Review of spin 1/2 fermions, Dirac, Weyl, Majorana.

$SO(3,1) \equiv SL(2,\mathbb{C})$ Lorentz group.

$$\boxed{x'_\mu = \Lambda_\mu{}^\nu x_\nu}$$

$$\mathbb{X} = \begin{pmatrix} x_0 + x_3 & x_1 - i x_2 \\ x_1 + i x_2 & x_0 - x_3 \end{pmatrix} = x_0 \mathbb{1} + x_j \sigma^j$$

$\mathbb{X}^+ = \mathbb{X}$, $\det \mathbb{X} = x_0^2 - x_3^2 - x_1^2 - x_2^2$

$\tilde{\mathbb{X}} = U \mathbb{X} U^+$, $U \in SL(2,\mathbb{C})$

$\tilde{\mathbb{X}}^+ = \tilde{\mathbb{X}}$ $\det \tilde{\mathbb{X}} = \det \mathbb{X}$ $\Rightarrow$ $SL(2,\mathbb{C})$ Lorentz transf..

spinor $\tilde{\xi} = U \xi$ left ; $U = e^{i \frac{\theta_i}{2} \sigma^i + \frac{\beta_i}{2} \sigma_i}$ rot boosts.

but also $\hat{\eta} = U^* \eta$ right

conjugate representation. Inequivalent rep.

$$SU(2) \Rightarrow \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} = U \qquad U^* = \begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix} = \underset{S}{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix} \underset{S^{-1}}{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

change of basis.

$$SL(2,\mathbb{C}) \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = U \qquad U^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}$$ no change of basis.

$$\tilde{\mathbb{X}}_{\alpha\dot{\alpha}} = U_{\alpha\beta} \mathbb{X}_{\beta\dot{\beta}} U_{\dot{\alpha}\dot{\beta}} = U_{\alpha\beta} U_{\dot{\alpha}\dot{\beta}} \mathbb{X}_{\beta\dot{\beta}}$$

ξ_α ; $\eta_{\dot{\alpha}}$, $\mathbb{X}_{\alpha\dot{\alpha}}$; $\xi_\alpha \to (\frac{1}{2}, 0)$ $\eta_{\dot{\alpha}} \to (0, \frac{1}{2})$ $\mathbb{X}_{\alpha\dot{\alpha}} \to (\frac{1}{2}, \frac{1}{2})$

$\xi, \eta \rightarrow$ Weyl ~~fermions~~/ spinors.

$\xi_L \; ; \; \eta_R$

$\tilde{\xi} = U\xi \qquad \tilde{\xi}^* = U^* \xi^* \quad \Rightarrow \quad \xi^*$ is a right ~~fermion~~. spinor

η^* is a left ~~fermion~~. spinor.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma_j^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_1 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2 \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_3 \end{cases}$$

$\sigma_2^* = -\sigma_2$

$\Rightarrow \; S\sigma_j S^{-1} = -\sigma_j^*$

$$U^* = e^{-i\frac{\theta_j}{2}(\sigma_j)^* + \frac{\beta_j}{2}\sigma_j^*} = S\, e^{i\frac{\theta_j}{2}\sigma_j - \frac{1}{2}\beta_j\sigma_j}\, S^{-1}$$

We can use $\tilde{U} = e^{i\frac{\theta_j}{2}\sigma_j - \frac{1}{2}\beta_j\sigma_j}$

$\eta \rightarrow \tilde{U}\eta$: more conventional.

Parity: $0 \rightarrow 0 \quad j \rightarrow -j$: rotations same boost $\rightarrow$ $-$boosts.

$U \rightleftarrows \tilde{U} \quad \Rightarrow \quad \boxed{L \rightleftarrows R}$

parity interchanges left and right representations.

We need both in a parity invariant theory.

Take a pair $(\xi, \tilde{\eta})$

•) Dirac fermion: a pair of independent spinors (left & right

$$\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \qquad \psi \to \begin{pmatrix} e^{i\theta_i \frac{\sigma_i}{2} + \beta_i \frac{\sigma_i}{2}} & 0 \\ 0 & e^{i\theta_i \frac{\sigma_i}{2} - \beta_i \frac{\sigma_i}{2}} \end{pmatrix} \psi .$$

•) Majorana fermion: $\tilde{\eta} = S\xi^*$. because ξ^* is right handed.

$(\xi, S\xi^*)$

At this level Majorana & Weyl are the same. The difference is in the Lagrangian. ~~If (ξ, ξ^*) appear then it is one~~

Kinetic term: $\cancel{(\xi^\alpha)\partial_{\alpha\dot\alpha}(\xi^{*\dot\alpha})}$ $\underbrace{\underbrace{\xi^\alpha \partial_{\alpha\dot\alpha} \xi^{*\dot\alpha}}_{\text{Weyl / Majorana}} - \eta^{\dot\alpha} \partial_{\dot\alpha\alpha} (\eta^*)^\alpha}_{\text{Dirac.}}$

$\cancel{\xi^\alpha \xi^{*\dot\alpha}}$

Mass term:

Dirac: $(\xi^*)^{\dot\alpha} \eta_{\dot\alpha} + (\xi)^\alpha (\eta^*)_\alpha$ ← invariant.

Majorana: $\xi^\alpha \xi_\alpha + (\xi^*)^{\dot\alpha} (\xi^*)_{\dot\alpha}$ ← cannot have charge. Not invariant

Weyl: no mass.

Charge

$\xi \to e^{i\alpha}\xi \quad \eta \to e^{i\alpha}\eta$

Also $SU(N) \to$ Not invariant

$SU(2)$ exception

States.

Massive Dirac: At rest $\uparrow\downarrow$ e^- $\uparrow\downarrow$ e^+

Massive Majorana At rest $\uparrow\downarrow$ antiparticle is the same.

Massless Weyl $k\uparrow \Uparrow$ antiparticle $\uparrow\Downarrow$

Dirac.
$$\psi = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(e^{-ikx}\sum_\sigma u_k^\sigma a_k^\sigma + \sum_\sigma e^{+ikx} v_k^\sigma b_{k,\sigma}^+\right)$$
$$\bar\psi = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\, e^{ikx}\sum_\sigma \bar u_{k,\sigma} a_{k,\sigma}^+ + \sum_\sigma e^{-ikx}\bar v_{k,\sigma} b_{k,\sigma})$$

Majorana
$$\psi = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(e^{-ikx}\sum_\sigma a_{k,\sigma}^\sigma u_\sigma + \sum_\sigma u_k^{*\sigma} e^{+ikx} a_{k,\sigma}^+\right)$$ reality cond.

$$\psi^* = \begin{pmatrix}\xi^* \\ \tilde\eta^*\end{pmatrix} = \begin{pmatrix} S^{-1}\tilde\eta \\ S\xi\end{pmatrix} = \begin{pmatrix} 0 & S^{-1} \\ S & 0\end{pmatrix}\begin{pmatrix}\xi \\ \tilde\eta\end{pmatrix}$$

$\tilde\eta = S\xi^*$

Weyl $\psi = \int \qquad a + \qquad b^+$

only one polarization

Symmetries in field theory.

Noether theorem → conserved current. e.g. $SU(N)$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_a)^\dagger \partial^\mu \phi^a - \frac{1}{2} m^2 \phi_a^\dagger \phi_a - \frac{\lambda}{4!}(\phi_a^\dagger \phi_a)^2$$

$$+ i \bar{\psi}_a \not{\partial} \psi_a - m \bar{\psi}_a \psi_a$$

$$\phi_a \to U\phi \qquad \psi \to U\psi \qquad U = e^{i\lambda_A t^A}$$

$$\delta\phi = i\lambda_A t^A \phi \qquad \delta\phi^* = \cancel{-i\lambda_A (t^A)} = -i\phi^\dagger \lambda_A t^A$$

$$\delta\psi = i\lambda_A t^A \psi$$

$$j^\mu = \frac{\delta h}{\delta \partial_\mu \phi_A} \delta\phi_A$$

$$\partial_\mu j^\mu = \partial_\mu \frac{\delta h}{\delta \partial_\mu \phi_A} \delta\phi_A + \frac{\delta h}{\delta \partial_\mu \phi_a} \delta \partial_\mu \phi_A$$

$$= \frac{\delta h}{\delta \phi_A} \delta\phi_A + \frac{\delta h}{\delta(\partial_\mu \phi_A)} \delta\partial_\mu \phi_A = 0 = \delta\mathcal{L}$$

