Review of QFT I

$$\text{Fields} \equiv \text{particles}$$

$$\hat{\phi}(\vec{x},t) \qquad\qquad a_{\vec{k},\sigma}\,,\ a^{\dagger}_{\vec{k},\sigma}$$

"Field" point of view: canonical quantization

Usual QM

$\{p_i, q_j\}_{P.B.} = -\delta_{ij} \rightarrow [p_i, q_j] = -i\hbar\,\delta_{ij}$

Find canonical variables $\left.\begin{array}{l}\phi(\vec{x}) \rightarrow \hat{\phi}(\vec{x}) \\ \pi(\vec{x}) \rightarrow \hat{\pi}(\vec{x})\end{array}\right\} \rightarrow [\pi(\vec{x}), \phi(\vec{y})] = -i\delta(\vec{x}-\vec{y})$

spacial only (not t)

Start from Lagrangian $\rightarrow (\phi, \pi) \longrightarrow$ find H

Lorentz invariance $\rightarrow$ construct $M_{\mu\nu}$ (Lorentz transf. generators)

Locality $\rightarrow [\mathcal{O}_1(\vec{x}_1, t_1), \mathcal{O}_2(\vec{x}_2, t_2)] = 0$ if $(\vec{x}_1, t_1)$ $(\vec{x}_2, t_2)$ space-like separated.

Heisenberg picture.

All the commutation relations are represented in terms of $a_{k,\sigma}$, $a^{\dagger}_{k,\sigma}$, creation & annihilation op. acting on a Hilbert space of particles.

"Particle" point of view:

The physics is described by a set of (weakly) interacting particles. The theory has Lorentz (and other) symmetries. It is also local.
The best way to ensure Lorentz symmetry is to construct field operators $\hat{\phi}(\vec{x},t)$, etc. and define a ~~local~~ Hamiltonian as an integral of local terms.

Particles:

spin 0 : scalar bosons
spin 1/2 : spinors, fermions
} we used them in QFT 1.
spin 1 : vector bosons.

Vector bosons: if massless we need a gauge symmetry to ~~reduce~~ eliminate the unphysical components.

massive vector bosons theories are generically non-renormalizable unless the mass comes from spontaneous symmetry breaking.

Perturbation theory

·) Define free fields $\equiv$ non-interacting particles

·) Write a quadratic Lagrangian:

$$\mathcal{L}_\phi = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2$$

$$\mathcal{L}_\psi = \bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$$

$$\mathcal{L}_A = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

·) Write interactions: bosonic, Lorentz inv., $\underbrace{\text{\& dim.} \leq 4}_{\text{if renormalizable}}$

dim $\phi = 1$, dim $A_\mu = 1$ dim $\psi = 3/2$

$\bar{\psi}\psi\phi$; ϕ^3 ; ϕ^4 ;

$\bar{\psi}\not{A}\psi$; $A_\mu A^\mu \phi^2$; $A_\mu \partial^\mu\phi\,\phi$

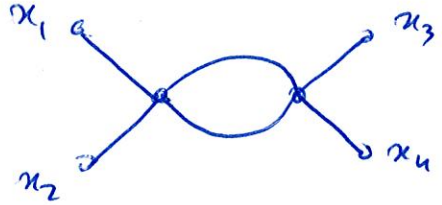
$A_\mu \partial^\mu \phi$.

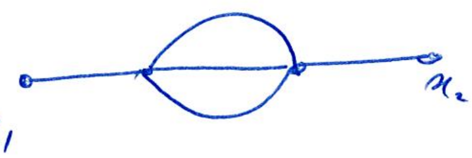
Non-ren.: $(\bar{\psi}\psi)^2$, ϕ^6, ...

·) Use pert. theory to compute Green functions.

$$\langle 0|\hat{T}\{\phi\phi\phi\phi\}|0\rangle \qquad \langle 0|\hat{T}\{\phi,\phi\}|0\rangle -$$

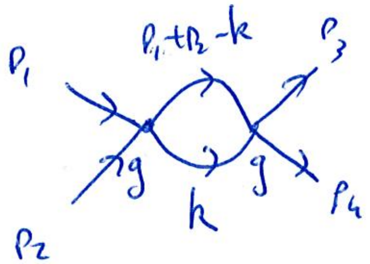
time-order.





expansion in Feynman diagrams.

Also in momentum space.



$$\frac{(-ig)^2}{2}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{(k^2-m^2+i\epsilon)\,((p_1+p_2-k)^2-m^2+i\epsilon)}$$

·) We need to regularise and renormalise (add counter terms) order by order

o) Green functions can be converted onto physical information such as cross sections and decay rates.

Gauge symmetry:

A local symmetry ⇒ redundancy in the description
not a physical symmetry (except global symmetry).

Local means that we can apply the symmetry independently at each point in space.

example: electromagnetism.

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (\not\partial - iq_\psi \not A) \psi + \frac{1}{2} |(\partial_\mu - iq_\phi A_\mu) \phi|^2 - \frac{1}{2} m^2 |\phi|^2$$

complex field.

$$\phi \to e^{i\alpha(x) q_\phi} \phi$$

$$\psi \to e^{i\alpha(x) q_\psi} \psi$$

$\phi_1 \bar{\psi}_2 \psi_3$ would be invariant if $q_1 + q_2 + q_3 = 0$

charge conservation.

what about $A_\mu \to A_\mu + \partial_\mu \alpha$.

$$\partial_\mu \psi \to \partial_\mu (e^{i\alpha q} \psi) = e^{i\alpha q} \partial \psi + iq \partial_\mu \alpha \psi$$

$$(\partial_\mu - iq A_\mu) \psi \to \partial_\mu (e^{i\alpha q} \psi) - iq A_\mu e^{i\alpha q} \psi - iq \partial_\mu \alpha \psi$$

$$e^{i\alpha q} (\partial_\mu - iq A_\mu) \psi \Rightarrow \underbrace{\bar{\psi} (\partial_\mu - iq A_\mu) \psi}_{\text{invariant}}$$

Same with ϕ.

