Gauge theories

- ·) spin 1 massless particles (2 polarizations instead of 3)
- ·) renormalizable (in general spin 1 massive is not renormalizable)
- ·) local symmetry, ensures Lorentz invariance w/ one polarization less
- ·) Describe EM, strong and weak interactions.

-) Abelian gauge theory : commuting gauge group $\underline{U(1)}$. $\underline{QED}$

Consider free fermion

$$\mathcal{L}_0 = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$U(1)$,
$$\left.\begin{aligned} \psi &\to e^{-i\alpha}\psi \\ \bar{\psi} &\to \bar{\psi} e^{i\alpha} \end{aligned}\right\} \alpha \text{ indep. of } x.$$

If we make $\alpha(x)$ (x dep.) then $-m\bar{\psi}\psi$ is still invariant but

$$i\bar{\psi}\gamma^\mu\partial_\mu\psi \to i\bar{\psi}e^{-i\alpha}\gamma^\mu\partial_\mu(e^{i\alpha}\psi) =$$

$$= i\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\alpha\,\bar{\psi}\gamma^\mu\psi$$

Introduce a gauge field A_μ (E.M. potential) and define $D_\mu\psi = (\partial_\mu + ieA_\mu)\psi$ <u>covariant</u> derivative

nice transf. properties. same as ψ.

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$$

$$D_\mu\psi \to \partial_\mu\left(e^{i\alpha}\psi\right) + ie\,e^{i\alpha}A_\mu\psi + i\,e^{i\alpha}\partial_\mu\alpha\,\psi$$

$$e^{i\alpha}\left(\partial_\mu\psi + \cancel{i\,\partial_\mu\alpha\,\psi} + ie A_\mu\psi \cancel{- i\,\frac{\psi\,\partial_\mu\alpha\,\psi}{\psi}}\right)$$

$$e^{i\alpha} D_\mu\psi$$

$$\mathcal{L} = i\bar\psi \not{D}\psi - m\bar\psi\psi$$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge inv. by direct comp.

$$\mathcal{L}_{QED} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Maxwell's Lagrangian}} + i\bar\psi\not{D}\psi - m\bar\psi\psi$$

(ψ: electron, μ, q, ...)

We can also check gauge invariance of $F_{\mu\nu}$ by doing.

$$D_\mu\psi = (\partial_\mu + ieA_\mu)\psi$$

$$D_\nu D_\mu\psi = (\partial_\nu + ieA_\nu)(\partial_\mu\psi + ieA_\mu\psi)$$

$$= \partial_{\nu\mu}\psi + ie\,\partial_\nu A_\mu\psi + ieA_\mu\partial_\nu\psi + ieA_\nu\partial_\mu\psi - e^2 A_\nu A_\mu\psi$$

$$(D_\nu D_\mu - D_\mu D_\nu)\psi = ie(\partial_\nu A_\mu - \partial_\mu A_\nu)\psi + ie\cancel{A_\mu\partial_\nu}\psi - \underline{ieA_\nu\partial_\mu\psi} + ie\cancel{A_\nu\partial_\mu}\psi - ie\cancel{A_\mu}\partial_\nu\psi = -ieF_{\mu\nu}\psi$$

Since

$$[D_\nu, D_\mu]\psi \to e^{i\alpha}\,[D_\nu, D_\mu]\psi$$

and $\psi \to e^{i\alpha}\psi$

then $F_{\mu\nu} \to F_{\mu\nu}$. gauge invariant
↳ different from covariant.

- ·) A_μ is massless $A_\mu A^\mu$ not gauge invariant.
- ·) as discussed before the magnetic moment of the particle is fixed (minimal coupling). ($g = 2 + \dots$)
  we can add $\bar\psi\, \Sigma_{\mu\nu}\, \psi F^{\mu\nu}$ but then non-renormalizable.
- ·) photon has no charge ($F_{\mu\nu}$ gauge inv, no self-coupling)
- ·) theory is renormalizable but base on photon propagator going as $\sim 1/k^2$ $k \to \infty$. Not true for massive vector bosons. Also not true in all gauges but one gauge is sufficient. More discussion when quantizing.
- ·) Local symmetry implies redundancy in the description
  We can eliminate one degree of freedom. (not a "real symmetry")
  ↳ maps equivalent descriptions of the same physics

Spin 1 should have 3 polarization → one goes away.
Requires massless particle. Otherwise in rest frame it should have three polarizations.