$$j_\mu = \frac{1}{2}\partial_\mu\phi^\dagger i\lambda_A t^A\phi - \frac{i}{2}\phi^\dagger\lambda_A t^A\partial_\mu\phi +$$

$$+ i\,\bar\psi\gamma^\mu i\lambda_A t^A\psi$$

$$j_\mu^A = \frac{i}{2}\left(\partial_\mu\phi_a^\dagger t^A_{ab}\phi_b - \phi_a^\dagger t^A_{ab}\partial_\mu\phi_b\right) - \bar\psi_a\gamma^\mu t^A_{ab}\psi_b$$

$$Q^A = \int d^3x\; j_0^A$$

$$j^0 = \frac{\delta h}{\delta\dot\phi_A}\,\delta\phi_a = \Pi_a\cdot\delta\phi_a$$

$$Q_A = \int d^3x\;\Pi_\phi\,\delta_A\phi \qquad ; \quad \delta_A\phi = i t^A\phi$$

$$[Q_A, Q_B] = \int d^3x\int d^3x'\;[\underbrace{\Pi_\phi\,\delta_A\phi}_{x}, \underbrace{\Pi_\phi\,\delta_B\phi}_{x'}]$$

$$= \int d^3x\int d^3x'\;[\Pi_\phi^a\, i t_A^{ab}\phi_b, \Pi_\phi^c\, i t_B^{cd}\phi_d]$$

$$= \int d^3x\int d^3x'\Big[(-i)\,\delta^{ad}\,\delta(x-x')\,\phi_b\,\Pi_c\,(-)\,t_A^{ab}\,t_B^{cd} +$$

$$+ (-)\,i\,\delta^{bc}\,\Pi^a\phi_d\,t_A^{ab}\,t_B^{cd}\Big]$$

$$= \int d^3x \; i \left(t_B^{\overset{a\,c}{ca}}\, t_A^{\overset{c}{ab}}\, \phi_b \Pi_{\underset{a}{c}} - t_A^{\overset{c}{ab}}\, t_B^{\overset{c\,b}{bd}}\, \Pi^a \phi_{\underset{b}{d}} \right)$$

$$= i \int d^3x \; [t_A, t_B]^{ab} \phi_b \Pi_a = i\, i f_{ABC}\, t^C_{ab} \int d^3x \; \Pi_a \phi_b$$

$$= f_{ABC}\, Q_C$$

charge algebra same as generators.

Current algebra.

same calc

$$[\rho^A, j_0^B] = \int d^3x \; [\quad] \; - - = f_{ABC}\, j_0^C$$

$$[Q_A, \phi] = -i\delta\phi$$

$$[Q_A, j_j^B] = \int d^3x \left[\Pi t^A \phi, \; \frac{i}{2}\left(\partial\phi^\dagger t\phi - \phi^\dagger t \partial\phi \right) \right]$$

$$= \int d^3x$$

$$[j^A_{0\,x}, \vec{J}^B_y] = \frac{i}{2}\frac{i}{2}\, t^A_{ab}\, t^B_{cd}\, [\pi_a \phi_b - \pi^*_b \phi^*_a,$$

$$\nabla\phi^*_c \phi_d - \phi^*_c \nabla\phi_d]$$

$$= +\frac{i}{4} t^A_{ab} t^B_{cd} \Big(\delta(x-y)\, \delta_{ad}\, \underset{x}{\phi^*_b}\, \underset{y}{\nabla\phi^*_c} - \nabla_y \delta(x-y)\, \delta_{ad}\, \underset{x}{\phi_b}\, \underset{y}{\phi^*_c} -$$

$$- \nabla_y \delta(x-y)\, \delta_{bc}\, \underset{x}{\phi^*_a}\, \underset{y}{\phi_d} + \delta_{bc}\, \delta(x-y)\, \underset{x}{\phi^*_a}\, \underset{y}{\nabla\phi_d} \Big)$$

$$= \frac{i}{4}\delta(x-y) \Big(\underbrace{t^A_{db} t^B_{cd}}_{(t^B t^A)_{cb}}\, \underset{x}{\phi_b}\, \underset{y}{\nabla\phi^*_c} + \underbrace{t^A_{ac} t^B_{cb}}_{(t^A t^B)_{ab}}\, \underset{x}{\phi^*_a}\, \underset{y}{\nabla\phi_b} \Big) -$$

$$- \frac{i}{4}\nabla_y \delta(x-y) \Big(t^A_{db} t^B_{ad}\, \underset{x}{\phi_b}\, \underset{y}{\phi^*_a} + t^A_{ac} t^B_{cb}\, \underset{x}{\phi^*_a}\, \underset{y}{\phi_b} \Big)$$

$$= \frac{i}{4}\delta^{(3)}(x-y)\, (t^B t^A)_{ab}\, \underset{y}{\nabla\phi^*_a}\, \underset{x}{\phi_b} + \frac{i}{4}\delta^{(3)}(x-y)\, (t^A t^B)_{ab}\, \underset{x}{\phi^*_a}\, \underset{y}{\nabla\phi_b}$$

$$- \frac{i}{4}\nabla_y \delta(x-y)\, (t^B t^A)_{ab}\, \underset{y}{\phi^*_a}\, \underset{x}{\phi_b} - \frac{i}{4}\nabla_y \delta(x-y)\, (t^A t^B)_{ab}\, \underset{x}{\phi^*_a}\, \underset{y}{\phi_b}$$

$$\int_y \nabla_y \delta(x-y)\, f(y)\, F(y) = -\nabla_y (fF)\Big|_b = -\underset{0}{\nabla_y f}\, \underset{0}{F} - \underset{0}{f}\, \underset{0}{\nabla_y F}$$

$$-\nabla f(x)\, \delta(x-y) + f(x)\nabla_y \delta(x-y)$$

$$\nabla_y \delta(x-y)\, f(x) = \nabla_y \delta(x-y)\, f(y) + \delta(x-y)\, \nabla f(x)$$

$$\nabla_y \delta(x-y) = -\nabla_x \delta(x-y)$$

$$[\underset{x}{j_0^A}, \underset{y}{\vec{j}^B}] = \frac{i}{4}\delta^{(3)}(x-y)\,(t^B t^A)_{ab}\,\nabla\underset{y}{\phi_a^*}\underset{y}{\phi_b} + \frac{i}{4}\delta^{(3)}(x-y)\,(t^A t^B)_{ab}\,\underset{x}{\phi_a^*}\nabla\underset{x}{\phi_b}$$

$$-\frac{i}{4}\delta^{(3)}(x-y)\,(t^B t^A)_{ab}\,\underset{y}{\phi_a^*}\nabla\underset{y}{\phi_b} - \frac{i}{4}\delta^{(3)}(x-y)\,(t^A t^B)_{ab}\,\nabla\underset{y}{\phi_a^*}\underset{y}{\phi_b}$$

$$+\frac{i}{4}\nabla_x\delta(x-y)\,(t^B t^A)_{ab}\underbrace{\phi_a^*\phi_b}_{y} + \frac{i}{4}\nabla_x\delta(x-y)\,(t^A t^B)_{ab}\,\underset{y}{\phi_a^*}\underset{y}{\phi_b}$$

$$= \frac{i}{4}\,i f^{BAC}\delta^{(3)}(x-y)\,t^C_{ab}\,\nabla\phi_a^*\phi_b + \frac{i}{4}\,i f^{ABC}\,t^C_{ab}\,\phi_a^*\nabla\phi_b$$

$$+\frac{i}{4}\nabla_x\delta(x-y)\,(t^B t^A + t^A t^B)_{ab}\underbrace{\phi_a^*\phi_b}_{y}$$

$$= -\frac{i}{2} f^{ABC}\,\vec{j}^C(y)\,\delta(x-y) + \underbrace{\frac{i}{4}\nabla_x\delta(x-y)\,(t^B t^A + t^A t^B)_{ab}\,\phi_a^*\phi_c}_{\int_x = 0.}$$

$$[Q^A, \vec{j}^B_y] = -\frac{i}{2} f^{ABC}\,\vec{j}^C.$$

Schwinger terms.