Massless spin 1 particle

We can always ~~boost~~ Lorentz transform a light-like vector to become $(1,0,0,1)$

$k_{0\mu} = (1,0,0,1)$: reference vector.

Little group: Leaves k_0 invariant.

$$\mathbb{K} = \begin{pmatrix} k_0+k_3 & k_1-ik_2 \\ k_1+ik_2 & k_0-k_3 \end{pmatrix} \qquad \det \mathbb{K} = k^2 \quad ; \mathbb{K}^\dagger = \mathbb{K}.$$

$$\tilde{\mathbb{K}} = U \mathbb{K} U^\dagger \; ; \qquad U \in SL(2,\mathbb{C})$$

$$\det \tilde{\mathbb{K}} = \det \mathbb{K} \qquad \tilde{\mathbb{K}}^\dagger = \tilde{\mathbb{K}}$$

$$\mathbb{K}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; ; \qquad \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u_1^* & u_3^* \\ u_2^* & u_4^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} u_1 & 0 \\ u_3 & 0 \end{pmatrix}\begin{pmatrix} u_1^* & u_3^* \\ u_2^* & u_4^* \end{pmatrix} = \begin{pmatrix} |u_1|^2 & u_1 u_3^* \\ u_3 u_1^* & |u_3|^2 \end{pmatrix} = \begin{pmatrix} |u_1|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\to |u_3| = 0 \Rightarrow u_3 = 0$$

$$|u_1|^2 = 1.$$

$$U = \begin{pmatrix} e^{i\varphi/2} & \beta \\ 0 & e^{-i\varphi/2} \end{pmatrix} = \begin{pmatrix} 1 & \tilde{\beta} \\ 0 & 1 \end{pmatrix} e^{i\varphi\sigma_3/2}$$

$\det = 1$

$$\begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \beta_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta_1+\beta_2 \\ 0 & 1 \end{pmatrix}$$

P_1, P_2, J_{12}.

$\sigma_+, \; i\sigma_+, \; \sigma_3$

$$[\sigma_3, \sigma_+] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$[\sigma_3, \sigma_+] = i \, i\sigma_+$$

$$[\sigma_3, i\sigma_+] = i\sigma_1$$

zero eigenvalue under $\sigma_+, i\sigma_+$

classified by eigenvalue of σ_3: ± 1 by parity we need both signs.

2 states $\quad |\vec{k}, \pm\rangle$ ← helicity

Take a standard map: $k_{0\mu} \to k_\mu$ by boosting & rotating. Denote as L_k $\quad e^{\beta\sigma_3/2}\begin{pmatrix}1&0\\0&0\end{pmatrix}e^{\beta\sigma_3/2} = \begin{pmatrix}e^{\beta}&0\\0&0\end{pmatrix} \xrightarrow{\text{rot.}}$

$$|\vec{k}, \pm\rangle = L_k\, |\vec{k}_0, \pm\rangle$$

$$U_\Lambda |\vec{k}, \pm\rangle = U_\Lambda L_k |\vec{k}_0, \pm\rangle = L_{\Lambda k} \underbrace{L_{\Lambda k}^{-1} U_\Lambda L_k}_{\text{little group}} |\vec{k}_0, \pm\rangle$$

$$= L_{\Lambda k}\, S(\beta) R_{\varphi_{\Lambda,k}} |\vec{k}_0, \pm\rangle = e^{\pm i\varphi_{\Lambda,k}} L_{\Lambda k} |k_0, \pm\rangle$$

$$= e^{\pm i\varphi_{\Lambda,k}} |\Lambda k, \pm\rangle$$

$$\boxed{U_\Lambda\, a^\dagger_{k,\sigma}\, U_\Lambda^{-1} = e^{i\varphi_{\Lambda,k}\sigma}\, a^\dagger_{\Lambda k,\sigma}}$$

↑ we can represent Lorentz transf. on massless states.

a photon polarized ~~along~~ parallel to $\vec{k}$, maps to a photon polarized along the new k, up to a phase.

Can we construct a vector field?

$$A_\mu = \int \frac{d^3k}{2k_0} \sum_{\sigma=1,2} \left(e^\sigma_\mu e^{ikx} a^+_{k,\sigma} + (e^\sigma_\mu)^* e^{-ikx} a_{k,\sigma} \right)$$

$\Box A_\mu = 0$ ✓

$k_0 = |\vec{k}|$ $\qquad \frac{d^3k}{k_0}$ rot. inv. Also boost inv.

e.g.
$$\begin{cases} k_3 = ch\beta\, \tilde{k}_3 - sh\beta\, \tilde{k}_0 \\ k_0 = ch\beta\, \tilde{k}_0 - sh\beta\, \tilde{k}_3 \\ k_{1,2} = \tilde{k}_{1,2} \end{cases}$$

$\rightarrow \tilde{k}_0 = \sqrt{\tilde{k}_1^2 + \tilde{k}_2^2 + \tilde{k}_3^2}$

$$\frac{\partial k_3}{\partial \tilde{k}_3} = ch\beta - \frac{\tilde{k}_3}{\tilde{k}_0} sh\beta = \frac{1}{\tilde{k}_0} k_0$$

$$\begin{vmatrix} \partial k_3/\partial \tilde{k}_3 & \partial k_3/\partial \tilde{k}_1 & \partial k_3/\partial \tilde{k}_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\frac{d^3k}{k_0} = \frac{k_0}{\tilde{k}_0} \frac{d^3\tilde{k}}{k_0} = \frac{d^3\tilde{k}}{\tilde{k}_0}$$: invariant measure.

