s_3 \sigma_3 s_4 \sigma_4 | s' \mu' \rangle \langle l' m' s' \mu' | j \sigma \rangle\, Y_{lm}(\hat{p}_3) \;\times$$

$$\times\; \langle s\mu | s_1 \sigma_1 s_2 \sigma_2 \rangle \langle j\sigma | l m s \mu \rangle\, Y^*_{lm}(\hat{p}_1) \cdot \mathcal{M}_{j\, l's'n'\, lsn}\, .$$

$$\boxed{\mathcal{M}_{\beta\alpha} = \frac{E}{\sqrt{|\vec{p}_1||\vec{p}_3|\, \varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4}} \sum_{\substack{j\sigma l s m\mu \\ l' s' m'\mu'}}{}' \langle s_3 \sigma_3 s_4 \sigma_4 | s' \mu' \rangle \langle l' m' s' \mu' | j \sigma \rangle\, Y_{lm}(\hat{p}_3) \cdot \langle s\mu | s_1 \sigma_1 s_2 \sigma_2 \rangle \langle j\sigma | l m s \mu \rangle\, Y^*_{lm}(\hat{p}_1) \cdot \mathcal{M}_{j\, l's'n'\, lsn}}$$

relation between matrix elements in $\vec{p}$ and j basis.

then:

$$f = -\frac{4\pi^2}{|\vec{p}_1|} \sum_{\substack{j\sigma l s m \mu \\ l' s' m' \mu'}} \langle s_3\sigma_3 s_4\sigma_4 | s'\mu'\rangle \langle l'm's'\mu' | j\sigma\rangle Y_{l'm'}(\hat{p}_3) \cdot$$
$$\cdot \langle s\mu | s_1\sigma_1 s_2\sigma_2\rangle \langle j\sigma | l m s \mu\rangle Y^*_{lm}(\hat{p}_1)\, \mathcal{M}_{j l' s' n' l s n}$$

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = |f|^2$$

to compute total unpolarized amplitude we integrate over final momenta, average over initial polarizations, sum over final polarizations

$$|f|^2 = \frac{(2\pi)^4}{|\vec{p}_1|^2} \sum_{\substack{j\sigma l s m \mu \\ l' s' m' \mu' \\ \tilde{j}\tilde{\sigma}\tilde{l}\tilde{s}\tilde{m}\tilde{\mu} \\ \tilde{l}'\tilde{s}'\tilde{m}'\tilde{\mu}'}} \langle s_3\sigma_3 s_4\sigma_4 | s'\mu'\rangle \langle \tilde{s}'\tilde{\mu}' | s_3\sigma_3 s_4\sigma_4\rangle Y_{l'm'}(\hat{p}_3)\, Y^*_{\tilde{l}'\tilde{m}'}(\hat{p}_3)$$
$$\langle l'm's'\mu' | j\sigma\rangle \langle \tilde{j}\tilde{\sigma} | \tilde{l}'\tilde{m}'\tilde{s}'\tilde{\mu}'\rangle$$
$$\langle s\mu | s_1\sigma_1 s_2\sigma_2\rangle \langle s_1\sigma_1 s_2\sigma_2 | \tilde{s}\tilde{\mu}\rangle \; Y^*_{lm}(\hat{p}_1)\, Y_{\tilde{l}\tilde{m}}(\hat{p}_1)$$
$$\langle j\sigma | l m s\mu\rangle \langle \tilde{l}\tilde{m}\tilde{s}\tilde{\mu} | \tilde{j}\tilde{\sigma}\rangle\, \mathcal{M}_{j l' s' n' l s n}\, \mathcal{M}^*_{\tilde{j}\tilde{l}'\tilde{s}'\tilde{n}'\tilde{l}\tilde{s}\tilde{n}}$$

$$\underbrace{\frac{1}{2s_1+1}\sum_{\sigma_1} \frac{1}{2s_2+1}\sum_{\sigma_2}}_{\text{average over initial pol.}} \to \frac{1}{(2s_1+1)(2s_2+1)}\, \delta_{s\tilde{s}}\, \delta_{\mu\tilde{\mu}}$$

$$\underbrace{\sum_{\sigma_3}\sum_{\sigma_4}}_{\text{sum over final pol}} \to \delta_{s'\tilde{s}'}\, \delta_{\mu'\tilde{\mu}'}$$

$$\underbrace{\int d^3\hat{p}_3 = \int d\Omega_3}_{\text{integral over angle}} \to \delta_{l'\tilde{l}'}\, \delta_{m'\tilde{m}'}$$

$$\frac{1}{(2s_1+1)}\frac{1}{(2s_2+1)}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}\int d\Omega_3\ |f|^2 =$$

$$\sum_{m'\mu'} \to \delta_{j\tilde{j}}\,\delta_{\sigma\tilde{\sigma}}$$

$$= \frac{1}{(2s_1+1)}\frac{1}{(2s_2+1)}\frac{(2\pi)^4}{|\vec{p}_1|^2}\sum_{\substack{j\sigma\tilde{j}\tilde{\sigma}\\ lsm\mu\\ l's'm'\mu'\\ \tilde{l}\tilde{m}}} \langle \tilde{j}\tilde{\sigma}\,|\,l'm's'\mu'\rangle\langle l'm's'\mu'\,|\,j\sigma\rangle \langle j\sigma\,|\,lms\mu\rangle\langle \tilde{l}\tilde{m}s\mu\,|\,\tilde{j}\tilde{\sigma}\rangle \cdot Y^*_{lm}(\hat{p}_1)\,Y_{\tilde{l}\tilde{m}}(\hat{p}_1)\cdot \mathcal{M}_{jl's'n'\,lsn}\,\mathcal{M}^*_{\tilde{j}l's'n'\tilde{l}sn}$$

$$\delta_{m\tilde{m}}$$

$$= \frac{1}{(2s_1+1)}\frac{1}{(2s_2+1)}\frac{(2\pi)^4}{|\vec{p}_1|^2}\sum_{\substack{j\sigma\\ lsm\mu\\ l's'\\ \tilde{l}\tilde{m}}} \langle \tilde{l}\tilde{m}s\mu\,|\,j\sigma\rangle\langle j\sigma\,|\,lms\mu\rangle \cdot Y^*_{lm}(\hat{p}_1)\,Y_{\tilde{l}\tilde{m}}(\hat{p}_1)\cdot \mathcal{M}_{jl's'n'\,lsn}\,\mathcal{M}^\dagger_{jl's'n'\tilde{l}sn}$$