If

$$[j_0^A, \vec{j}_i^B] = f^{ABC} j_i^C$$

$$[j_0^A, \partial_0 j_i^B] = f^{ABC}\partial_i j_i^C$$

$$[j_0^A, -\partial_0 j_0^B] = f^{ABC}\partial_i j_i^C$$

$$[j_0^A, \partial_0 j_0^A] = 0$$

$$\cancel{\sum_n \langle 0|j_0\partial_0 j_i|n\rangle\langle n|\partial_i j_i j_0|0\rangle}$$

$$\langle 0|j_0\partial_i j_0|0\rangle - \langle 0|\partial_0 j_0 j_0|0\rangle$$

$$\sum_n \langle 0|j_0|n\rangle\langle n|\partial_0 j_0|0\rangle - \langle 0|\partial_0 j_0|n\rangle\langle n|j_0|0\rangle$$

$$\partial_0 j_0 = -i[H, j_0]$$

$$\langle 0|\partial j_0|n\rangle = +iE_n\langle 0|j_0|n\rangle$$

$$\langle n|\partial_0 j_0|0\rangle = -iE_n\langle n|j_0|0\rangle$$

$$\langle 0| j_0^A \partial_x j_1^A |0\rangle = \sum_n 2i\, E_n \, |\langle 0| j_0 |n\rangle|^2 \geq 0.$$

$x \to y$

No renormalization of currents.

Spontaneous symmetry breaking (global symmetry)

$$U H_0 U^\dagger = H_0$$

$$U \phi_A U^\dagger = \phi_B$$

$$|A\rangle = \phi_A |0\rangle \qquad |B\rangle = \phi_B |0\rangle$$

$$U|A\rangle = U\phi_A U^{-1} U|0\rangle = \phi_B U|0\rangle$$

if $|0\rangle$ is invariant $\quad U|0\rangle = |0\rangle$

then

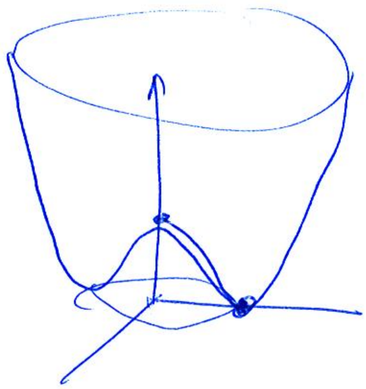
$$U|A\rangle = \phi_B|0\rangle = |B\rangle$$

$$\langle A|H_0|A\rangle = \underbrace{\langle A|U^\dagger}_{\langle B|}\underbrace{U H_0 U^\dagger}_{H_0}\underbrace{U|A\rangle}_{|B\rangle} = \langle B|H_0|B\rangle$$

multiplets of particles in rep. of the symmetry

if $U|0\rangle \neq |0\rangle$ then we relate states constructed around different vacua. $U|0\rangle$ has same energy as $|0\rangle$ and ~~so does~~ $U|A\rangle$ has same energy as $|A\rangle$ but they represent fluctuations around different vacua.

Goldstone theorem: There are massless scalar fields that represent the local fluctuations of one vacuum towards the other.



$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 - V(\sigma^2+\pi^2)$$

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4$$

O(2) symmetry $\begin{pmatrix}\sigma\\ \pi\end{pmatrix} \to \begin{pmatrix} c\alpha & s\alpha \\ -s\alpha & c\alpha\end{pmatrix}\begin{pmatrix}\sigma\\ \pi\end{pmatrix}$

$$\frac{\delta V}{\delta \phi} = -\mu^2\phi + \lambda\phi^3 = 0$$

$\phi = 0$

$\mu^2 = \lambda\phi^2$ $\quad \phi = \frac{\mu}{\sqrt{\lambda}}$

$$\boxed{\sigma^2+\pi^2 = \frac{\mu^2}{\lambda}}$$ v

pick $\langle 0|\sigma|0\rangle = \mu/\sqrt{\lambda} = v$ $\quad \langle 0|\pi|0\rangle = 0$

$$\sigma = v + \tilde{\sigma}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\tilde{\sigma})^2 + \frac{1}{2}(\partial_\mu\pi)^2 - V(v^2 + 2v\tilde{\sigma} + \tilde{\sigma}^2 + \pi^2)$$

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{\mu^2}{\lambda}\right)^2 - \frac{\mu^4}{4\lambda} = \frac{\lambda}{4}\left(2v\tilde{\sigma} + \tilde{\sigma}^2 + \pi^2\right)^2 - \frac{\mu^4}{4\lambda}$$

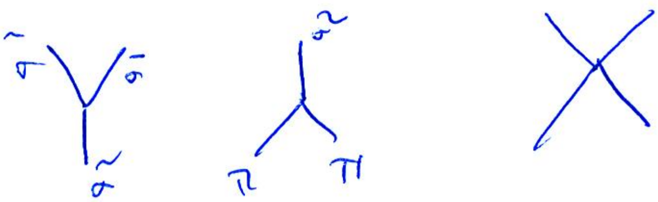
$$\phi^4 - \frac{2\mu^2}{\lambda}\phi^2 + \frac{\mu^4}{\lambda^2}$$

$$= \frac{\lambda}{4}\left(4v^2\tilde{\sigma}^2 + 4v\tilde{\sigma}^3 + 4v\tilde{\sigma}\pi^2 + (\tilde{\sigma}^2+\pi^2)^2\right) - \mu^4/4\lambda$$

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu \tilde{\sigma})^2 + \frac{1}{2}(\partial_\mu \pi)^2 - \lambda v^2 \tilde{\sigma}^2$$

$$- \lambda v \tilde{\sigma}^3 - \lambda v \tilde{\sigma} \pi^2 - \frac{\lambda}{4}(\tilde{\sigma}^2 + \pi^2)^2 - \frac{\mu^4}{4\lambda}$$

$\boxed{m_\sigma^2 = 2\lambda v^2}$ $\boxed{m_\pi = 0}$



$\boxed{m_\sigma^2 = 2\mu^2}$

$$J_\mu(x) = (\partial_\mu \pi\, \sigma - \partial_\mu \sigma\, \pi)$$

$$Q = \int J_0 = \int d^3x\, (\partial_0 \pi\, \sigma - \partial_0 \sigma\, \pi)$$

Formally $\quad 0 = \int d^3x\, [\partial_\mu J^\mu, \phi(0)] = \partial^0 \int d^3x\, [J_0, \phi(0)] + \underbrace{\oint_{\partial S} d\vec{S}\cdot[\vec{J}, \phi(0)]}_{\to 0.}$

$$\frac{d}{dt}[Q(t), \phi(0)] = 0.$$

if $\langle 0|[Q(t), \phi(0)]|0\rangle \equiv \eta \neq 0 \Rightarrow$ symmetry breaking.

$$\sum_n (2\pi)^3 \delta^{(3)}(p_n) \{\langle 0| J_0(0)|n\rangle \langle n|\phi(0)|0\rangle e^{-iE_n t} - \langle 0|\phi(0)|n\rangle\langle n|J_0(0)|0\rangle . e^{iE_n t}\} = \eta$$

we need $p_n = 0$ but also $E_n = 0 \Rightarrow$ time indep

$$\langle 0|Q(t)|\vec{p}, \nu\rangle = \int d^3x\, e^{-i\vec{p}\vec{x} + iEt} \langle 0|j^0(0)|\vec{p}, \nu\rangle = \delta^{(3)}(\vec{p})\, e^{iEt} \langle 0|j^0(0)|\vec{p}, \nu\rangle$$

there is a state $|n\rangle$; $E_n = 0 \cdot P_n = 0$.

such that $\langle n | \phi(0) | 0 \rangle \neq 0$

$\langle 0 | J_0(0) | n \rangle \neq 0$

→ created by J_0 should be rotationally invariant or spin 0.

Here $|n\rangle = |\pi\rangle$

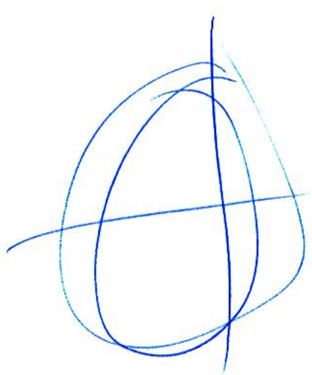
$$\langle 0 | J_0(0) | \pi(p) \rangle = i v p_0$$

$$\langle 0 | \pi(0) | \pi(p) \rangle = 1 \qquad (\langle 0 | J_\mu(0) | \pi(p) \rangle = i v p_\mu)$$

$$\langle 0 | \partial^\mu J_\mu(0) | \pi(p) \rangle = v m_\pi^2$$

$$v = \langle 0 | \sigma(0) | 0 \rangle = 0 \quad \text{or} \quad \underbrace{m_\pi = 0}_{\text{Goldstone boson.}}$$

G/H → one Goldstone boson for each broken generator

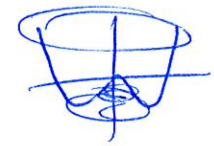


$SO(n)/SO(n-1)$

low energy interaction given by symmetry. ⇒ effective action

Properties of vacua & difference w/ quantum mechanics

In Q.M. the vacuum would have $J=0$



that violates cluster decomposition pple.