We want

$$U_\Lambda A_\mu(x) U_\Lambda^{-1} = \Lambda^{-1}{}_\mu{}^\nu A_\nu(\Lambda x)$$

why Λ^{-1}

$$U_{\Lambda_2} U_{\Lambda_1} A_\mu(x) U_{\Lambda_1}^{-1} U_{\Lambda_2}^{-1} = \Lambda^{-1}_{1\mu}{}^\nu U_{\Lambda_2} A_\nu(\Lambda_1 x) U_{\Lambda_2}^{-1}$$

$$= \Lambda_1^{-1} \Lambda_2^{-1} A(\Lambda_2\Lambda_1 x) = (\Lambda_2\Lambda_1)^{-1} A(\Lambda_2\Lambda_1 x)$$

group mult. OK.

Also $\Lambda^t \eta \Lambda = \eta$ $\quad \Lambda^t \eta = \eta \Lambda^{-1}$ $\quad (\Lambda^{-1})^t \eta = \eta \Lambda$ (4)

$$U_\Lambda A_\mu(x) U_\Lambda^{-1} = \int \frac{d^3k}{2k_0} \sum_{\sigma=1,2} \left(e_\mu^\sigma(k) e^{i\varphi_{\Lambda,k} \cdot \sigma} e^{ik^\ell x} a^+_{\Lambda k,\sigma} + \dots \right)$$

$$\Lambda^{-1}{}_\mu{}^\nu A_\nu(\Lambda x) = \Lambda^{-1}{}_\mu{}^\nu \int \frac{d^3k}{2k_0} \sum_{\sigma=1,2} \left(e_\nu^\sigma(k) e^{ik^t \Lambda x} a^+_{k,\sigma} + \dots \right.$$

$$= \Lambda^{-1}{}_\mu{}^\nu \int \frac{d^3k}{2k_0} \sum_{\sigma=1,2} \left(e_\nu^\sigma(\Lambda k) e^{ikx} a^+_{\Lambda k,\sigma} + \dots \right.$$

$k \to \Lambda k$

We need $\boxed{e_\mu^\sigma(k) e^{i\varphi_{\Lambda,k}\sigma} \overset{?}{=} \Lambda^{-1}{}_\mu{}^\nu e_\nu^\sigma(\Lambda k)}$

if Λ is a rotation around $k \Rightarrow \Lambda k = k$.

So e_μ^σ is a vector that, under a rot. around k transforms with a phase. e_μ^σ should be the usual circular polarizations

if $k = k_0 = (1,0,0,1)$ $\quad e^\pm = (0, 1, \pm i, 0)$. [Clearly it's not going to work for boosts]

Define: $e_\mu^\sigma(k) = (L_k)_\mu{}^\nu e_\nu^\sigma(k_0)$

$$e^\sigma e^{i\varphi_{\Lambda,k}\cdot\sigma} = L_k \underbrace{e^{i\varphi_{\Lambda,k}\cdot\sigma} e_0^\sigma}_{R^{-1}_{\varphi_{\Lambda,k}} e_0^\sigma .}$$

$(0, 1, \pm i, 0)$

polarizations are contracted opposite

$$L_k R^{-1}_{\varphi_{\Lambda,k}} e_0^\sigma \overset{?}{=} \Lambda^{-1} L_{\Lambda k} e_0^\sigma$$

$$R^{-1}_{\varphi_{\Lambda k}} e_0^\sigma \overset{?}{=} L_k^{-1}\Lambda^{-1} L_{\Lambda k} e_0^\sigma = (L^{-1}_{\Lambda k}\Lambda L_k)^{-1} e_0^\sigma$$

$$\overset{?}{=} S^{-1}(\beta)\, R^{-1}_{\varphi_{\Lambda,k}} e_0^\sigma$$

just a phase.

NO! not possible.

should do nothing (as before)

But this is not possible on a vector.

We need to compute $S^{-1}(\beta)\, e_0^\sigma$.

$$e = (0, 1, \pm i, 0) \to e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta^* & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta^* & 1 \end{pmatrix} = \begin{pmatrix} \beta^* & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \beta^*\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S^{-1}(\beta)\cdot e_0^\sigma = e_0^\sigma + \beta^* \mathbb{K}_0$$

$$S^{-1}(\beta)_\mu{}^\nu e^\sigma_{0\nu} = e^\sigma_{0\mu} + \beta^* k_{0\mu}$$

What happens then?

$$\left(L_k^{-1}\Lambda^{-1}L_{\Lambda k}\right)e_0^\sigma = R^{-1}_{\varphi,k}\, e_0^\sigma + e^{i\varphi_{\Lambda,k}\sigma}\beta^i k_\mu$$

$L_k \cdot k = k.$

$$\Lambda^{-1}L_{\Lambda k}\, e_0^\sigma = L_k R^{-1}_{\varphi,k}\, e_0^\sigma + e^{i\varphi_{\Lambda,k}\sigma}\beta^i k_\mu$$

$$\Lambda^{-1} e^\sigma_{\Lambda k} = e^{i\varphi_{\Lambda,k}} e_k^\sigma + f(k)\, k_\mu$$

$$U_\Lambda A_\mu(x) U_\Lambda^{-1} = \Lambda^{-1}{}_\mu{}^\nu A_\nu(\Lambda x) + \underbrace{\int \frac{d^3k}{2k_0}\sum_{\sigma=1,2} f(k)\, k_\mu\, e^{ikx} a^+_{\Lambda k,\sigma}}_{}$$

$$+\, \partial_\mu \Omega(x)$$

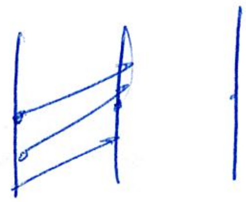
$$\boxed{U_\Lambda A_\mu(x) U_\Lambda^{-1} = \Lambda^{-1}{}_\mu{}^\nu A_\nu(\Lambda x) + \partial_\mu \Omega(x)}$$

"almost" a vector $\Rightarrow$ we need a gauge theory to have gauge invariance.