(Take $\hat{p}_1 \parallel \hat{z} \Rightarrow Y_{lm}(\hat{z}) = \delta_{m0}\sqrt{\frac{2l+1}{4\pi}}$

Formula:

$$\sum_{\sigma\mu}{}' \langle \tilde{l}\,\underset{\tilde{m}}{0}\,s\mu\,|\,j\sigma\rangle\langle j\sigma\,|\,l\,\underset{m}{0}\,s\mu\rangle = \frac{2j+1}{2l+1}\,\delta_{l\tilde{l}}$$
)

$$= \frac{1}{(2s_1+1)} \frac{1}{(2s_2+1)} \frac{(2\pi)^4}{|\vec{p}_1|^2} \sum_{\substack{j \\ lsn \\ l's'n'}} \frac{\cancel{2l+1}}{4\pi} \frac{2j+1}{\cancel{2l+1}} \left| \mathcal{M}_{j l's'n' lsn} \right|^2$$

$$= \frac{4\pi^3}{(2s_1+1)(2s_2+1)} \frac{1}{|\vec{p}_1|^2} \sum_{j l s l' s'} (2j+1) \left| \mathcal{M}_{j l's'n' lsn} \right|^2$$

$$S_{\substack{jlsn \\ l's'n'}} = \delta_{ll'} \delta_{ss'} \delta_{nn'} - 2\pi i \mathcal{M}^{j}_{\substack{lsn \\ l's'n'}}(E) \qquad (\text{Definition of } \mathcal{M}).$$

$$\Rightarrow |\mathcal{M}_j|^2 = \frac{1}{4\pi^2} \left| \delta_{ll'} \delta_{ss'} \delta_{nn'} - S^{j}_{\substack{lsn \\ l's'n'}} \right|^2$$

$$\sigma(n \to n', E) = \frac{1}{(2s_1+1)} \frac{1}{(2s_2+1)} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \int d\Omega_3 \frac{d\sigma_{\alpha \to \beta}}{d\Omega}$$

$$\boxed{\sigma_{\text{total}}(n \to n', E) = \frac{\pi}{(2s_1+1)(2s_2+1)} \frac{1}{|\vec{p}_1|^2} \sum_{\substack{jls \\ l's'}}' (2j+1) \left| \delta_{ll'} \delta_{ss'} \delta_{nn'} - S^{j}_{\substack{lsn \\ l's'n'}} \right|^2}$$

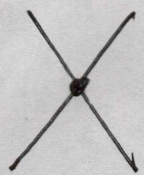
$2 \to 2$ scattering $n \to n'$ channel

As a matrix S^j is unitary if only 2 particle channels are opened.

$$\langle l's'n' | S^j | lsn \rangle = S^{j}_{\substack{lsn \\ l's'n'}}$$

Example

Cross section for ϕ^4 theory



$-i\lambda$ $\leftarrow$ as in the book

However change in normalization

In the book they normalize

$$\langle p' | p \rangle = 2E_p \, (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{p}\,')$$

Here we used

$$\langle p' | p \rangle = \delta^{(3)}(\vec{p} - \vec{p}\,')$$

$$\Rightarrow \quad |p\rangle_{book} = \sqrt{2E_p}\, (2\pi)^{3/2} \, |p\rangle_{here}.$$

Also the book uses

$$S_{\beta\alpha} = \delta_{\beta\alpha} + i \mathcal{M}_{book} (2\pi)^4 \delta^{(4)}(P_f - P_i)$$

Here

$$S_{\beta\alpha} = \delta_{\beta\alpha} - 2\pi i \, \mathcal{M}_{here} \, \delta^{(4)}(P_f - P_i)$$

$$\left.\right\} \quad \mathcal{M}_{book} = \frac{\mathcal{M}_{here}}{(2\pi)^3}$$

then
$$\langle p_3 p_4 | \mathcal{M}_{here} | p_1 p_2 \rangle_{here} = (2\pi)^3 \frac{1}{\sqrt{2E_1 2E_2 2E_3 2E_4}\,(2\pi)^6} \overbrace{\langle p_3 p_4 | \mathcal{M} | p_1 p_2 \rangle_{book}}^{-i\lambda}$$

$$= -\frac{i\lambda}{(2\pi)^3} \frac{1}{\sqrt{2E_1 2E_2 2E_3 2E_4}}$$

then

$$\frac{d\sigma_{\alpha\gamma\beta}}{d\Omega} = (\cancel{2}\pi)^4 \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3\mathcal{E}_4}{E^2} \frac{\lambda^2}{(2\pi)^6 \, \cancel{2}\mathcal{E}_1\cancel{2}\mathcal{E}_2\cancel{2}\mathcal{E}_3\cancel{2}\mathcal{E}_4}$$

$$= \frac{1}{64\pi^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{\lambda^2}{E_{cm}^2}$$

$\leftarrow$ total CM energy

identical particles $|\vec{p}_3| = |\vec{p}_1|$

$$\frac{d\sigma_{\alpha\gamma\beta}}{d\Omega} = \frac{\lambda^2}{64\pi^2 E_{cm}^2} = \frac{\lambda^2}{64\pi^2 s}$$

$s = (p_1 + p_2)^2$

$$\sigma = \frac{1}{2}\int d\Omega \, \frac{\lambda^2}{64\pi^2 s} = \frac{2\pi\lambda^2}{64\pi^2 s} = \frac{\lambda^3}{32\pi s} \qquad \left(\underset{s\to\infty}{\sim} 1/s\right)$$

total

identical particles

s includes rest mass

$s \geq 4m^2$

Partial waves

no spin

$$\mathcal{M}_{\beta\alpha} = \frac{E}{\sqrt{|\vec{p}_1||\vec{p}_3|\,\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4}} \sum_{\substack{j\sigma\, \ell m \\ \ell' m'}} \langle \ell' m' | j\sigma\rangle\, Y_{\ell' m'}(\hat{p}_3)\, \langle j\sigma | \ell m\rangle\, Y^*_{\ell m}(\hat{p}_1)$$