•)  spin chain example.

$$|\sigma(\theta,\phi)\rangle = c\tfrac{\theta}{2}\, e^{i\phi/2}\,|\uparrow\rangle + s\tfrac{\theta}{2}\, e^{-i\phi/2}\,|\downarrow\rangle$$

$\langle\sigma(0)|\sigma(\theta,0)\rangle = c\theta/2$ in general $\langle\sigma(\hat n_1)|\sigma(\hat n_2)\rangle = \cos\left(\frac{\theta(\hat n_1,\hat n_2)}{2}\right)$

$$\langle\sigma(\theta,\phi)|\sigma(\theta,\phi)\rangle = 1$$

Vacuum $|\psi(\theta,\phi)\rangle = \prod_i |\sigma(\theta,\phi)\rangle_i$

i ← sites

$$\langle\psi|\psi\rangle = 1$$

but $\langle\psi(\hat n_1)|\psi(\hat n_2)\rangle = \prod_i \cos\left(\frac{\theta(\hat n_1,\hat n_2)}{2}\right) = \left(\cos\left(\frac{\theta(\hat n_1, n_2)}{2}\right)\right)^N \to 0$

$N\to\infty$

they are orthogonal in the continuum limit.

•) The vacuum (or vacua) is a discrete state with $E=0$ $\vec P=0$ up to constant
i.e. normalizable.

$\langle 0|0\rangle = 1$ $\langle E,P_1|E,P_2\rangle = \delta^{(3)}(P_1-P_2)$

normalized to δ function
continuum of multiparticle states.

$$\mathbb{1} = \sum_{|v\rangle} |v\rangle\langle v| + \sum_{\nu} \int dE\, d^3\vec{P}\ |E,\vec{P},\nu\rangle\langle E,\vec{P},\nu|$$

$|v\rangle$ vacua

ν sum or integral.

ν other quantum numbers

Consider local operators. $A(x) = e^{-iP^\mu x_\mu} A(0)\, e^{iP^\mu x_\mu}$

$$\langle v_1 | A(x) B(0) | v_2\rangle = \sum_{|w\rangle} \langle v_1 | \underbrace{A(x)}_{= A(0)} | w\rangle \langle w | B(0) | v_2\rangle +$$

$$+ \sum_{\nu} \int dE\, d^3\vec{P}\ \underbrace{e^{iP^\mu x_\mu}}_{e^{iEt - i\vec{P}\vec{x}}} \langle v_1 | A(0) | E, P, \nu\rangle \langle E, P, \nu | B(0) | v_2\rangle$$

for a smooth function $\int d^3P\, e^{-i\vec{P}\vec{x}} f(\vec{P}) \xrightarrow[|\vec{x}|\to\infty]{} 0$

$|\vec{x}| \to \infty$

$$\langle v_1 | A(x) B(0) | v_2\rangle = \sum_{|w\rangle} \langle v_1 | A(0) | w\rangle \langle w | B(0) | v_2\rangle$$

so it is given by the product of the matrices on the vacua.

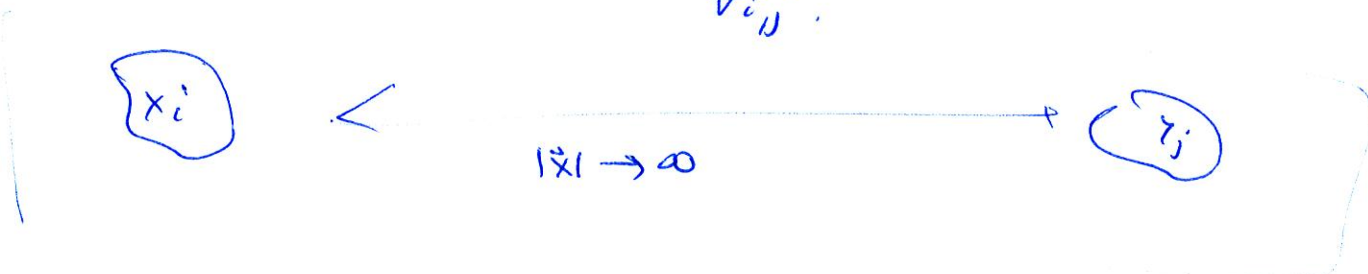
But $[A(x), B(0)] = 0$ for spacelike x.

$\Rightarrow$ matrices on vacua commute and can be diagonalized simultaneously for <u>all</u> local operators.

Such bases satisfies

·) Cluster decomposition pple

$$\langle 0|T\{\phi(x_1)\cdots\phi(x_n)\phi(y_1)\cdots\phi(y_m)\}|0\rangle \xrightarrow[\forall i,j.]{|x_i-y_j|\to\infty} \langle 0|T\{\phi(x_1)\cdots\phi(x_n)\}|0\rangle \cdot \langle 0|T\{\phi(y_1)\cdots\phi(y_m)\}|0\rangle$$



·) ~~It~~ they are stable under perturbation.

Having many vacua is intrinsically unstable since any small perturbation needs to be diagonalized and changes the vacuum

All off diagonal elements of local operators are zero

$\Rightarrow$ vacua are <u>stable</u> under local perturbations.

$|\phi\rangle$ where ϕ is well defined satisfy that property.

for example something like

$$|J\rangle = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\phi \; e^{iJ\phi} \, |\phi\rangle \qquad J \text{ integer.}$$

does not

$$\langle J_2 | \phi | J_1 \rangle = \int_0^{2\pi} d\phi \; e^{-i(J_2 - J_1)\phi} \, \phi \neq 0$$

so it is not diagonal.

$$\int_0^{2\pi} d\phi \; e^{-iJ\phi} \phi = i \frac{\partial}{\partial J} \left. \frac{e^{-iJ\phi}}{-iJ} \right|_0^{2\pi} = i \frac{\partial}{\partial J} \frac{e^{-i2\pi J} - 1}{-iJ}$$

$$= -\left(\frac{-2\pi i \, e^{-i\pi J}}{J} - \frac{1}{J^2} \left(e^{-2\pi i J} - J \right) \right) = \frac{2\pi i}{J}$$

J integer

Does not satisfy cluster decomposition pple. and not stable under local perturbations.

$$\langle J | \phi(x) \phi(0) | J \rangle = \frac{1}{2\pi} \int_0^{2\pi} d\phi_1 d\phi_2 \, e^{-iJ\phi_1} \langle \phi_1 | \phi(x) \phi(0) | \phi_2 \rangle \, e^{iJ\phi_2}$$

$\phi_1^2 \langle \phi_1 | \phi_2 \rangle$

$$= \frac{1}{2\pi} \int_0^{2\pi} d\phi \, \phi^2 \neq 0$$

indep. of position

Massive vector field

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + \frac{m^2}{2}A_\mu A^\mu = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{m^2}{2}A_\mu A^\mu$$

e.o.m.

$$\frac{\delta L}{\delta \partial_\mu A_\nu} = - F_{\mu\nu}$$

$$-\partial_\mu F^{\mu\nu} = m^2 A^\nu$$

$$-\partial_\mu^2 A_\nu + \partial_\mu \partial_\nu A_\mu - m^2 A^\nu = 0$$

propagator.