• •) <u>Non-abelian</u>, Yang-Mills theory, non-commuting group.

simplest case $SU(2)$. Simple generalization to $SU(N)$ or any Lie group (continuous group)

take two fermions. ψ_1, ψ_2 same mass.

$$\mathcal{L} = \sum_{a=1}^{2} \bar{\psi}_a (i\not{\partial} - m) \psi_a$$

$$\psi_a \to U_{ab} \psi_b$$

U 2×2 unitary matrix.

Det = 1 $SU(2)$

overall phase. is $U(1)$.

$$\bar{\psi}_a \to U^*_{ab} \bar{\psi}_b$$

$$\sum_{a=1}^{2} \bar{\psi}_a (i\not{\partial} - m) \psi_a \to \sum_a U^*_{ab} \bar{\psi}_b (i\not{\partial} - m) U_{ac} \psi_c = \sum_b \bar{\psi}_b (i\not{\partial} - m) \psi_c$$

$$\sum_{a=1}^{2} \underbrace{U^*_{ab} U_{ac}}_{(U^\dagger U)_{bc}} = \delta_{bc}$$

works for <u>$U(N)$</u> also.

Make symmetry local.

First write a column vector $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to U \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$$\psi \to U\psi$$

$$\bar\psi \to \bar\psi U^\dagger \qquad \text{(Now we have to be careful in the ordering)}$$

$$i\bar\psi \not\partial \psi \to i\bar\psi U^\dagger \gamma^\mu \partial_\mu (U\psi) =$$

$$= i\bar\psi \not\partial \psi + i\bar\psi (U^\dagger \partial_\mu U)\gamma^\mu \psi$$

Define $\not D\psi \to U \not D \psi$

$$\not D \psi = \gamma^\mu(\partial_\mu \psi - i g A_\mu \psi) \quad ; \quad A_\mu \to \tilde A_\mu$$

$$D_\mu \psi \to U\partial_\mu \psi + \partial_\mu U \psi - i g \tilde A_\mu U \psi =$$

$$= U\partial_\mu \psi - i g\, U A_\mu \psi$$

$$\partial_\mu U - i g \tilde A_\mu U = -i g\, U A_\mu$$

$$\tilde A_\mu U = U A_\mu + \frac{1}{ig}\partial_\mu U = U A_\mu - \frac{i}{g}\partial_\mu U$$

$$\tilde A_\mu = U A_\mu U^\dagger - \frac{i}{g}\partial_\mu U U^\dagger$$

$$\boxed{A_\mu \to U A_\mu U^\dagger - \frac{i}{g}\partial_\mu U U^\dagger}$$

(Notice $\partial_\mu U U^\dagger = -U\partial_\mu U^\dagger$)

What we did works for any $SU(N)$.

Look a little closer to $SU(2)$.

$$U = e^{-i\theta_a \frac{\sigma^a}{2}}$$

; isospin, similar to spin 1/2
but it is an <u>internal</u> symmetry
<u>not</u> space-time.

Recall

$$U = \sum_{n=0}^{\infty} \frac{(-i\theta_a \sigma^a)^n}{n!\, 2^n} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i\theta}{2}\right)^n (\hat{\theta}_a \sigma^a)^n$$

$\hat{\theta}_a$ unit vector. $\theta = |\theta_a|$

$$(\hat{\theta}_a \sigma^a)^2 = \hat{\theta}_a \hat{\theta}_b \sigma^a \sigma^b = \hat{\theta}_a \hat{\theta}_b (\delta^{ab} + \underbrace{i\epsilon^{abc}\sigma^c}_{0}) = \hat{\theta}^2 = 1.$$

$$U = \sum_{n \text{ even}} \frac{1}{n!} \left(-\frac{i\theta}{2}\right)^n + \sum_{n \text{ odd}} \frac{1}{n!} \left(-\frac{i\theta}{2}\right)^n (\hat{\theta}_a \sigma^a)$$

$$= \cos\frac{\theta}{2} - i \sin\frac{\theta}{2} (\hat{\theta}_a \sigma^a)$$

$$\partial_\mu U = -\frac{1}{2} \sin\frac{\theta}{2}\, \partial_\mu \theta - \frac{i}{2} \cos\frac{\theta}{2}\, \partial_\mu \theta\, (\hat{\theta}_a \sigma^a) - i\, s\frac{\theta}{2}\, (\partial_\mu \hat{\theta}_a \sigma^a)$$