$\underline{j = \ell,\ \sigma = m}$ $\qquad \mathcal{M}_{j\ell'\ell}$

$$-\frac{i\lambda}{(2\pi)^3}\frac{1}{\sqrt{2\cancel{\varepsilon_1}2\cancel{\varepsilon_2}2\cancel{\varepsilon_3}2\cancel{\varepsilon_4}}} = \frac{E}{\sqrt{p_1^2\,\cancel{\varepsilon_1}\cancel{\varepsilon_2}\cancel{\varepsilon_3}\cancel{\varepsilon_4}}} \sum_{j\sigma} Y_{j\sigma}(\hat{p}_3)\, Y^*_{j\sigma}(\hat{p}_1)\, \mathcal{M}_j$$

$$\underbrace{-\frac{i\lambda}{(2\pi)^3}\frac{1}{4}} = \frac{E}{p_1}\sum_{j\sigma} Y_{j\sigma}(\hat{p}_3)\, Y^*_{j\sigma}(\hat{p}_1)\, \mathcal{M}_j$$

Does not depend on $\hat{p}_1, \hat{p}_3 \Rightarrow j=0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$$-\frac{i\lambda}{(2\pi)^3}\frac{1}{\cancel{4}} = \frac{E}{p_1}\frac{1}{\cancel{4}\pi}\mathcal{M}_0$$

$$\mathcal{M}_0 = -\frac{i\lambda}{8\pi^3}\,\pi\,\frac{p_1}{E} = -\frac{i\lambda}{4\pi^2}\frac{p_1}{E}$$ only <u>s-wave</u>.

scattering happens <u>only</u> for the s-wave. (at lowest order)

point like interaction. Higher partial waves vanish at <u>r=0</u>

Changing normalization to the book:

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = (2\pi)^4 \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{\varepsilon_1 \varepsilon_2 \varepsilon_3 \varepsilon_4}{\underbrace{E^2}_{s}} \frac{1}{2\varepsilon_1 2\varepsilon_2 2\varepsilon_3 2\varepsilon_4} \frac{|\mathcal{M}^{book}_{\beta\alpha}|^2}{(2\pi)^6}$$

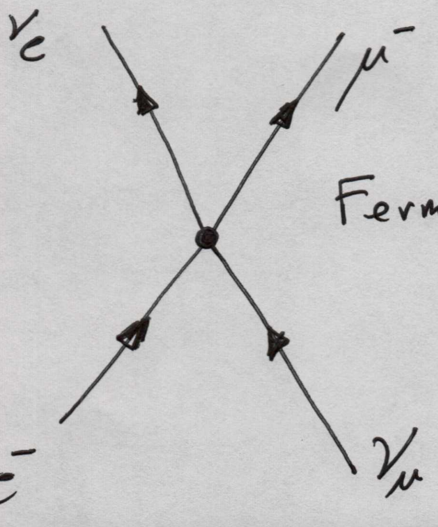
$$= \frac{1}{4\pi^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{1}{16s} |\mathcal{M}^{book}_{\beta\alpha}|^2$$

$$\boxed{\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{1}{64\pi^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{1}{s} |\mathcal{M}^{book}_{\beta\alpha}|^2 \; ; \quad 2\to 2 \text{ Scattering}}$$

---

4 fermions interactions

$$\nu_\mu + e^- \to \mu^- + \nu_e \qquad \text{(Inverse muon decay)}$$



Fermi interaction

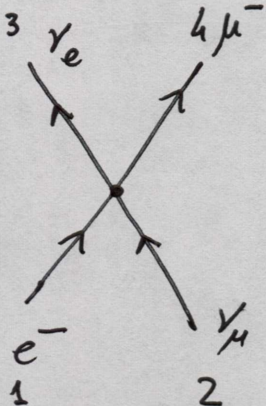
$$\mathcal{L}_{ef} = -\frac{1}{\sqrt{2}} \underset{\text{dim } M^{-2}}{G_F} \underbrace{J^+_\mu J^\mu}_{\text{dim } 6}$$

$$\underset{\text{dim}=3}{J^\alpha} = \bar{\nu}_e \gamma^\alpha (1-\gamma_5) e + \bar{\nu}_\mu \gamma^\alpha (1-\gamma_5) \mu$$

$$\mathcal{L}_{eff} = -\frac{1}{\sqrt{2}} G_F \left(\bar{\nu}_e \gamma^\alpha (1-\gamma_5) e\right)\left(\bar{\mu} (1+\gamma_5) \gamma_\alpha \nu_\mu\right)$$

Using $\quad (\bar{\nu}_\mu \gamma^\alpha (1-\gamma_5)\mu)^\dagger = \mu^\dagger (1-\gamma_5) \underbrace{\gamma_0 \gamma^\alpha \gamma_0}_{(\gamma^\alpha)^\dagger} \gamma_0 \nu_\mu$

$$= \bar{\mu} (1+\gamma_5) \gamma^\alpha \nu_\mu \qquad (\{\gamma_5, \gamma_\mu\} = 0)$$



$$-\frac{iG_F}{\sqrt{2}} \Big(\underset{\nu_e}{\bar{u}_3^{(\sigma_3)}} \gamma^\alpha (1-\gamma_5) \underset{e^-}{u_1^{(\sigma_1)}}\Big)\Big(\underset{\mu^-}{\bar{u}_4^{(\sigma_4)}} (1+\gamma_5)\gamma_\alpha \underset{\nu_\mu}{u_2^{(\sigma_2)}}\Big)$$

$(\not{p} - m) u = 0$ solution of Dirac eqn. with the appropriate mass.

$$\mathcal{M}_{\beta\alpha} = -\frac{iG_F}{\sqrt{2}} \left(\bar{u}_3 \gamma^\alpha (1-\gamma_5) u_1\right)\left(\bar{u}_4 (1+\gamma_5) \gamma_\alpha u_2\right)$$