$$\left(-\partial^2 \eta_{\nu\alpha} + \partial_\alpha \partial_\nu - m^2 \eta_{\nu\alpha}\right) G_{\alpha\beta} = \delta(x-x')\,\eta_{\nu\beta}$$

$$\left(k^2 \eta_{\nu\alpha} - k_\alpha k_\nu - m^2 \eta_{\nu\alpha}\right) G_{\alpha\beta}(k) = \eta_{\nu\beta}$$

$$\mathbb{P}_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$

$$\left(k^2 \mathbb{P}_{\nu\alpha} - m^2 \mathbb{P}_{\nu\alpha} - m^2 \frac{k_\nu k_\alpha}{k^2}\right) G_{\alpha\beta} = \eta_{\nu\beta}$$

$$\left((k^2 - m^2)\mathbb{P}_{\nu\alpha} - m^2 \frac{k_\nu k_\alpha}{k^2}\right) G_{\alpha\beta} = \eta_{\nu\beta}$$

$$(A\mathbb{P} + B\, k\otimes k).(C\mathbb{P} + D\, k\otimes k) = AC\mathbb{P} + BDk^2\, k\otimes k = \eta$$

$$C = 1/A \quad ; \quad \mathbb{P} + BDk^2\, k\otimes k = \eta \qquad -\frac{k\otimes k}{k^2} + BDk^2\, k\otimes k = 0 \qquad \boxed{D = \frac{1}{k^4 B}}$$

$$G_{\alpha\beta} = \frac{1}{k^2-m^2}\, \mathbb{P}_{\alpha\beta} - \frac{\cancel{k^2}}{m^2 k^2}\, k_\alpha k_\beta$$

$$G_{\alpha\beta} = \frac{1}{k^2-m^2}\left(\eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right) - \frac{1}{m^2}\,\frac{k_\alpha k_\beta}{k^2}$$

$$= \frac{1}{k^2-m^2}\left(\eta_{\alpha\beta} \underbrace{- \frac{k_\alpha k_\beta}{k^2} - \frac{k^2-m^2}{k^2 m^2} k_\alpha k_\beta}\right)$$

$$1 + \frac{k^2-m^2}{m^2} = \frac{k^2}{m^2}$$

$$\boxed{G_{\alpha\beta} = \frac{1}{k^2-m^2}\left(\eta_{\alpha\beta} - \frac{k_\alpha k_\beta}{m^2}\right)}$$

$G(k) \sim 1$ (More appropriately $G(\xi k) \sim \xi^0$)
$k^2 \to \infty$ $\qquad \xi \to \infty$

not good for renormalizability!

theories with massive vector fields are
not renormalizable.

Exception: if mass comes from S.S.B. $\Rightarrow$ gauge symmetry controls divergences

Spontaneous breaking of gauge theories

Abelian Higgs model

$$\phi = \sigma + i\pi$$

$$\mathcal{L} = |D_\mu \phi|^2 - V(|\phi|^2) - \frac{1}{4g} F_{\mu\nu} F^{\mu\nu}$$

$$\phi = \frac{1}{\sqrt{2}} (v+\eta)\, e^{i\xi/v} \qquad \boxed{\xi(x)\,;\ \eta(x)}$$

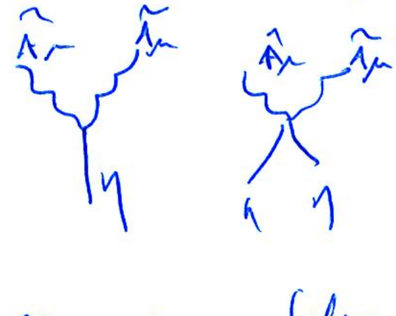
$$D_\mu \phi = \frac{1}{\sqrt{2}} \partial_\mu \eta \, e^{i\xi/v} + \frac{i}{v} \partial_\mu \xi \frac{1}{\sqrt{2}} (v+\eta)\, e^{i\xi/v} - ieA_\mu \frac{(v+\eta)}{\sqrt{2}} e^{i\xi/v}$$

$$= \frac{1}{\sqrt{2}} e^{i\xi/v} \left(\partial_\mu \eta + i \partial_\mu \xi (1+\eta/v) - ieA_\mu v (1+\eta/v) \right)$$

$$|D_\mu \phi|^2 = \frac{1}{2} (\partial_\mu \eta)^2 + \frac{1}{2} (1+\eta/v)^2 \underbrace{(\partial_\mu \xi - eA_\mu v)^2}_{ev\,(\underbrace{A_\mu - \frac{1}{ev}\partial_\mu \xi}_{\tilde{A}_\mu})}$$

$$= \frac{1}{2} (\partial_\mu \eta)^2 + \frac{e^2 v^2}{2} \left(1 + \frac{2\eta}{v} + \frac{\eta^2}{v^2}\right) \tilde{A}_\mu \tilde{A}^\mu + \frac{1}{4g} (\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)^2$$

$$m_{\tilde{A}} = ev$$



— m_η = same as before.

$$\mathcal{L} = \frac{1}{2}(D_\mu \Phi^\dagger) D_\mu \Phi - V(\phi^\dagger \phi) - \frac{1}{4g} F_\mu F^\mu$$

<u>SU(2)</u>

Φ : fundamental. $\quad D_\mu \phi = \partial_\mu \phi - i A_\mu \phi$

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

$$\Phi = \frac{1}{\sqrt{2}} U \begin{pmatrix} 0 \\ v+\eta \end{pmatrix}$$

SU(2) rot. → gives all Φ. → gauge away.

$$\cancel{D_\mu \Phi = \frac{1}{\sqrt{2}} \frac{1}{2} \frac{1}{2} (-i A_\mu \Phi)}$$

$$\cancel{(D_\mu \Phi)^\dagger (D_\mu \Phi)} =$$

$$D_\mu \Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \partial_\mu \eta \end{pmatrix} + \frac{1}{\sqrt{2}} i A_\mu \begin{pmatrix} 0 \\ v+\eta \end{pmatrix}$$

$$(D_\mu \Phi)^\dagger = \frac{1}{\sqrt{2}} (0 \ \partial_\mu \eta) - \frac{i}{\sqrt{2}} (0 \ \ v+\eta) A_\mu.$$

$$(D_\mu \Phi)^\dagger D_\mu \Phi = \frac{1}{2} (\partial_\mu \eta)^2 + \frac{i}{2} (0 \ \partial_\mu \eta) A_\mu \begin{pmatrix} 0 \\ v+\eta \end{pmatrix} -$$

$$- \frac{i}{2} (0 \ \ v+\eta) A_\mu \begin{pmatrix} 0 \\ \partial_\mu \eta \end{pmatrix} + \frac{1}{2} (0 \ \ v+\eta) A_\mu A^\mu \begin{pmatrix} 0 \\ v+\eta \end{pmatrix}$$

②

$$A_\mu = A_\mu^1 \sigma_1 + A_\mu^2 \sigma_2 + A_\mu^3 \sigma_3$$

$$(0 \ \partial_\mu \eta)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ v+\eta \end{pmatrix} = (\partial_\mu \eta \ \ 0)\begin{pmatrix} 0 \\ v+\eta \end{pmatrix} = 0$$

$$\begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix} \qquad = (i\partial_\mu \eta \ \ 0)\begin{pmatrix} 0 \\ v+\eta \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad = (0 \ \ -\partial_\mu \eta)\begin{pmatrix} 0 \\ v+\eta \end{pmatrix} = -v\partial_\mu \eta + \eta \partial_\mu \eta$$

$$(0 \ \ v+\eta)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ \partial_\mu \eta \end{pmatrix} = (0 \ \ -v-\eta)\begin{pmatrix} 0 \\ \partial_\mu \eta \end{pmatrix} = -(v+\eta)\partial_\mu \eta$$

$0.$

$$\cancel{A_\mu A^\mu = (A_\mu^1)^2}$$

$$(0 \ \ v+\eta)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 \\ v+\eta \end{pmatrix} = ((v+\eta)$$

$$(0 \ \ 1)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\gamma \ \ \delta)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \delta$$

$$(0 \ \ v+\eta)\ \sigma_3 \begin{pmatrix} 0 \\ v+\eta \end{pmatrix} = -(v+\eta)^2$$

$$\{\sigma_i, \sigma_j\} = 2$$

$$A_\mu^1 A_\mu^2$$

$$-\frac{1}{2}(v+\eta)^2 \qquad A_\mu^i A_\mu^j \ \sigma_i \sigma_j$$

$$+\frac{1}{2}(v+\eta)^2 \ (A_\mu^1 A^\mu_1 + A_\mu^2 A^\mu_2 + A_\mu^3 A_\mu^3)$$

$$-\frac{1}{2}(v+\eta)^2$$

ϕ : adjoint (vector of $SO(3)$)

$\boxed{SO(3)/SO(2)}$

vector rep.