$$U^\dagger = c\frac{\theta}{2} + i\, s\frac{\theta}{2}\, (\hat{\theta}_a \sigma^a)$$

$$U^\dagger \partial_\mu U = \cancel{-\frac{1}{2} s\frac{\theta}{2} c\frac{\theta}{2} \partial_\mu \theta} - \frac{i}{2} c^2\frac{\theta}{2}\, \partial_\mu \theta\, (\hat{\theta}\cdot\sigma) - i\, s\frac{\theta}{2} c\frac{\theta}{2}\, (\partial_\mu \hat{\theta} \sigma)$$

$$- \frac{i}{2} s^2\frac{\theta}{2}\, \partial_\mu \theta\, (\hat{\theta}\sigma) + \cancel{\frac{1}{2} s\frac{\theta}{2} c\frac{\theta}{2} \partial_\mu \theta} + s^2\frac{\theta}{2}\, (\hat{\theta}\sigma)(\partial_\mu \hat{\theta} \sigma)$$

$$U^\dagger \partial_\mu U = -\frac{i}{2}\partial\theta\,(\hat{\theta}\vec{\sigma}) + s^2\frac{\theta}{2}\;\hat{\theta}_a\partial_\mu\hat{\theta}_b\;(\delta_{ab} + i\varepsilon_{abc}\sigma_c)$$

$$\hat{\theta}_a\partial_\mu\hat{\theta}_a = \frac{1}{2}\partial_\mu\underbrace{(\hat{\theta}_a\hat{\theta}_a)}_{1} = 0.$$

$$U^\dagger\partial_\mu U = \left(-\frac{i}{2}\partial_\mu\theta\;\hat{\theta}_c + i\,\epsilon_{abc}\;s^2\frac{\theta}{2}\;\hat{\theta}_a\partial_\mu\hat{\theta}_b\right)\sigma_c$$

$$= i\underbrace{f_c(\theta_a)}_{\text{real.}}\sigma_c$$

↑ linear combination of the generators.

$$A_\mu \rightarrow U A_\mu U^\dagger + \frac{1}{g} f_c(\theta_a)\,\sigma_c$$

$$A_\mu = \sum_a A^a_\mu \frac{\sigma^a}{2} + \hat{A}_\mu \cdot 1\!\!1_{2\times 2}$$

$U A_\mu U^\dagger$; $U\sigma^a U^\dagger$ rotates the index a.

$U\,1\!\!1\,U^\dagger$ invariant. so $\hat{A}_\mu$ is gauge inv.

we can drop it ($U(1)$ part).

$$A_\mu = \sum_a A^a_\mu \frac{\sigma^a}{2} \longrightarrow \sum_a A^a_\mu\, \underbrace{\frac{U\sigma^a U^\dagger}{2}}_{R_{ab}\sigma^b} + \frac{1}{g} f_c(\theta)\,\sigma_c$$

a rotation (3d)

$A^{a=1\ldots3}_\mu$: 3 gauge fields, one for each infinitesimal generator.

$F_{\mu\nu}$:

$$D_\mu \psi = \partial_\mu \psi - ig A_\mu \psi$$

$$D_\nu D_\mu \psi = (\partial_\nu - ig A_\nu)(\partial_\mu \psi - ig A_\mu \psi)$$

$$= \cancel{\partial_\nu \partial_\mu \psi} \underline{- ig\, \partial_\nu A_\mu \psi} \cancel{- ig A_\mu \partial_\nu \psi} \cancel{- ig A_\nu \partial_\mu \psi} - g^2 A_\nu A_\mu \psi$$

$$D_\mu D_\nu \psi = \cancel{\partial_\mu \partial_\nu \psi} \underline{- ig\, \partial_\mu A_\nu \psi} \cancel{- ig A_\nu \partial_\mu \psi} \cancel{- ig A_\mu \partial_\nu \psi} - g^2 A_\mu A_\nu \psi$$

cancellation in

$$[D_\mu, D_\nu]\psi = -ig\,(\partial_\mu A_\nu - \partial_\nu A_\mu)\psi - g^2 (A_\mu A_\nu - A_\nu A_\mu)\psi$$

$$\equiv -ig\, F_{\mu\nu} \psi \qquad (-ig)(-ig)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig \underbrace{[A_\mu, A_\nu]}_{\text{as matrices}}$$

(this is classical)

$$[D_\mu, D_\nu]\psi \to U [D_\mu, D_\nu]\psi = -ig\, U F_\mu \psi$$

$$\tilde{F}_{\mu\nu} U = U F_{\mu\nu}$$

$$-ig F_\mu \psi \to -ig \tilde{F}_\mu U \psi$$

$$\tilde{F}_\mu = U F_\mu U^\dagger \qquad F_{\mu\nu} \to U F_{\mu\nu} U^\dagger$$

F_μ is not gauge invariant any more → but it is covariant.