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{1}{64\pi^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{1}{s} \left|\mathcal{M}_{\beta\alpha}\right|^2$$

$$(\bar{u}_3 \gamma^\alpha (1-\gamma_5) u_1) \underbrace{(\bar{u}_3 \gamma^\beta (1-\gamma_5) u_1)^*}$$

$$(\bar{u}_3 \gamma^\beta (1-\gamma_5) u_1)^+$$

$$\bar{u}_1 (1+\gamma_5) \gamma^\beta u_3$$

← cyclic permutation

$$\mathrm{Tr}\left[(\bar{u}_3 \gamma^\alpha (1-\gamma_5) u_1)(\bar{u}_1 (1+\gamma_5) \gamma^\beta u_3)\right]$$

$$\mathrm{Tr}\left[u_3^{(\sigma_3)} \bar{u}_3^{(\sigma_3)} \gamma^\alpha (1-\gamma_5) u_1^{(\sigma_1)} \bar{u}_1^{(\sigma_1)} (1+\gamma_5) \gamma^\beta\right]$$

$$\underbrace{\frac{1}{2} \sum_{\sigma_1 = 1,2}}_{\text{average over initial pol.}} \quad \underbrace{\sum_{\sigma_3 = 1,2}}_{\text{sum over final}} \qquad \Bigg\| \qquad \sum_{\sigma_3 = 1,2} u_3^{\sigma_3} \bar{u}_3^{\sigma_3} = \not{p}_3 + m_3 \quad \downarrow \; m_3 = 0$$

We get

$$\frac{1}{2} \mathrm{Tr}\left[\not{p}_3 \gamma^\alpha (1-\gamma_5)(\not{p}_1 + \cancel{m_e})(1+\gamma_5) \gamma^\beta\right] \quad \cancel{m_e} \to 0 \text{ (odd \# of } \gamma\text{-matrices)}$$

$$\frac{1}{2} \mathrm{Tr}\left[\not{p}_3 \gamma^\alpha (1-\gamma_5) \underbrace{\not{p}_1 (1+\gamma_5)}_{= (1-\gamma_5)\not{p}_1} \gamma^\beta\right] = \mathrm{Tr}\left[\not{p}_3 \gamma^\alpha (1-\gamma_5) \not{p}_1 \gamma^\beta\right]$$

$$(1-\gamma_5)^2 = 1 - 2\gamma_5 + \gamma_5^2 = 2(1-\gamma_5)$$

$$= 4\left(P_3^{\alpha} P_1^{\beta} - (P_1 P_3)\,\eta^{\alpha\beta} + P_3^{\beta} P_1^{\alpha} + i\,\varepsilon^{\rho\alpha\sigma\beta} P_{3\rho} P_{1\sigma} \right)$$

We also need

$$\sum_{pol} \left(\bar{u}_4 (1+\gamma_5)\gamma_\alpha u_2\right)\left(\bar{u}_2 \gamma_\beta (1-\gamma_5) u_4\right)$$

$$= \sum_{pol} \left(\bar{u}_4 \gamma_\alpha (1-\gamma_5) u_2\right)\left(\bar{u}_2 (1+\gamma_5)\gamma_\beta u_4\right)$$

same as before $3 \to 4$ $1 \to 2$

$$= 4\left(P_{4\alpha} P_{2\beta} - (P_2 P_4)\,\eta_{\alpha\beta} + P_{4\beta} P_{1\alpha} + i\,\varepsilon_{\rho\alpha\sigma\beta} P_4^{\rho} P_2^{\sigma} \right)$$

multiplying

$$16\left[P_1^{\alpha} P_3^{\beta} + P_1^{\beta} P_3^{\alpha} - (P_1 P_3)\,\eta^{\alpha\beta} + i\,\varepsilon^{\overset{\substack{3\\2}}{\rho}\overset{0}{\alpha}\overset{\substack{2\\3}}{\sigma}\overset{1}{\beta}} P_{3\rho} P_{1\sigma} \right]$$

$$\left[P_{2\alpha} P_{4\beta} + P_{2\beta} P_{4\alpha} - (P_2 \cdot P_4)\,\eta_{\alpha\beta} + i\,\varepsilon_{\underset{\substack{3\\2}}{\rho'}\underset{0}{\alpha}\underset{\substack{2\\3}}{\sigma'}\underset{1}{\beta}} P_4^{\rho'} P_2^{\sigma'} \right]$$

Notice $\varepsilon^{0123} = -\,\varepsilon_{0123}$ (3 (−) signs).

$$\varepsilon^{\rho\alpha\sigma\beta}\,\varepsilon_{\rho'\alpha\sigma'\beta} = -2\,\delta^{\rho}_{\rho'}\,\delta^{\sigma}_{\sigma'} + 2\,\delta^{\rho}_{\sigma'}\,\delta^{\sigma}_{\rho'}$$

$$16\Big[2(p_1p_2)(p_3p_4) + \cancel{2(p_1\cdot p_4)(p_2\cdot p_3)} - 2\cancel{(p_2p_4)(p_1\cdot p_3)}$$

$$-\cancel{(p_1p_3)(p_2p_4)} - \cancel{(p_1p_3)(p_2p_4)} + 4\,\cancel{(p_1p_3)(p_2p_4)}$$

$$+2(p_3p_4)(p_1\cdot p_2) - 2\,\cancel{(p_2p_3)(p_1p_4)}\Big]$$

$$= 64\,(p_1p_2)(p_3p_4).$$

$$\Rightarrow \quad \frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{1}{\cancel{64}\pi^2}\frac{|\vec{p}_3|}{|\vec{p}_1|}\frac{1}{s}\frac{G_F^2}{2}\cancel{64}(p_1p_2)(p_3p_4)$$

$$= \frac{G_F^2}{2\pi^2}\frac{|\vec{p}_3|}{|\vec{p}_1|}\frac{1}{s}(p_1\cdot p_2)(p_3p_4)$$