$$D_\mu \phi_a = \partial_\mu \phi_a + g\,\epsilon_{abc} A_\mu^b \phi_c$$

$$\phi_3 = v$$

$$\vec{\phi} = R_{\theta\varphi}\begin{pmatrix}0\\0\\v+\eta\end{pmatrix}$$

$$D_\mu \phi_1 = g\,\epsilon_{1b3} A_\mu^b (v+\eta) = g\,A_\mu^{(2)}(v+\eta)$$

$$D_\mu \phi_2 = g\,\epsilon_{213} A_\mu^1 (v+\eta) = -g\,A_\mu^{(1)}(v+\eta)$$

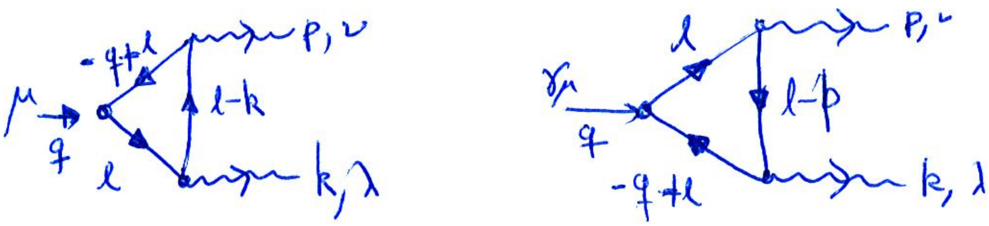
$$D_\mu \phi_3 = \partial_\mu \eta$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)^2 + \frac{1}{2} g^2 (v+\eta)^2\left((A_\mu^1)^2 + (A_\mu^2)^2\right)$$

Axial anomaly

$$\int d^4x \; e^{-iqx} \langle p,k | j_5^{\mu}(x) | 0 \rangle =$$

$$= (2\pi)^4 \, \delta^{(4)}(p+k-q) \; \epsilon^*_{\nu}(p) \; \epsilon^*_{\lambda}(k) \; \mathcal{M}^{\mu\nu\lambda}(p,k)$$



$$(-)\int \frac{d^4l}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^{\lambda} \frac{i}{\not{l}-m} \gamma_5 \gamma_{\mu} \frac{i}{-\not{q}+\not{l}-m} \gamma^{\nu} \frac{i}{\not{l}-\not{k}-m} \right\} +$$

$$+(-)\int \frac{d^4l}{(2\pi)^4} \mathrm{tr}\left\{ \gamma^{\nu} \frac{i}{\not{l}-m} \gamma_5 \gamma_{\mu} \frac{i}{-\not{q}+\not{l}-m} \gamma^{\lambda} \frac{i}{\not{l}-\not{p}-m} \right\}$$

$$(iq_{\mu}) \cdot \; ; \; -\gamma_5 \not{q} = (\not{l}-m)\gamma_5 + \gamma_5(\not{l}-\not{q}-m) + 2m\gamma_5$$

$$\not{l}\gamma_5 = -\gamma_5 \not{l}$$

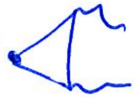
$$\int \frac{d^4l}{(2\pi)^4} \mathrm{Tr}\left\{ \gamma^{\lambda} \frac{i}{(\not{l}-m)} \left((\not{l}-m)\gamma_5 + \gamma_5 (\not{l}-\not{q}-m) + 2m\gamma_5 \right) \frac{i}{(\not{l}-\not{q}-m)} \gamma^{\nu} \frac{i}{\not{l}-\not{k}-m} \right\}$$

$$\int \frac{d^4l}{(2\pi)^4} \mathrm{tr}\left\{ i \gamma^{\lambda} \gamma_5 \frac{i}{\not{l}-\not{q}-m} \gamma^{\nu} \frac{i}{\not{l}-\not{k}-m} + \gamma^{\lambda} \frac{i}{\not{l}-m} \gamma_5 \, i \gamma^{\nu} \frac{i}{\not{l}-\not{k}-m} \right\}$$

$$+2m \int \frac{d^4l}{(2\pi)^4} \mathrm{tr}\left\{ \gamma^{\lambda} \frac{i}{\not{l}-m} \gamma_5 \frac{i}{\not{l}-\not{q}-m} \gamma^{\nu} \frac{i}{\not{l}-\not{k}-m} \right\}$$

$$= i \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\Big\{ \gamma^\lambda \gamma_5 \frac{1}{\not l - \not q - m} \gamma^\nu \frac{1}{\not l - \not k - m} + \gamma^\lambda \frac{1}{\not l - m} \gamma_5 \gamma_\nu \frac{1}{\not l - \not k - m} +$$

$$+ \gamma^\nu \gamma_5 \frac{1}{\not l - \not q - m} \gamma^\lambda \frac{1}{\not l - \not p - m} + \gamma^\nu \frac{1}{\not l - m} \gamma_5 \gamma_\lambda \frac{1}{\not l - \not p - m} \Big\}$$

$+ 2m\, P$ 

$p + k = q \qquad k - q = -p$

$f(l-k) - f(l)$

$$-i \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\Big\{ \gamma^\nu \frac{1}{\not l - \not k - m} \gamma^\lambda \gamma_5 \frac{1}{\not l - \not q - m} - \gamma^\nu \frac{1}{\not l - m} \gamma_\lambda \gamma_5 \frac{1}{\not l - \not p - m} \Big\}$$

$l + k - k \qquad l + k - q - m$ ← cancels after shift

$$-i \int \frac{d^4 l}{(2\pi)^4} \mathrm{Tr}\Big\{ \gamma^\lambda \frac{1}{\not l - m} \gamma_5 \gamma_\nu \frac{1}{\not l - \not k - m} - \gamma^\lambda \frac{1}{\not l - \not p - m} \gamma^\nu \gamma_5 \frac{1}{\not l - \not q - m} \Big\}$$

$l + p - p \qquad l + p - q$

$l - k.$

Shift of linearly divergent integral → dangerous.

$$\Delta(a) = \int d^n r \left(f(\vec r + \vec a) - f(\vec r) \right) = \int d^n r \; a^\mu \partial_\mu f + \tfrac{1}{2} a^\mu a^\nu \partial_\mu \partial_\nu f$$

$$= a^\mu \oint f \, dS_\mu \qquad\qquad \frac{1}{\not l} = \frac{\not l}{l^2}$$

$$\frac{-i k^\mu}{(2\pi)^4} \oint_S \mathrm{Tr}\Big\{ \gamma^\nu \frac{1}{\not l} \gamma^\lambda \gamma_5 \frac{1}{\not l} \Big\} dS_\mu$$

$$\mathrm{Tr}(\gamma^\nu \gamma^\alpha \gamma^\lambda \gamma_5 \gamma^\beta) \oint \frac{l_\alpha l_\beta}{l^4} \, l^2 \, dS_\mu$$

$$= -\frac{2\pi^2 i k^\mu}{16 \pi^4} \epsilon_{\nu\alpha\lambda\beta}$$

Define

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$l = \underset{4d}{l_{\parallel}} + \underset{-2\epsilon}{l_{\perp}}$$

$$-\gamma_5 \not{q} = (\not{l} - m)\gamma_5 + \gamma_5(\not{l} - \not{q} - m) + 2m\gamma_5 - 2\not{l}_{\perp}\gamma_5$$

$$-i\int\frac{d^dl}{(2\pi)^d}\,\mathrm{Tr}\left\{\gamma^\lambda \frac{(\not{l}_{\perp}+m)}{l^2-m^2}(-2\not{l}_{\perp}\gamma_5)\frac{(\not{l}-\not{q}+m)}{(l-q)^2-m^2}\gamma^\nu\frac{(\not{l}-\not{k}+m)}{(l-k)^2-m^2}\right\}$$

$-i \quad (\lambda \leftrightarrow \nu)(p \leftrightarrow k)$

$$\mathrm{Tr}(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma_5) = -4i\,\varepsilon^{\alpha\beta\mu\nu}$$

$\not{a}\not{a} = \frac{1}{2}q_\mu a_\nu\{\gamma^\mu,\gamma^\nu\} = a^2$

$$-i\int\frac{d^dl}{(2\pi)^d}\,(-2)\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\,\frac{2\,\mathrm{Tr}\{(\not{l}-\not{q}+m)\gamma^\nu(\not{l}-\not{k}+m)\gamma^\lambda(\not{l}+m)\not{l}_{\perp}\gamma_5\}}{\left((l-\alpha q-\beta k)^2+\underbrace{\alpha(1-\alpha)q^2+\beta(1-\beta)k^2-2\alpha\beta qk-m^2}_{-\Delta^2}\right)^3}$$