$(e^\sigma_\mu k^\mu) = 0 \quad \Rightarrow$ a boost will preserve this.

However the boosted e^σ_μ can have a component parallel to $k_\mu \Rightarrow \partial_\mu \Omega$ term.

Mathematical point of view → fiber bundles.



at each point there is a fiber (e.g. phase of ϕ)

But we cannot compare the value of the phase at two different points if we do not introduce a way to "transport" a fiber to a neighbouring one. We allow for a shift;

$$\begin{matrix} x_1 & & x_2 \\ \alpha_1 & \rightarrow & \alpha_2 \end{matrix} \qquad \alpha_2 \equiv \alpha_1 \quad \text{if} \quad \alpha_2 = \alpha_1 + A_\mu \delta x^\mu$$

parallel transport of the fiber.

A_μ: a structure that allows the comparison of fibers at different points.

gauge symmetry: we can shift the fibers independently if we also change. A_μ: how we compare them.

$$\alpha_1 \to \alpha_1 + \delta\alpha_1 \qquad \alpha_2 - \alpha_1 \to \alpha_2 - \alpha_1 + \delta\alpha_2 - \delta\alpha_1 = \alpha_2 - \alpha_1 + \partial_\mu \delta\alpha\, \delta x^\mu$$

$$\alpha_2 \to \alpha_2 + \delta\alpha_2 \qquad \boxed{A_\mu \to A_\mu + \partial_\mu \delta\alpha} \qquad = A_\mu \delta x^\mu + \partial_\mu \delta\alpha\, \delta x^\mu$$

$$D_\mu \alpha = \partial_\mu \alpha + A_\mu \qquad D_\mu(\Phi) = D_\mu(e^{i\alpha}\phi_0) = \partial_\mu \phi_0 e^{i\alpha} + i(\partial_\mu \alpha + A_\mu) e^{i\alpha}\phi$$

$$= \partial_\mu \Phi + i A_\mu \Phi$$

$$A_\mu(x) = A_\mu(x_0) + \partial_\nu A_\mu \, \delta x^\nu + \frac{1}{2} \partial_{\nu_1 \nu_2} A_\mu \, \delta x^{\nu_1} \delta x^{\nu_2} + \dots$$

$x(\xi) = x_0 + \delta x(\xi)$; $\xi : 0 \to 2\pi$ $\quad \delta x$: periodic

$$\oint A_\mu \, dx^\nu = \oint A_\mu \, \partial_\xi \delta x^\mu \, d\xi$$

$$\oint_\xi \delta x^\nu d\xi \to 0 \qquad \partial_\nu A_\mu \oint \delta x^\nu(\xi) \, \partial_\xi \delta x^\mu \, d\xi$$

$$\oint \delta x^\nu(\xi) \, \partial_\xi \delta x^\mu d\xi = \underbrace{\oint \partial_\xi (\quad)}_{0} - \oint \delta x^\mu \partial_\xi \delta x^\nu \, d\xi$$

antisym.

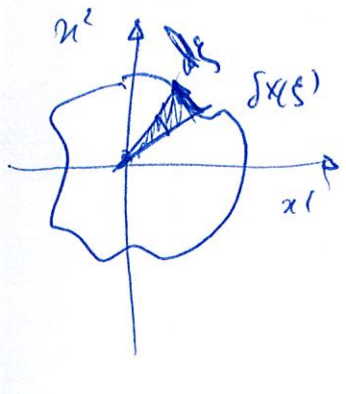
$$\oint A_\mu \, dx^\mu = \frac{1}{2} (\partial_\nu A_\mu - \partial_\mu A_\nu) \oint \delta x^\nu(\xi) \, \partial_\xi \delta x^\mu d\xi + \mathcal{O}(\delta x^3)$$

$$= \frac{1}{2} F_{\nu\mu} \underbrace{\oint \delta x^\nu \partial_\xi \delta x^\mu \, d\xi}_{\text{area}} = \frac{1}{2} F_{\nu\mu} A^{\mu\nu}$$

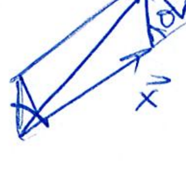
area projected onto (μ, ν)

plane μ, ν

$$\frac{1}{2} \oint (\delta x^1 \partial_\xi \delta x^2 - \delta x^2 \partial_\xi \delta x^1) \, d\xi = \underline{\text{Area}}$$



$$x v_y - y v_x = L$$



$$\vec{x} \times \vec{v} = |x| \cdot |v| \cdot \sin\theta$$

Non-abelian case

example $SU(N)$; $N \times N$ unitary matrices

$$\begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \to U \cdot \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix}$$

we make it dependent of spacetime point.

$$U = e^{i \theta_a T^a}$$

basis of traceless hermitian matrices

invariant terms: $\phi^\dagger \phi$, $\bar{\psi} \psi$ $\qquad (U^\dagger U = \mathbb{1})$