Mandelstam variables

$$s=(p_1+p_2)^2 = m_e^2 + 2p_1\cdot p_2 \qquad p_1\cdot p_2 = \frac{s-m_e^2}{2}$$

$$t=(p_1-p_3)^2 = m_e^2 - 2p_1\cdot p_3$$

$$u=(p_1-p_4)^2 = m_e^2 + m_\mu^2 - 2p_1\cdot p_4$$

$p_1+p_2 = p_3+p_4$

$s=(p_3+p_4)^2 = m_\mu^2 + 2p_3 p_3$

$p_3p_4 = \frac{s-m_\mu^2}{2}$

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{G_F^2}{2\pi^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{1}{s} \frac{(s-m_e^2)(s-m_\mu^2)}{4}$$

$$(\varepsilon_1, \vec{p}_1) + (\varepsilon_2, -\vec{p}_1) \to (\varepsilon_3, \vec{p}_3) + (\varepsilon_4, -\vec{p}_3)$$

e^- $\quad \nu_\mu$ $\quad \nu_e$ $\quad \mu^-$

$m_\nu \simeq 0$

$\varepsilon_2 = |\vec{p}_1|$

$s = (\varepsilon_1 + \varepsilon_2)^2$

$\varepsilon_1^2 - \vec{p}_1^{\,2} = m_e^2 \Rightarrow (\varepsilon_1^2 - \varepsilon_2^2) = m_e^2$

$(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 + \varepsilon_2) = m_e^2$

$\varepsilon_1 + \varepsilon_2 = \sqrt{s}$;

$\varepsilon_1 - \varepsilon_2 = \dfrac{m_e^2}{\sqrt{s}}$

$2\varepsilon_2 = \sqrt{s} - \dfrac{m_e^2}{\sqrt{s}} = \dfrac{s-m_e^2}{\sqrt{s}}$; $\quad \varepsilon_2 = \dfrac{s-m_e^2}{2\sqrt{s}} = |\vec{p}_1|$

in the same way $\quad \varepsilon_3 = |\vec{p}_3| = \dfrac{s-m_\mu^2}{2\sqrt{s}}$

$$\frac{|\vec{p}_1|}{|\vec{p}_3|} = \frac{s-m_e^2}{s-m_\mu^2} \quad \Bigg\| \quad \boxed{\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{G_F^2}{8\pi^2} \frac{(s-m_\mu^2)^2}{s}}$$

$$\boxed{\sigma_{\alpha\to\beta} = \frac{G_F^2}{2\pi}\,\frac{(s-m_\mu^2)^2}{s}} = \int d\Omega\, \frac{d\sigma_{\alpha\to\beta}}{d\Omega}.$$

$\sigma \sim s$ when $s \to \infty$ $(s \gg m_\mu^2)$

partial waves:

$$\mathcal{M}_{\beta\alpha} = -\frac{iG_F}{\sqrt{2}} \left(\bar u_3 \gamma^\alpha (1-\gamma_5) u_1\right)\left(\bar u_4 \gamma_\alpha (1-\gamma_5) u_2\right)$$

<u>Fierz identity</u>

$P_L = \frac{1-\gamma_5}{2}$ $\quad P_L u_1 = u_{1L}$ left part (top part in chiral rep.)

$$\begin{pmatrix} u_{1L} \\ u_{1R} \end{pmatrix}$$

$$P_L^2 = \frac{(1-\gamma_5)^2}{4} = \frac{1-\gamma_5}{2} = P_L \quad ✓$$

$$\mathcal{M}_{\beta\alpha} = -\frac{4iG_F}{\sqrt{2}} \left(\bar u_{3L} \gamma^\alpha u_{1L}\right)\left(\bar u_{4L} \gamma_\alpha u_{2L}\right)$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} \qquad \sigma^\mu = (1, \vec\sigma) \qquad \bar\sigma^\mu = (1, -\vec\sigma)$$

$$(u_{3L}^\dagger \ 0)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}\begin{pmatrix} u_{1L} \\ 0 \end{pmatrix} = (0 \ u_{3L}^\dagger)\begin{pmatrix} 0 \\ \bar\sigma^\mu u_{1L} \end{pmatrix} = u_{3L}^\dagger \bar\sigma^\mu u_{1L}$$

$$\left(\underset{a}{u^\dagger_{3L}} \underset{ab}{\bar\sigma^\alpha} \underset{b}{u_{1L}}\right)\left(\underset{c}{u^\dagger_{4L}} \underset{cd}{\bar\sigma_\alpha} \underset{d}{u_{2L}}\right) = 2 u^\dagger_{3La} u_{1Lb} u^\dagger_{4Lc} u_{2Ld} \epsilon_{ac}\epsilon_{bd}$$

$$\underset{ab}{(\bar\sigma^\mu)_{\alpha\beta}} \underset{cd}{(\bar\sigma_\mu)_{\gamma\delta}} = 2 \underset{ac}{\epsilon_{\alpha\gamma}} \underset{bd}{\epsilon_{\beta\delta}} \qquad \Big| \quad = 2 u^\dagger_{3La} u^\dagger_{4Lc} \epsilon_{ac}\, u_{1Lb} u_{2Ld} \epsilon_{bd}$$

(−)

$$\left(u^\dagger_{3a} \underset{ab}{\bar\sigma^\alpha} u_{2Lb}\right)\left(u^\dagger_{4Lc} \bar\sigma_{\alpha cd} u_{1Ld}\right) = 2 \underbrace{u^\dagger_{3La} u^\dagger_{4Lc} \epsilon_{ac}}_{\text{factorized.}} u_{2Lb} u_{1Ld} \epsilon_{bd}$$

(26)

$$\mathcal{M}_{\beta\alpha} = -\frac{4iG_F}{\sqrt{2}}\, 2\, \underbrace{\left(u^\dagger_{3La}\,\epsilon_{ac}\,u^\dagger_{4Lc}\right)}_{\hat{p}_3} \underbrace{\left(u_{1Lb}\,\epsilon_{bd}\,u_{2Ld}\right)}_{\hat{p}_1}$$

$$u^{(\sigma_1)}_{1Lb}\,\epsilon_{bd}\,u^{(\sigma_2)}_{2Ld}.$$

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix} \qquad S_3 = \frac{1}{2}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}$$