$l \to l + \alpha q + \beta k$

$$4i\int\frac{d^dl}{(2\pi)^d}\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\,\frac{\mathrm{Tr}\left((\not{l}+(\alpha-1)\not{q}+\beta\not{k}+m)\gamma^\nu(\not{l}+\alpha\not{q}+(\beta-1)\not{k}+m)\gamma^\lambda(\not{l}+\alpha\not{q}+\beta\not{k}+m)\not{l}_{\perp}\gamma_5\right)}{(l^2-\Delta^2)^3}$$

④

$$\mathrm{Tr}\left\{\not{l}_\perp \gamma^\nu (\alpha \not{q} + (\beta-1)\not{k}) \gamma^\lambda (\alpha \not{q} + \beta \not{k}) \not{l}_\perp \gamma_5\right\} = \cancel{l}_\perp^2 (-4i)\, \varepsilon_{\nu\alpha\lambda\beta} \left(\cancel{\gamma}^\nu (\alpha q + (\beta-1)k)^\alpha \cancel{\gamma} (\alpha q + \beta k)^\beta\right)$$

$$= -4i\, l_\perp^2\, \varepsilon_{\nu\tilde{\alpha}\lambda\tilde{\beta}} \left(\alpha\beta\, q^{\tilde{\alpha}} k^{\tilde{\beta}} + (\beta-1)\alpha\, k^\alpha q^\beta\right)$$

$$= -4i\, l_\perp^2\, \varepsilon_{\nu\rho\lambda\sigma}\, q^\rho k^\sigma\, \alpha$$

$$\mathrm{Tr}\left\{((\alpha-1)\not{q} + \beta \not{k}) \gamma^\nu (\alpha \not{q} + (\beta-1)\not{k}) \gamma^\lambda \not{l}_\perp \not{l}_\perp \gamma_5\right\} = l_\perp^2 (-4i)\, \varepsilon_{\rho\nu\sigma\lambda}\, q^\rho k^\sigma.$$

$$((\alpha-1)(\beta-1) - \alpha\beta)$$

$$= -4i\, l_\perp^2\, \varepsilon_{\rho\nu\sigma\lambda}\, q^\rho k^\sigma (-\alpha - \beta + 1)$$

$$\mathrm{Tr}\left\{((\alpha-1)\not{q} + \beta \not{k}) \gamma^\nu \underbrace{\not{l}_\perp \gamma^\lambda (\alpha \not{q} + \beta \not{k})}_{\text{anticommutes}} \not{l}_\perp \gamma_5\right\} = -4i\, l_\perp^2\, \varepsilon_{\rho\nu\lambda\sigma}\, q^\rho k^\sigma$$

$$((\alpha-1)\beta - \alpha\beta)$$

$$= -4i\, l_\perp^2\, \varepsilon_{\rho\nu\lambda\sigma}\, q^\rho k^\sigma (-\beta)$$

In total:

$$4i \int \frac{d^d l}{(2\pi)^d} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \frac{(-4i\, l_\perp^2\, \varepsilon_{\nu\rho\lambda\sigma}\, q^\rho k^\sigma)(\cancel{\alpha} \cancel{-\alpha} \cancel{-\beta} + 1 \cancel{+\beta})}{(l^2 - \Delta^2)^3}$$

$$16\, \varepsilon_{\nu\rho\lambda\sigma}\, q^\rho k^\sigma \int \frac{d^d l}{(2\pi)^d} \frac{l_\perp^2}{(l^2-\Delta^2)^3} \underbrace{\int_0^1 d\alpha \int_0^{1-\alpha} d\beta}_{1/2} \quad (1-\alpha) \to \alpha - \frac{\alpha^2}{2}\Big|_0^1$$

$d = 4 - 2\varepsilon$

$2 - d/2 = 2 - d + \varepsilon =$

$$l_a^2 \underset{d}{\leftrightarrow} \frac{(-2\varepsilon)}{d} \int \frac{d^d l}{(2\pi)^d} \frac{l^2}{(l^2-\Delta^2)^3} = \frac{(-\cancel{2}\varepsilon)}{\cancel{d}} \frac{1}{(4\pi)^{d/2}}\, i\, \frac{\cancel{d}}{\cancel{2}} \frac{\Gamma(2-d/2)}{\Gamma(3)} \frac{1}{\Delta^{2-d/2}} \overset{\varepsilon\to 0}{\simeq} -\frac{i\varepsilon}{2(4\pi)^{d/2}} \frac{1}{\cancel{\varepsilon}} \frac{1}{\Delta^\varepsilon} = -\frac{i}{32\pi^2}$$

$$= -\frac{i}{4\pi^2}\,\varepsilon_{\nu\rho\lambda\sigma}\, q^\rho k^\sigma$$

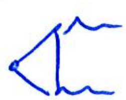
We add $\nu \leftrightarrow \lambda$ $k \leftrightarrow p$ $\qquad q = p+k \quad p = q-k$

$$-\frac{i}{4\pi^2}\,\varepsilon_{\lambda\rho\nu\sigma}\, q^\rho p^\sigma = -\frac{i}{4\pi^2}\,\varepsilon_{\nu\rho\lambda\sigma}\, q^\rho k^\sigma$$

<u>Finally:</u>

$$-\frac{i}{2\pi^2}\,\varepsilon_{\nu\rho\lambda\sigma}\, q^\rho k^\sigma$$

vertex

iq^μ  $= -\frac{i\,i e^2}{2\pi^2}\,\epsilon_{\nu\rho\lambda\sigma}\, q^\rho k^\sigma = -\frac{i\,i e^2}{2\pi^2}\,\epsilon_{\nu\rho\lambda\sigma}\, p^\rho k^\sigma$

e^2

$$= \frac{e^2}{2\pi^2}\,\epsilon_{\nu\rho\lambda\sigma}\, p^\rho k^\sigma$$

$$\langle p,k|\partial_\mu j^{\mu 5}|0\rangle = -\frac{e^2}{2\pi^2}\,\epsilon_{\nu\rho\lambda\sigma}\,(-ip^\rho)(-ik^\sigma)\,\epsilon^*_\nu(p)\,\epsilon^*_\lambda(k)$$

$$= -\frac{e^2}{16\pi^2}\,\langle p,k|\,\epsilon_{\nu\rho\lambda\sigma}F_{\nu\rho}F_{\lambda\sigma}\,|0\rangle$$

$$(\partial_\nu A_\rho - \partial_\rho A_\nu)(\partial_\lambda A_\sigma - \partial_\sigma A_\lambda)$$

$$\boxed{\Rightarrow \partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\,\epsilon_{\nu\rho\lambda\sigma}F_{\nu\rho}F_{\lambda\sigma}}$$

$$2\times 4\,\partial_\rho A_\nu\,\partial_\sigma A_\lambda$$

π : Goldstone boson of chiral anomaly

$$\langle 0 | j^{\mu 5 a}(x) | \pi^b(p) \rangle = -ip^\mu f_\pi \delta^{ab} e^{-ipx}$$

$$\langle 0 | \underbrace{\partial_\mu j^{\mu 5 a}(x)}_{\text{"pion field"}} | \pi^b(p) \rangle = -i \underset{m_\pi^2 \neq 0}{p^2} f_\pi \delta^{ab} e^{-ipx}$$

$$\langle \gamma_1 \gamma_2 | \pi \rangle = \langle \gamma\gamma | \partial_\mu j^\mu | 0 \rangle \to \pi^0 \to \gamma + \gamma$$

(p)

$$D_\mu \phi = \left(\partial_\mu - i g A^a_\mu \tau^a - \tfrac{i}{2} g' B_\mu \right) \phi$$

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \qquad Y = 1/2$$

$$\phi \to e^{i\alpha^a \tau^a} e^{i\frac{\beta}{2}} \phi \qquad \tau^a = \sigma^a/2$$

$$e^{i\alpha \sigma_3/2} e^{i\beta/2}$$

$$\alpha_3 = \beta \qquad \alpha_{1,2} = 0$$

unbroken.