Derivatives: $\partial_\mu \psi \to \partial_\mu (U \psi) = \partial_\mu U \psi + U \partial_\mu \psi$

$$= U \left(U^{-1} \partial_\mu U \psi + \partial_\mu \psi \right)$$

$$(\partial_\mu - i A_\mu) \psi \to U \left(U^{-1} \partial_\mu U \psi + \partial_\mu \psi - i U^{-1} \tilde{A}_\mu U \psi \right)$$

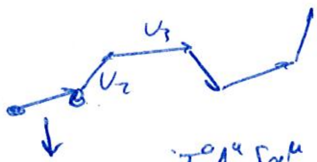
$$= U (\partial_\mu - i A_\mu) \psi$$

$$\tilde{A}_\mu = U A_\mu U^{-1} \cancel{+} - i \partial_\mu U U^{-1}$$

$$A_\mu \to \underbrace{U A_\mu U^{-1}}_{A_\mu \text{ is "charged"}} - i \partial_\mu U U^{-1}$$

gauge field.

Lifting of a path is more complicated now



$$U_1 = e^{iT^a A_\mu^a \delta x^\mu} \simeq 1 + i A_\mu^a T^a \delta x^\mu + \dots$$

$$\psi(x_\mu(\xi)) \neq \quad \boxed{\partial_\xi \psi = i A_\mu^a (x_\mu(\xi))\, T^a \frac{dx^\mu}{d\xi} \psi(x_\mu(\xi))}$$

we need to solve a differential eqn.

$$\psi = \hat{P}\, e^{i\int_0^1 A_\mu^a T^a \frac{dx^\mu}{d\xi} d\xi} \psi(\xi=0)$$

path order

very similar to Sch. eqn for time dep. Hamiltonian

Wilson loop $\mathrm{Tr}\left(\hat{P}\, e^{i\oint A_\mu^a T^a \frac{dx^\mu}{d\xi} d\xi}\right)$ along a closed path. (holonomy).

$$[D_\mu, D_\nu]\psi = (\partial_\mu - iA_\mu)(\partial_\nu - iA_\nu)\psi - \dots$$

$$= \partial_\mu \psi - i\partial_\mu A_\nu \psi - iA_\nu \partial_\mu \psi - iA_\mu \partial_\nu \psi -$$

$$- A_\mu A_\nu \psi - \dots$$

$F \to UFU^{-1}$

$$= -i(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi - [A_\mu, A_\nu]\psi$$

$$= -i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu])\psi = -iF_{\mu\nu}\psi$$

$$\partial_\xi \psi = i M(\xi)\psi$$

$$M(\xi) = A_\mu(\cancel{x}(\xi))\, T^a \dot{x}^\mu(\xi)$$

$$\cancel{M(\xi) = M_0(\xi) +}$$

$$\cancel{A_\mu(\xi) = A_\mu(0) + \partial_\xi A_\mu\, \xi + \frac{1}{2}\partial_\xi^2 A_\mu\, \xi^2 + \ldots}$$

$$= \cancel{A_\mu(0)} + \partial_\nu A_\mu$$

$$A_\mu(x(\xi)) = \bar{A}_\mu + \partial_\nu A_\mu x^\nu + \partial_{\nu\rho} A_\mu x^\nu x^\rho + \ldots$$

$$\partial_\xi \psi = i \bar{A}_\mu T^a \dot{x}^\mu(\xi)\psi + i\, \partial_\nu A_\mu x^\nu(\xi)\, \dot{x}^\mu(\xi)\psi$$

$$\hat{T} e^{i\int_0^1 M(\xi)d\xi} = 1 + i\int_0^1 M(\xi)d\xi - \frac{1}{2}\int_0^1 d\xi_1 d\xi_2\, M(\xi_1)M(\xi_2) - \ldots$$

$$= 1 + i\int_0^1 \bar{A}_\mu \dot{x}^\mu d\xi + i\int_0^1 \partial_\nu \bar{A}_\mu x^\nu \dot{x}^\mu d\xi +$$

$$+ i\oint_0^1 \quad - \frac{1}{2}\int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2\, \bar{A}_\mu \dot{x}^\mu \bar{A}_\nu \dot{x}^\nu d\xi.$$

$$- \bar{A}_\mu \bar{A}_\nu \int_0^1 d\xi_1 \int_0^{\xi_1} d\xi_2\, \dot{x}^\mu \dot{x}^\nu d\xi$$

$$\int_0^1 d\xi_1\, \dot{x}^\mu(\xi_1)\left(x^\nu(\xi_1) - x^\nu(0)\right) =$$

$$= \int_0^1 d\xi\; \dot{x}^\mu_\xi\, x^\nu_\xi$$

↑ area same as before. y

Important matrix $U_\sigma(x,y) = \hat{P}\, e^{i\int_{\sigma\, x}^{y} A_\mu dx^\mu}$ $\quad \mathrm{Tr}\, U_\sigma(x,x) = WL$.