<u>on u_L</u> $\boxed{S_3 = \frac{1}{2}\sigma_3}$

$$\not{p}\,u = m u \qquad \begin{pmatrix}0 & p_\mu\sigma^\mu\\ p_\mu\bar{\sigma}^\mu & 0\end{pmatrix}\begin{pmatrix}u_L\\ u_R\end{pmatrix} = m\begin{pmatrix}u_L\\ u_R\end{pmatrix}$$

$$\begin{pmatrix}p_0+p_3 & p_1-ip_2\\ p_1+ip_2 & p_0-p_3\end{pmatrix}\begin{pmatrix}u_R\\ 0\end{pmatrix} = \begin{pmatrix}mu_L\\ 0\end{pmatrix}$$

$$p_\mu\sigma^\mu u_R = m u_L$$

$$p_\mu\bar{\sigma}^\mu u_L = m u_R$$

$$\begin{pmatrix}p_0-p_3 & -p_1+ip_2\\ -p_1-ip_2 & p_0+p_3\end{pmatrix}\begin{pmatrix}u_L\\ 0\end{pmatrix} = \begin{pmatrix}mu_R\\ 0\end{pmatrix}$$

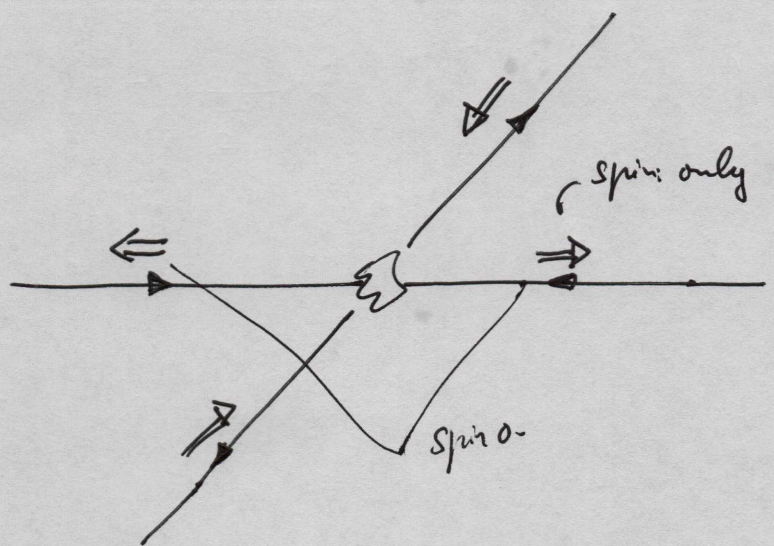
$$(p_0+p_3)u_R = m u_L$$

$$(p_1+ip_2)u_R = 0$$

$$u(p) = \gamma^0 u(\tilde{p}) \qquad u^{t\,(\sigma_1)}_{1L}\,\epsilon\, u^{(\sigma_2)}_{1R}$$

$$(u^\uparrow, u^\downarrow)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}u^\uparrow\\ u^\downarrow\end{pmatrix} = (u^\downarrow\ \ -u^\uparrow)\begin{pmatrix}u^\uparrow\\ u^\downarrow\end{pmatrix} = u^\downarrow u^\uparrow - u^\uparrow u^\downarrow$$

spin 0



$$\sigma_T = \frac{\pi}{\cancel{4}\,2} \frac{1}{|\vec{p}_1|^2} \sum_j (2j+1) \left| \delta_{\ell\ell'} \delta_{ss'} \delta_{nn'} - S^j_{\substack{\ell s n \\ \ell' s' n'}} \right|^2$$

only s-wave. $n \neq n'$

$$= \frac{\pi}{2} \frac{1}{|\vec{p}_1|^2} \underbrace{\left| S^0 \right|^2}_{\lesssim 1} \lesssim \frac{\pi}{2|\vec{p}_1|^2} = \frac{\pi \cancel{4}^2 s}{\cancel{2}\,(s-m_e^2)^2} \sim 1/s \quad s \to \infty$$

then

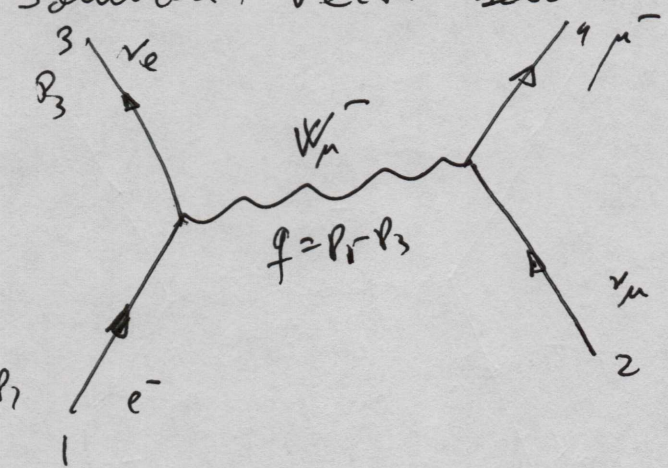
$\sigma_T \sim s \quad s \to \infty$ violates unitarity.

$$\frac{G_F^2}{2\pi} \frac{(s-m_\mu^2)^2}{\cancel{s}} \lesssim \frac{2\pi}{\cancel{s}}$$

$$s - m_\mu^2 \lesssim \frac{2\pi}{G_F} \qquad s \gtrsim \frac{2\pi}{G_F} \quad \text{problem with unitarity.}$$

$G_F \sim M^{-2}$ ✓.