$$\frac{v^2}{2} (0 \ \cancel{1}) \left(g A^a_\mu \tau^a + \tfrac{1}{2} g' B_\mu \right) \left(g A^{b\mu} \tau^b + \tfrac{1}{2} g' B^\mu \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\frac{1}{2} v^2 (0 \ 1) \left(g^2 A^a_\mu A^a_\mu \frac{1}{4} + \frac{\cancel{2}}{4} g g' A^a_\mu B^\mu \tau^{a,3} + \frac{1}{4} g'^2 B_\mu B^\mu \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{8} v^2 \left(g^2 A^a_\mu A^a_\mu - 2 g g' A^3_\mu B^\mu + g'^2 B_\mu B^\mu \right)$$

$$= \frac{1}{8} v^2 \left(g^2 A^h_\mu A^h_\mu + g^2 A^3_\mu A^3_\mu + (g A^3_\mu - g' B_\mu)^2 \right)$$

$$W^\pm_\mu = \frac{1}{\sqrt{2}} (A^1_\mu \pm i A^2_\mu) \qquad Z^0_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g A^3_\mu - g' B_\mu \right)$$

$$A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left(g' A^3_\mu + g B_\mu \right) \ ; \quad m_W = g \frac{v}{2} \qquad m_\gamma = 0$$

$$m_Z = \sqrt{g^2 + g'^2} \ \frac{v}{2}$$

$$g A^3_\mu - g' B_\mu = \sqrt{g^2+g'^2}\; Z^0_\mu$$

$$g' A^3_\mu + g B_\mu = \sqrt{g^2+g'^2}\; A_\mu$$

$$A^3_\mu = \frac{1}{\sqrt{g^2+g'^2}} \left(g Z_\mu + g' A_\mu \right)$$

$$B_\mu = \frac{1}{\sqrt{g^2+g'^2}} \left(g A_\mu - g' Z_\mu \right)$$

$$D_\mu = \partial_\mu - i g A^a_\mu T^a - i g' Y B_\mu$$

$$- i g A^3_\mu T^3 - i g' Y B_\mu$$

$$\frac{-ig}{\sqrt{\ }} (g Z_\mu + g' A_\mu) T^3 - \frac{ig'}{\sqrt{\ }} Y (g A_\mu - g' Z_\mu)$$

$$-\frac{igg'}{\sqrt{\ }} A_\mu T^3 - \frac{ig'gY}{\sqrt{\ }} A_\mu$$

$$\boxed{-\frac{igg'}{\sqrt{g^2+g'^2}} A_\mu \underbrace{(T^3+Y)}_{Q}.}$$

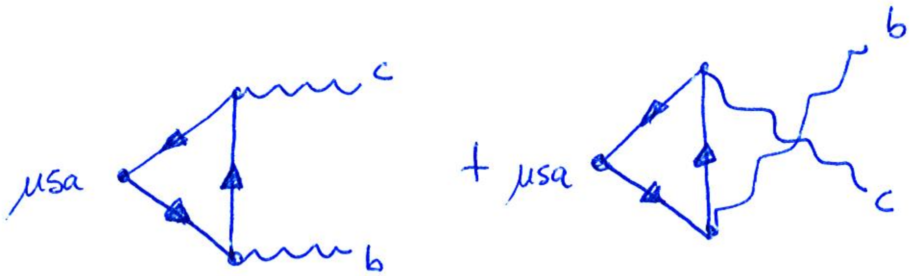
$$e = \frac{gg'}{\sqrt{g^2+g'^2}}$$

$Q = -1$ for e^-

Standard model anomalies.

$U(1)_Y \otimes SU(2)_L \otimes SU(3)_c$

$Q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L \qquad L_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L \qquad \begin{matrix} u_R \\ d_R \end{matrix} \qquad e_R$



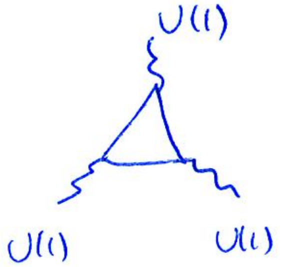
$j^{\mu a 5} = \bar{\psi}_\alpha \gamma_5 \gamma^\mu \psi_\beta \, t^a_{\alpha\beta}$

$Tr(t^a t^b t^b) \qquad Tr(t^a t^c t^b)$

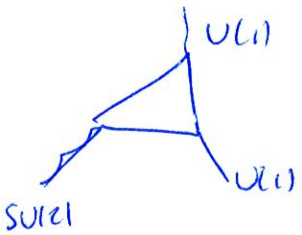
$Tr(\gamma_5 t^a \{t^b, t^c\})$

$Q = T^3 + Y$

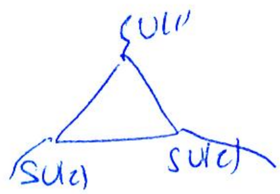
	Y	T_3	Q	$SU(3)$
u_R	$2/3$	0	$2/3$	**3**
d_R	$-1/3$	0	$-1/3$	**3**
u_L	$1/6$	$1/2$	$2/3$	**3**
d_L	$1/6$	$-1/2$	$-1/3$	**3**
ν_L	$-1/2$	$1/2$	0	**1**
e_L	$-1/2$	$-1/2$	-1	**1**
e_R	-1	0	-1	**1**



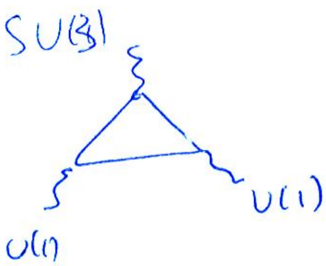
$\mathrm{tr}(t_a t_b t_c) \to \sum_i y_i^3$



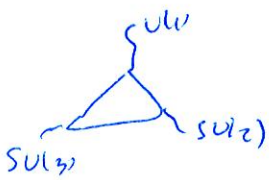
$\mathrm{Tr}(\sigma_a Y_b Y_c) = 0 \qquad \mathrm{Tr}(\sigma_a) = 0$



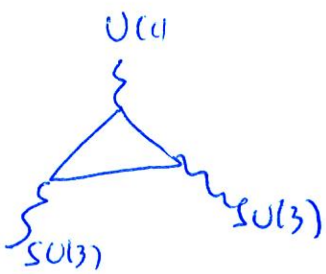
$\mathrm{Tr}(Y_i \underbrace{\sigma^a \sigma^b}_{2\delta^{ab}}) \to \sum_{I=1/2} y_i = 3\times\frac{1}{6} + 3\times\frac{1}{6} - \frac{1}{2} - \frac{1}{2}$



$\mathrm{Tr}(t^A Y Y) \qquad \mathrm{tr}\, t^A = 0.$

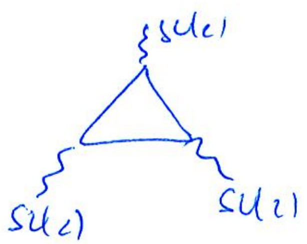


$\to 0$

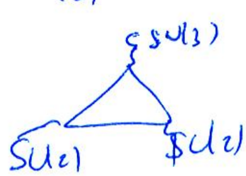


$\mathrm{Tr}(\{t^A t^B\} Y) = \sum_{i \in \mathbf{3}} Y_i = 3\times\frac{2}{3} + 3\left(-\frac{1}{3}\right) - 3\frac{1}{6} - 3\frac{1}{6}$

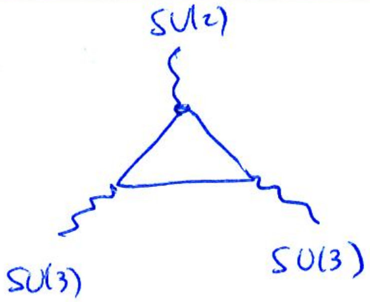
$= 3\left(\frac{2}{3} - \frac{1}{3} - \frac{1}{3}\right) = 0.$



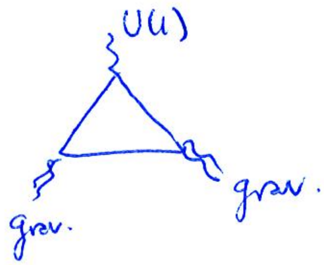
$\mathrm{Tr}(\underbrace{\{\sigma^a \sigma^b\}}_{\delta^{ab}} \sigma^c) = 0$



$\mathrm{tr}(\{\sigma^a \sigma^b\} t^A) = 0$



$\mathrm{tr}\,\sigma^a = 0.$



$$\sum y_i =$$

$$= 3\left(\tfrac{2}{3}\right) - 3\tfrac{1}{3} - \tfrac{3}{6} - \tfrac{3}{6} + \tfrac{1}{2} + \tfrac{1}{2} - 1$$

$$= \cancel{2} - \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} = 0.$$