Actions

$\psi \to U\psi \qquad \bar{\psi}\psi$ invariant

$\psi^\dagger \to \psi^\dagger U^\dagger$

$\phi \to U\phi \qquad - \phi^\dagger\phi$ invariant

vector of complex scalars

$D\psi = (\partial_\mu - iA_\mu)\psi \qquad D\psi \to UD\psi$

$$\mathcal{L} = \bar{\psi}\not{D}\psi + \frac{1}{2}|D\phi|^2 + m\bar{\psi}\psi - \frac{1}{2}m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$$

$(D\phi)^\dagger D\phi$

Adjoint fields: $\phi \to U\phi U^\dagger$

matrix $N\times N$

$\bar{\psi}\phi\psi$ invariant (Yukawa interaction)

Covariant derivative?

$$\partial_\mu \phi \to \partial_\mu (U \phi U^\dagger) = \partial_\mu U \phi U^\dagger + U \partial_\mu \phi U^\dagger + U \phi \partial_\mu U^\dagger$$

$$= U (U^\dagger \partial_\mu U \phi + \partial_\mu \phi + \phi \partial_\mu U^\dagger U) U^\dagger$$

$$A_\mu \phi \to U A_\mu \phi U^\dagger + i \partial_\mu U U^\dagger$$
$$+ i U \partial_\mu U^\dagger$$

$$i A_\mu \phi \to i U A_\mu \phi U^\dagger - U \partial_\mu U^\dagger \phi U^\dagger$$
$$= U (i A_\mu \phi - \partial_\mu U^\dagger U \phi) U^\dagger$$
$$= U (i A_\mu \phi + U^\dagger \partial_\mu U \phi) U^\dagger$$

$$i \phi A_\mu \to i U \phi A_\mu U^\dagger + U \phi \partial_\mu U^\dagger$$
$$= U (i \phi A_\mu - \phi \partial_\mu U^\dagger U) U^\dagger$$

$$\partial_\mu \phi - i A_\mu \phi + i \phi A_\mu$$

$$D_\mu \phi = \partial_\mu \phi - i [A_\mu, \phi]$$ adjoint.

$$\mathrm{Tr}\left[(D_\mu \phi)^\dagger (D_\mu \phi) \right].$$

Action for gauge field.

$F_{\mu\nu} \to$

$$A_\mu \to U A_\mu U^{-1} - i\partial_\mu U U^{-1}$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu U A_\nu U^{-1} + U\partial_\mu A_\nu U^{-1} + U A_\nu \partial_\mu U^{-1}$$

$$- \partial_\nu U A_\mu U^{-1} - U\partial_\nu A_\mu U^{-1} - U A_\mu \partial_\nu U^{-1}$$

$$-i\cancel{\partial_{\mu\nu} U U^{-1}} - i\partial_\nu U \partial_\mu U^{-1}$$

$$+i\cancel{\partial_{\mu\nu} U U^{-1}} + i\partial_\mu U \partial_\nu U^{-1}$$

$$A_\mu A_\nu - A_\nu A_\mu \to U[A_\mu, A_\nu]U^{-1} - i\partial_\mu U \cancel{U^{-1}U} A_\nu U^{-1} - iU A_\mu U^{-1}\partial_\nu U U^{-1}$$

$$- \partial_\mu U U^{-1}\partial_\nu U U^{-1} + i\partial_\nu U \cancel{U^{-1}U} A_\mu U^{-1} + iU A_\nu U^{-1}\partial_\mu U U^{-1}$$

$$+ \partial_\nu U U^{-1}\partial_\mu U U^{-1}$$

$$-i[A_\mu, A_\nu] \to -iU[A_\mu, A_\nu]U^{-1} - \partial_\mu U A_\nu U^{-1} + U A_\nu \partial_\mu U^{-1}$$

$$+i\partial_\mu U \partial_\nu U^{-1} + \partial_\nu U A_\mu U^{-1} - U A_\nu \partial_\mu U^{-1}$$

$$+i\partial_\nu U \partial_\mu U^{-1}$$

$F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$

$\mathrm{Tr}\, F_{\mu\nu}^2$.

$$\mathcal{L} = \frac{1}{4g_{YM}} \mathrm{Tr}\, F^2 + |D\phi|^2 + \mathrm{Tr}\,(D\Phi)^\dagger (D\Phi) +$$
$$+ \bar{\psi} \not{D} \psi + \lambda \bar{\psi} \Phi \psi - \frac{1}{2} m^2 \phi^\dagger \phi$$
$$- \frac{1}{2} m_\Phi^2 \mathrm{Tr}\, \Phi^\dagger \Phi - \lambda_\phi (\phi^\dagger \phi)^2 - \lambda_\Phi \mathrm{Tr}(\Phi^\dagger \Phi \Phi^\dagger \Phi)$$

Notice $\phi\phi^\dagger \to$ adjoint $\sim \Phi$

e.o.m.

$$\delta\left(\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}\right)$$

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu - i\,[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i\, A^{af}_\mu A^{fb}_\nu + i A^{af}_\nu A^{fb}_\mu$$

$$\frac{\delta F^{ab}_{\mu\nu}(x)}{\delta A^{cd}_\alpha(y)} =$$

by parts

$$\partial_\mu A^{ab}_\nu F^{ba}$$

$$\int d^4x\; A^{ab}_\nu \partial_\mu F^{ba\mu\nu} - A^{ab}_\mu \partial_\nu F^{ba\mu\nu}$$

$$-i \int d^4x\; \mathrm{Tr}\, A_\mu A_\nu F^{\mu\nu} + i \int d^4x\; \mathrm{Tr}(A_\nu A_\mu F^{\mu\nu})$$

$$2 \int d^4x\;\; \partial_\mu A_\nu F^{\mu\nu} - \cancel{2} i\, A_\mu A_\nu F^{\mu\nu}$$

$$-\cancel{2} \int d^4x\; A_\nu \partial_\mu F^{\mu\nu}\; \cancel{- i A_\nu F^{\mu\nu} A_\mu - i A_\mu A_\nu F^{\mu\nu}}$$

$$\frac{\delta}{\delta A_\alpha} \qquad -\cancel{2}\, \partial_\mu F^{\mu\alpha} - i\, A_\nu F^{\alpha\nu} - i\, F^{\mu\alpha} A_\mu$$

$$-\partial_\mu F^{\mu\alpha} + i\, A_\nu F^{\nu\alpha} - i\, F^{\mu\alpha} A_\mu$$

$$-\left(\partial_\mu F^{\mu\alpha} - i\,[A_\mu, F^{\mu\alpha}]\right) = -D_\mu F^{\mu\alpha}$$

$$\frac{\delta}{\delta A_\mu(x)}\left(\frac{1}{4g_{ym}^2}\int d^4x\, F_{\mu\nu}F^{\mu\nu}\right) = \frac{1}{g_{ym}^2} D_\mu F^{\mu\nu}$$