Solution: vector boson



$$\mathcal{M}_{\beta\alpha} = \lambda^2 (\bar{u}_3 \gamma^\alpha (1-\gamma_5) u_1) \frac{(-i)\left(\eta_{\alpha\beta} - \frac{q_\alpha q_\beta}{M_W^2}\right)}{q^2 - M_W^2} (\bar{u}_4 \gamma^\beta (1-\gamma_5) u_2)$$

$$q^2 = (p_1 - p_3)^2 = t$$

In the propagator we have $\dfrac{q_\alpha q_\beta}{M_W^2}$

this gives

$$(\bar{u}_3 \not{q} (1-\gamma_5) u_1) = (\bar{u}_3 (\not{p}_1 - \not{p}_3) (1-\gamma_5) u_1) =$$

$$= \bar{u}_3 (1+\gamma_5) \underbrace{\not{p}_1 u_1}_{m_e u_1} - \underbrace{\bar{u}_3 \not{p}_3}_{0} (1-\gamma_5) u_1 \qquad \not{p}_3 u_3 = 0$$

does not grow with p_1. Can be ignored for $s \gg m_e$.

then we get

$$\mathcal{M}_{\beta\alpha} = -i\lambda^2 \frac{(\bar{u}_3 \gamma^\alpha (1-\gamma_5) u_1)(\bar{u}_4 \gamma_\alpha (1-\gamma_5) u_2)}{(t - M_W^2)}$$

We get (similar calculation as before)

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{\lambda^4}{4\pi^2}\frac{(s-m_\mu^2)^2}{s}\frac{1}{(t-M_W^2)^2}$$

for $s \ll M_W^2$

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} \simeq \frac{\lambda^4}{4\pi^2}\frac{(s-m_\mu^2)^2}{s M_W^4} \; ; \quad \boxed{\frac{1}{\sqrt{2}}G_F = \frac{\lambda^2}{M_W^2}} \; (\sim M^{-2}\checkmark)$$

For large energy $(s \gg m_\mu^2)$

$$t = (p_1-p_3)^2 = m_e^2 - 2p_1\cdot p_3 \simeq -2p_1p_3 = -2(\varepsilon_1\varepsilon_3 - \vec{p}_1\cdot\vec{p}_3)$$

$$= -2(\varepsilon_1\varepsilon_3 - |\vec{p}_1||\vec{p}_3| c\theta) \underset{m_e\simeq 0}{\simeq} -2\varepsilon_1\varepsilon_3(1-c\theta) = -4\varepsilon_1\varepsilon_3 s^2\tfrac{\theta}{2}$$

$1-c^2\theta/_2 + s^2\theta/_2 = 2s^2\theta/_2$

$\varepsilon_1 \simeq \frac{\sqrt{s}}{2} \simeq \varepsilon_3$

$$t \simeq -4\frac{s}{4}s^2\frac{\theta}{2} = -s\cdot s^2\frac{\theta}{2}$$

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{\lambda^4}{4\pi^2}\frac{(s^{\cancel{2}})}{\cancel{s}}\frac{1}{(M_W^2+s\cdot s^2\frac{\theta}{2})^2} \quad \Big\downarrow \int_0^{2\pi}d\varphi\int_0^\pi s\theta\, d\theta$$

$$\sigma_{\alpha\to\beta} = \frac{\lambda^4}{\cancel{4\pi^2}}s\frac{\cancel{2\pi}\cdot\cancel{2}}{M_W^2(M_W^2+s)} = \frac{\lambda^4 s}{\pi M_W^2(s+M_W^2)}$$

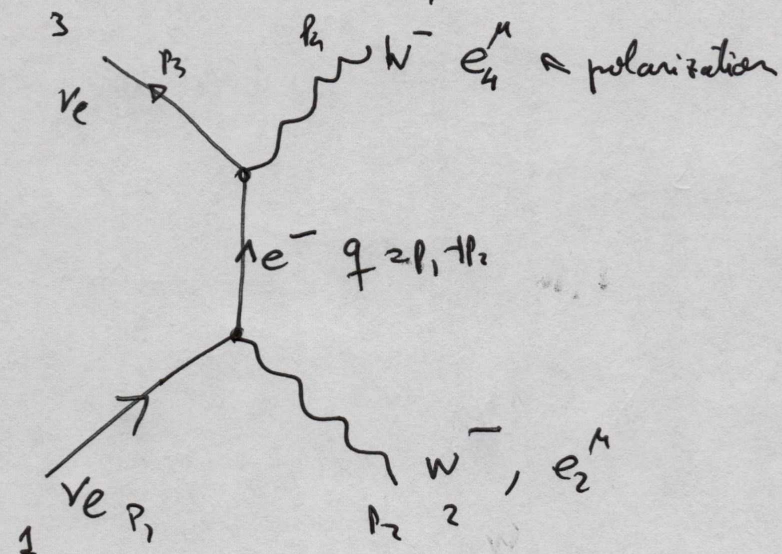
$$\sigma_{\alpha\to\beta} = \frac{\lambda^4}{\pi M_W^2} \frac{s}{s+M_W^2} \qquad (\sim \ell^2 \checkmark)$$

$$s \to \infty \qquad \sigma_{\alpha\to\beta} \to \frac{\lambda^4}{\pi M_W^2} = \frac{1}{2\pi} G_F^2 M_W^2$$

Does not diverge anymore.

However : new problem. Consider

$$\nu_e + W^- \longrightarrow \nu_e + W^-$$



We use $m_e \approx 0$

$$\mathcal{M}_{\beta\alpha} = -i\lambda^2 \bar{u}_3 \gamma^\nu (1-\gamma_5) \frac{\not{q}}{q^2} \gamma^\mu (1-\gamma_5) u_1 \; e_\mu^{(2)} \; e_\nu^{*(4)}$$

Enhancement for longitudinal polarization.

suppose $\quad p_2 = (\mathcal{E}_2, 0, 0, p_2) \qquad \mathcal{E}_2^2 - p_2^2 = M_W^2$

polarizations $\quad \underset{(1)}{(0,1,0,0)} \qquad \underset{(2)}{(0,0,1,0)} \qquad \underset{(3)}{(p_2, 0, 0, \mathcal{E}_2)} \frac{1}{M_W}$

$(e^\mu)^2 = -1 \qquad e^\mu p_\mu = 0$ $\qquad$ ↗ normalization

For high energy $\quad \mathcal{E}_2 \approx p_2 \quad \Rightarrow \quad e_\mu^{(3)} \simeq \dfrac{p_\mu}{M_W}$
$\mathcal{E} \gg M_W$

If $\epsilon_\mu \sim p_\mu$ does not cancel (and it does not) then
. it can produce bad high energy behavior (unitarity violation)