Transforms in the adjoint representation.

<u>SU(2)</u>

$$A_\mu = A_\mu^a \frac{\sigma^a}{2}$$

$$\mathbb{F}_{\mu\nu} = \partial_\mu A_\nu^a \frac{\sigma^a}{2} - \partial_\nu A_\mu^a \frac{\sigma^a}{2} - ig A_\mu^a A_\nu^b \overbrace{\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right]}^{i\epsilon^{abc}\sigma^c/2}$$

$$= \left(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g A_\mu^a A_\nu^b \epsilon^{abc}\right)\frac{\sigma^c}{2}$$

$$= F_{\mu\nu}^c \sigma^c/2$$

$$\boxed{F_{\mu\nu}^c = \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g A_\mu^a A_\nu^b \epsilon^{abc}}$$

$\mathrm{Tr}\, \mathbb{F}_{\mu\nu}$ invariant but $\mathrm{Tr}\, \mathbb{F}_{\mu\nu} = 0$

$\mathrm{Tr}\, \mathbb{F}_{\mu\nu} \mathbb{F}_{\mu\nu}$ invariant $\rightarrow$ $\mathrm{Tr}\, \cancel{U} \mathbb{F}_{\mu\nu} \cancel{U^\dagger} U \mathbb{F}_{\mu\nu} \cancel{U^\dagger}$ invariant

$$\mathcal{L} = -\mathrm{Tr}\, \mathbb{F}_{\mu\nu} \mathbb{F}_{\mu\nu} + i\bar{\psi} \not{D} \psi - m\bar{\psi}\psi$$

$$\mathrm{Tr}\, \mathbb{F}_{\mu\nu}\mathbb{F}_{\mu\nu} = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^b \,\mathrm{Tr}\,(\underbrace{\sigma^a \sigma^b}_{\delta^{ab} + \underbrace{i\epsilon^{abc}\sigma^c}_{\mathrm{Tr}=0}}) = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a$$

Infinitesimal transformations

$$U = e^{-i\frac{\sigma^a}{2}\theta_a} \underset{\theta_a \to 0}{\approx} 1 + \frac{i\sigma^a}{2}\theta_a + \mathcal{O}(\theta^2)$$

$$A_\mu \to U A_\mu U^\dagger - \frac{i}{g}\partial_\mu U U^\dagger$$

$$A_\mu \to \left(1 - i\frac{\sigma^a}{2}\theta_a\right) A_\mu \left(1 + \frac{i\sigma^a}{2}\theta_a\right) - \frac{i}{g}\left(-i\frac{\sigma^a}{2}\partial_\mu\theta_a\right)$$

$$= A_\mu - i\frac{\theta_a}{2}\left(\sigma^a A_\mu - A_\mu\sigma^a\right) - \frac{1}{g}\partial_\mu\theta_a\frac{\sigma^a}{2}$$

$$A_\mu = A^b_\mu \frac{\sigma^b}{2}$$

$$A^b_\mu\frac{\sigma^b}{2} \to A^b_\mu\frac{\sigma^b}{2} - i\frac{\theta_a}{2}A^b_\mu \underbrace{\left(\sigma^a\frac{\sigma^b}{2} - \frac{\sigma^b}{2}\sigma^a\right)}_{i\varepsilon^{abc}\sigma^c} - \frac{1}{g}\partial_\mu\theta_a\frac{\sigma^a}{2}$$

$$A^c_\mu\frac{\sigma^c}{2} + \theta_a^a A^b_\mu\varepsilon^{abc}\frac{\sigma^c}{2} - \frac{1}{g}\partial_\mu\theta_c\frac{\sigma^c}{2}$$

$$A^c_\mu \to A^c_\mu \underbrace{- \frac{1}{g}\partial_\mu\theta^c}_{\text{same as abelian}} + \underbrace{\varepsilon^{abc}\theta^a A^b_\mu}_{\text{new term}}$$