$A_\mu^\dagger = A_\mu$

$$\int d^4x\, (D_\mu\phi)^\dagger (D_\mu\phi) = \int d^4x\, (\partial_\mu\phi + iA_\mu\phi)^\dagger(\partial_\mu\phi + iA_\mu\phi)$$

$$\int d^4x\, (\partial_\mu\phi)^\dagger(\partial_\mu\phi) + \int d^4x\, i\left[(\partial_\mu\phi)^\dagger A_\mu\phi - i\phi^\dagger A_\mu\partial_\mu\phi\right]$$

$(\partial_\mu\phi)^*_a A_\mu^{ab}\phi_b$

$$+\int d^4x\, \phi^\dagger_a A_\mu{}_{ab} A^\mu{}_{bc}\phi_c$$

$$\frac{\delta}{\delta A_\mu^{ab}} = i\left(\phi\cdot\partial_\mu\phi^\dagger - \partial_\mu\phi\,\phi^\dagger\right) + 2A^\mu\phi\phi^\dagger + \\ + \phi\phi^\dagger A_\mu$$

$$= i\phi(\partial_\mu\phi^\dagger - i\phi^* A_\mu) - i(\partial_\mu\phi + iA^\mu\phi)\phi^\dagger$$

$$= i\phi D_\mu\phi^\dagger - iD_\mu\phi\,\phi^\dagger$$

$$\frac{1}{g_{ym}^2} D_\mu F^{\mu\nu} + i\left(\phi D_\mu\phi^\dagger - D_\mu\phi\,\phi^\dagger\right) = 0$$

non-abelian current

gauge field $\bar\psi_a A^\mu_{ab}\gamma_\mu\psi_b$

$\bar\psi_a\gamma_\mu\psi_b$

$$D_\mu F^{\mu\nu} = i g^2_{YM} \left(\phi D_\nu \phi^\dagger - D_\nu \phi\, \phi^\dagger \right)$$

$$D_\nu D_\mu F^{\mu\nu} = \partial_\nu \left(D_\mu F^{\mu\nu} \right) - i[A_\nu, D_\mu F^{\mu\nu}]$$

$$= \underbrace{\partial_{\nu\mu}}_{\text{symm.}} \overset{0}{\cancel{F^{\mu\nu}}} - i\, \partial_\nu \left([A_\mu, F^{\mu\nu}] \right) - i[A_\nu, \partial_\mu F^{\mu\nu}] -$$

$$- [A_\nu, [A_\mu, F^{\mu\nu}]]$$

$$= -i [\partial_\nu A_\mu, F^{\mu\nu}] - i\, \cancel{[A_{\mu}, \partial_{\nu} F^{\mu\nu}] - i [A_\nu, \partial_\mu F^{\mu\nu}]}$$

$$- [A_\nu, [A_\mu, F^{\mu\nu}]]$$

$$= -\frac{i}{2} [\partial_\nu A_\mu - \partial_\mu A_\nu, F^{\mu\nu}]$$

$$[A_\nu [A_\mu, F^{\mu\nu}]] + [F^{\mu\nu} [A_\nu A_\mu]] + [A_\mu [F^{\mu\nu} A_\nu]] = 0$$

$$[A_\nu [A_\mu, F^{\mu\nu}]] + [A_\nu, [A_\mu, F^{\nu\mu}]] = + [[A_\nu, A_\mu], F^{\mu\nu}]$$

$$[A_\nu [A_\mu, F^{\mu\nu}]] = \frac{1}{2} [[A_\nu, A_\mu], F^{\mu\nu}]$$

$$D_\nu D_\mu F^{\mu\nu} = -\frac{i}{2} \left[(\partial_\nu A_\mu - \partial_\mu A_\nu + i [A_\nu, A_\mu]), F^{\mu\nu} \right]$$

$$= -\frac{i}{2} [F_{\nu\mu}, F^{\mu\nu}] = \frac{i}{2} [F_{\mu\nu}, F^{\mu\nu}] = 0.$$

$\Rightarrow D_\mu J^\mu = 0 \qquad J^\mu = i g_{ym}^2 \left(\phi D_\mu \phi^\dagger - D_\mu \phi\, \phi^\dagger \right)$

$D_\mu J^\mu = \partial_\mu J^\mu - i \left[A_\mu, J^\mu \right]$

$J^\mu \to U J^\mu U^\dagger \quad \Rightarrow$ adjoint!