As usual we get

$$|\mathcal{M}_{\beta\alpha}|^2 = \frac{\lambda^4}{s^2} (\bar{u}_3 \gamma^\nu (1-\gamma_5) \not{q} \gamma^\mu (1-\gamma_5) u_1)(\bar{u}_1 (1+\gamma_5) \gamma^{\mu'} \cdot \not{q} (1+\gamma_5) \gamma^{\nu'} u_3) \cdot e_\mu^{(2)} e_\nu^{*(4)} \cdot e_{\mu'}^{(2)*} e_{\nu'}^{(4)}$$

$\boxed{q^2 = s}$

For longitudinal W:

$$|\mathcal{M}_{\beta\alpha}|^2 = \frac{\lambda^4}{s^2 M_W^4} \mathrm{Tr}\Big(\underbrace{u_3 \bar{u}_3}_{\not{p}_3} \not{p}_4 (1-\gamma_5) \not{q} \not{p}_2 (1-\gamma_5) \not{p}_1 (1+\gamma_5) \not{p}_2 \not{q} \cdot (1+\gamma_5) \not{p}_4 \Big)$$

$$= \frac{8\lambda^4}{s^2 M_W^4} \mathrm{Tr}\big(\not{p}_3 \not{p}_4 \not{q} \not{p}_2 \not{p}_1 \not{p}_2 \not{q} \not{p}_4 (1-\gamma_5)\big)$$

$$\not{p}_2 \not{p}_1 \not{p}_2 = p_{2\alpha} p_{1\beta} p_{2\rho} \gamma^\alpha \gamma^\beta \gamma^\rho = - p_{2\alpha} p_{1\beta} p_{2\rho} \gamma^\beta \underbrace{\gamma^\alpha \gamma^\rho}_{\to \{\gamma^\alpha \gamma^\rho\} = 2\eta^{\alpha\rho}} + 2(p_1 \cdot p_2) \not{p}_2$$

$$= -2 \overset{= M_W^2}{p_2^2} \not{p}_1 + 2(p_1 \cdot p_2) \not{p}_2$$

$$= \frac{16\lambda^4}{s^2 M_W^4} \mathrm{Tr}\Big(- \not{p}_3 \not{p}_4 \underbrace{\not{q} \not{p}_1 \not{q}}_{-2q^2 \not{p}_1 + 2(p_1 \cdot q)\not{q}} \not{p}_4 (1-\gamma_5) M_W^2 + \not{p}_3 \not{p}_4 \underbrace{\not{q} \not{p}_2 \not{q}}_{-2q^2 \not{p}_2 + 2(p_2 \cdot q)\not{q}} \not{p}_4 (1-\gamma_5)(p_1 \cdot p_2)$$

symmetry antisymmetry

$$= \frac{32\lambda^4}{s^2 M_w^4} \mathrm{Tr}\Big[s M_w^2\, \not p_3 \not p_4 \not p_1 \not p_4 (1-\cancel{\gamma_5}) - (p_1\cdot q) M_w^2\, \not p_3 \not p_4 \not q \not p_4 (1-\cancel{\gamma_5}) -$$

$$- s(p_1\cdot p_2) \not p_3 \not p_4 \not p_2 \not p_4 (1-\cancel{\gamma_5}) + (p_2 q)(p_1\cdot p_2) \not p_3 \not p_4 \not q \not p_4 (1-\cancel{\gamma_5}) \Big]$$

$$= \frac{128\lambda^4}{s^2 M_w^4} \Big[s M_w^2 \big((p_3\cdot p_4)(p_1 p_4) - (p_1 p_3) p_4^2 + (p_3 p_4)(p_1 p_4) \big)$$

$$- M_w^2 (p_1 q) \big((p_3 p_4)(p_4 q) - (p_3 q)(p_4^2) + (p_3 p_4)(p_4 q) \big)$$

$$- s(p_1 p_2) \big((p_3 p_4)(p_2 p_4) - (p_2 p_3) p_4^2 + (p_3 p_4)(p_2 p_4) \big)$$

$$+ (p_2 q)(p_1 p_2) \big((p_3 p_4)(q p_4) - (p_3 q) p_4^2 + (p_3 p_4)(p_4 q) \big) \Big]$$

use $p_4^2 = M_w^2$ and look for highest power of s.

$$= \frac{128\lambda^4}{s^2 M_w^4} \Big\{ - s(p_1 p_2)\, 2 (p_3 p_4)(p_2 p_4) + (p_2 \underset{p_1+p_2}{q})(p_1 p_2)\, 2 (p_3 p_4)(\underset{p_3+p_4}{q} p_4) \Big\}$$

$$= \frac{\cancel{128}\,256\,\lambda^4}{s^2 M_w^4} \Big\{ - s(p_1 p_2)(p_3 p_4)(p_2 p_4) + \underbrace{(p_1 p_2)(p_1 p_2)(p_3 p_4)(p_3 p_4)}_{s^4/16} \Big\}$$

$(p_1+p_2)^2 = 2p_1 p_2 \quad \rightarrow p_1 p_2 = s/2 \qquad p_2 p_4 = p_2(-p_1 - p_2 - p_3) =$

$(p_3+p_4)^2 = 2 p_3 p_4 \quad \rightarrow p_3 p_4 = s/2 \qquad = -p_1 p_2 - p_2^2 - p_2 p_3$

$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 = -2 p_2 p_4 \qquad p_2 p_4 = -t/2$

$$= \frac{256\lambda^4}{s^2 M_W^4} \left\{ + \frac{s^3}{4}\frac{t}{2} + \frac{s^4}{16} \right\}$$

$$= \frac{32\lambda^4}{s^2 M_W^4} s^3 \left\{ t + s/2 \right\} = \frac{16\lambda^4}{M_W^4} s \left\{ s + 2t \right\}$$

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{1}{64\pi^2} \frac{|\vec{p}_3|}{|\vec{p}_1|} \frac{1}{\cancel{s}} \frac{16\lambda^4}{M_W^4} \cancel{s} (s+2t)$$

massless limit $|p_3| \approx |p_1|$

$$\frac{d\sigma_{\alpha\to\beta}}{d\Omega} = \frac{\lambda^4}{4\pi^2} \frac{1}{M_W^4} (s+2t)$$

violates unitarity again.