$$= A^c_\mu - \frac{1}{g}\underbrace{\left(\partial_\mu\theta^c - g\,\varepsilon^{abc}\theta^a A^b_\mu\right)}_{D_\mu\theta^c}$$

$$\psi \to \psi - i\theta_a \frac{\sigma^a}{2}\psi$$

$$\mathbb{F}_{\mu\nu} \to U\mathbb{F}_{\mu\nu}U^\dagger$$

$$F^c_{\mu\nu} \to F^c_{\mu\nu} + \varepsilon^{abc}\theta^a F^b_{\mu\nu}$$

$$D_\mu\psi = \partial_\mu\psi - igA^a_\mu\frac{\sigma^a}{2}\psi$$

$$D_\mu\theta^c = \partial_\mu\theta^c - g\epsilon^{abc}\theta^a A^b_\mu$$

different rep.

We can have fields in the adjoint

example scalar field

$$\Phi^a \to \Phi = \phi^a\frac{\sigma^a}{2}$$

$\Phi \to U\Phi U^\dagger$ (by definition.

$$D_\mu\Phi \to UD_\mu\Phi U^\dagger$$

$$\partial_\mu\Phi \to \partial_\mu(U\Phi U^\dagger) = \partial_\mu U\Phi U^\dagger + U\Phi\partial_\mu U^\dagger + \underbrace{U\partial_\mu\Phi U^\dagger}_{\checkmark}$$

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$$A_\mu \Phi \to U A_\mu \cancel{U^\dagger} \cancel{U} \Phi U^\dagger - \frac{i}{g} \partial_\mu U \cancel{U^\dagger} \cancel{U} \Phi U^\dagger$$

$$\partial_\mu \Phi - ig A_\mu \Phi \to U \partial_\mu \Phi U^\dagger + \cancel{\partial_\mu U \Phi U^\dagger} + \underbrace{U \Phi \partial_\mu U^\dagger}$$

$$- ig U A_\mu \Phi U^\dagger - \cancel{\partial_\mu U \Phi U^\dagger}$$

we have to cancel this

$$\partial_\mu \Phi - ig A_\mu \Phi + ig \Phi A_\mu \to$$

$$\to U \partial_\mu \Phi U^\dagger - ig U A_\mu \Phi U^\dagger + U \Phi \partial_\mu U^\dagger$$

$$+ ig U \Phi \cancel{U^\dagger} (\cancel{U} A_\mu U^\dagger - \frac{i}{g} \underbrace{\partial_\mu U U^\dagger}_{-U \partial_\mu U^\dagger})$$

$$U \partial_\mu \Phi U^\dagger - ig U A_\mu \Phi U^\dagger + \cancel{U \Phi \partial_\mu U^\dagger}$$

$$+ ig U \Phi A_\mu U^\dagger - \cancel{U \Phi \partial_\mu U^\dagger}$$

$$= U (\partial_\mu \Phi - ig A_\mu \Phi + ig \Phi A_\mu) U^\dagger$$

$$D_\mu \Phi \to U D_\mu \Phi U^\dagger \qquad \boxed{D_\mu \Phi = \partial_\mu \Phi - ig [A_\mu, \Phi]}$$

adjoint.

$$\mathcal{L} = -\mathrm{Tr}\,\mathbb{F}_{\mu\nu}\mathbb{F}_{\mu\nu} + i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi + \frac{1}{2}\mathrm{Tr}\,D_\mu\Phi^\dagger D^\mu\Phi - \frac{m^2}{2}\underbrace{\mathrm{Tr}(\Phi^\dagger\Phi)}_{\geq 0\ \checkmark}$$

example of ψ in fundamental and scalar in adjoint

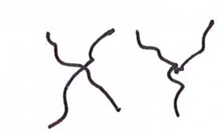
$\Phi \to U\Phi U^\dagger$

$\Phi^\dagger \to U\Phi^\dagger U^\dagger$ ✓

Fermions can be in adjoint or scalar in fundamental also.

·) g has to be the same for all fields because it is in $\mathbb{F}_{\mu\nu}$ also. They cannot have different coupling. Their "charge" is the representation they are in.