Take $a = \phi \cdot \tilde{\phi}^\dagger$

$$D_\mu a = \partial_\mu \phi\, \tilde{\phi}^\dagger + \phi\, \partial_\mu \tilde{\phi}^\dagger - i A_\mu \phi\, \tilde{\phi}^\dagger + i \phi\, \tilde{\phi}^\dagger A_\mu$$

$$= \left(\partial_\mu \phi - i A_\mu \phi \right) \tilde{\phi}^\dagger + \phi \left(\partial_\mu \tilde{\phi}^\dagger + i \tilde{\phi}^\dagger A_\mu \right)$$

$$= D_\mu \phi\, \tilde{\phi}^\dagger + \phi \left(D_\mu \tilde{\phi} \right)^\dagger$$

$\uparrow$ Leibniz rule!

$$D_\mu J^\mu = i g_{ym}^2 \left(\cancel{D_\mu \phi D^\mu \phi^\dagger} + \phi D_\mu D^\mu \phi^\dagger - \cancel{D_\mu \phi D^\mu \phi^\dagger} - D_\mu D^\mu \phi\, \phi^\dagger \right)$$

e.o.m. ϕ:

$$\int d^4x \left(D_\mu \phi \right)^\dagger D_\mu \phi - \int d^4x\, m^2 \phi^\dagger \phi$$

$$D_\mu D^\mu \phi = m^2 \phi \qquad D_\mu \frac{\delta L}{\delta D^\mu \phi} = \frac{\delta L}{\delta \phi}.$$

Let's see:

$$\int d^4x \; (\partial_\mu \phi^\dagger + i\, \phi^\dagger A_\mu)\, D_\mu \phi \; - \int d^4x\, m^2 \phi^\dagger \phi$$

$\downarrow$

int. by part

$$\int d^4x \left(-\phi^\dagger \partial_\mu D_\mu \phi + i\, \phi^\dagger A_\mu D_\mu \phi\right) - \int d^4x\, m^2 \phi^\dagger \phi$$

$$- \phi^\dagger \big(\underbrace{\partial_\mu D_\mu \phi - i A_\mu D_\mu \phi}_{D_\mu D^\mu \phi} \big)$$

$$S = -\int d^4x \;\; \phi^\dagger \; (D_\mu D^\mu \phi + m^2 \phi)$$

$$\boxed{D_\mu D^\mu \phi + m^2 \phi = 0}$$

$$\Rightarrow \; D_\mu J^\mu = i g_{YM}^2 \;\; m^2 (- \phi \phi^\dagger + \phi \phi^\dagger) = 0 \; \checkmark$$

$$\partial_\mu J^\mu - i\, [A_\mu, J^\mu] = 0$$

$$\partial_\mu J^\mu = i\, [A_\mu, J^\mu] \; \neq 0.$$

But $\quad \partial_\mu F^{\mu\nu} - i\, [A_\mu, F^{\mu\nu}] = J^\nu \quad \Rightarrow \quad \tilde{J}^\nu = \overbrace{i\, [A_\mu, F^{\mu\nu}]}^{\text{gauge field contributes to the current!!}} + J^\nu$

$\partial_\nu \tilde{J}^\nu = 0.$

Energy momentum

Coupling to an external metric.

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$g = \det g_{\mu\nu}$

$$S_{YM} = \frac{1}{4g^2_{YM}} \int d^4x \sqrt{g}\; g^{\mu\nu} g^{\alpha\beta}\, \mathrm{Tr}(F_{\mu\alpha} F_{\nu\beta})$$

Interesting : $g_{\mu\nu} \to e^{\Omega} g_{\mu\nu}$ $\qquad g^{\mu\nu} \to e^{-\Omega} g_{\mu\nu}$

$\det g \to e^{4\Omega} g_{\mu\nu}$ $\qquad \sqrt{g} \to e^{2\Omega}\sqrt{g}$.

$\Rightarrow$ S_{YM} is invariant! (only classically).

$$T_{\mu\nu} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}}$$ gives a gauge inv. $T_{\mu\nu}$.

Also canonical.

$$T_{\mu\nu} = \frac{\delta S}{\delta \partial_\mu A_\alpha} \partial_\nu A^\alpha - \eta_{\mu\nu} \mathcal{L}.$$

$$\frac{\delta S}{\delta \partial_\mu A_\alpha} = \frac{1}{g^2_{YM}} F^{\mu\alpha} \Rightarrow T_{\mu\nu} = \frac{1}{g^2_{YM}} F_{\mu\alpha} \partial_\nu A^\alpha - \eta_{\mu\nu} \mathcal{L}.$$

$$\tilde{T}_{\mu\nu} = \frac{1}{g^2_{YM}} \mathrm{Tr}\left(F_{\mu\alpha} \partial_\nu A^\alpha - \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) \to \text{not gauge inv!}$$

But D_ν $\partial_\nu A^\alpha - \partial_\alpha A_\nu + i[A_\nu, A_\alpha] = F_{\nu\alpha}$.

would give

$$\boxed{T_{\mu\nu} = \frac{1}{g^2_{YM}} \operatorname{Tr}\left(F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4}\eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right)}$$

we need extra term.

$$-\frac{1}{g^2_{YM}} \operatorname{Tr}\left(F_{\mu\alpha}\partial_\alpha A_\nu + i F_{\mu\alpha}[A_\nu, A_\alpha]\right)$$

$$F_{\mu\alpha} A_\nu A_\alpha - F_{\mu\alpha} A_\alpha A_\nu$$

$$A_\alpha F_{\mu\alpha} A_\nu - F_{\mu\alpha} A_\alpha A_\nu$$

$$[A_\alpha, F_{\mu\alpha}] A_\nu$$

$$-\frac{1}{g^2_{YM}} \partial_\alpha \operatorname{Tr}\left(F_{\mu\alpha} A_\nu\right) + \frac{1}{g^2_{YM}} \operatorname{Tr}\left(\partial_\alpha F_\mu{}^\alpha A_\nu - i[A_\alpha, F_{\mu\alpha}]A_\nu\right)$$

$$\operatorname{Tr}(D_\alpha F_\mu{}^\alpha A_\nu)$$
$$\overset{\|}{0} \quad (\text{e.o.m.})$$

we are adding a total

derivative!