·) the gauge field is charged. $\mathbb{F}_{\mu\nu}$ transforms in adjoint

$\mathrm{Tr}\,\mathbb{F}_{\mu\nu}\mathbb{F}_{\mu\nu}$ has a cubic and quartic coupling 

·) gauge fields are massless. No $\underbrace{\mathrm{Tr}\,\mathbb{A}_\mu\mathbb{A}_\mu}$ is not gauge inv. for $\mathbb{A}_\mu$ (It is for Φ)

infinitesimal:

$$\Phi \to e^{-i\frac{\sigma^a}{2}\theta^a}\,\Phi\,e^{i\frac{\sigma^b}{2}\theta^b} \underset{\theta^a\to 0}{\simeq} \Phi - i\theta^a \underbrace{\left[\frac{\sigma^a}{2},\Phi\right]}_{i\varepsilon^{abc}\frac{\sigma^c}{2}}$$

$$\Phi = \phi^b\frac{\sigma^b}{2}$$

$$\phi^k\frac{\sigma^k}{2} \to \phi^c\frac{\sigma^c}{2} + \theta^a\varepsilon^{abc}\frac{\sigma^c}{2}\phi^b$$

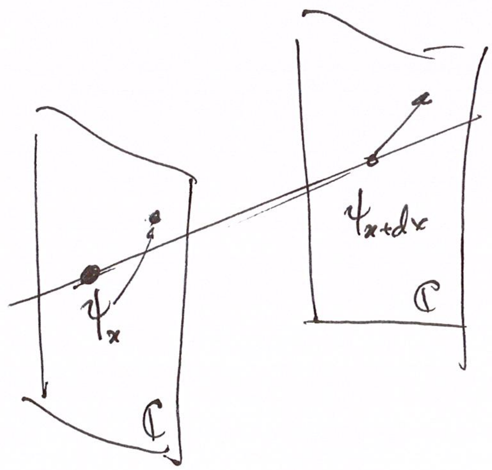
$$\boxed{\phi^c \to \phi^c + \theta^a\varepsilon^{abc}\phi^b}$$ adjoint rep.

$$(D_\mu\phi)^c\frac{\sigma^c}{2} = \partial_\mu\phi^c\frac{\sigma^c}{2} - ig\underbrace{\left[A^a_\mu\frac{\sigma^a}{2},\ \phi^b\frac{\sigma^b}{2}\right]}_{i\varepsilon^{abc}\frac{\sigma^c}{2}}$$

$$= \left(\partial_\mu\phi^c + \varepsilon^{abc}A^a_\mu\phi^b\right)\sigma^c/2$$

$$\boxed{(D_\mu\phi)^c = \partial_\mu\phi^c + \varepsilon^{abc}A^a_\mu\phi^b}$$

Geometric interpretation

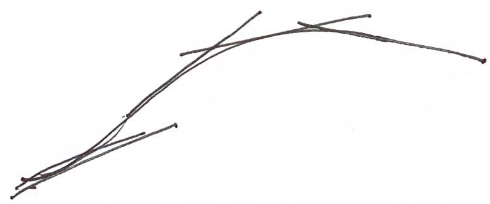


$\phi(x) \in \mathbb{C}$

$\phi : \mathbb{R}^{3,1} \to \mathbb{C}$?

Better $\phi(x)$ belongs to a different $\mathbb{C}$ at each point

(example tangents to a curve



How do we compare $\phi(x+dx)$ and $\psi(x)$?

We need to map $\mathbb{C}_x \to \mathbb{C}_{x+dx}$

this is called parallel transport. It should preserve (structure) properties of the space. We require it to be an element of the gauge group.

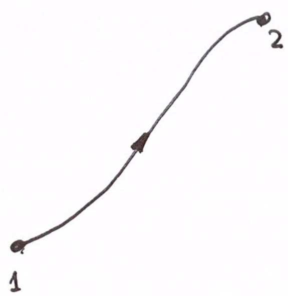
$$\psi_{x+dx} \overset{?}{\approx} \quad \psi_x \to \underbrace{e^{i A_\mu dx^\mu g} \psi_x}_{\text{in } x+dx} \simeq \psi_x + i g \underset{\text{connection}}{A_\mu} dx^\mu \psi_x$$

Now we compare:

$$\Delta\psi = \psi_{x+dx} - \psi_x - i g A_\mu dx^\mu \psi_x$$

$$= \left(\partial_\mu \psi - i g A_\mu \psi\right) dx^\mu = D_\mu \psi \, dx^\mu$$

finite distance



parallel transport along a path

$$\psi_1 \to \underbrace{\left(\hat{P} e^{ig\int_1^2 A_\mu dx^\mu}\right)\psi_1}$$

this is an element in space around 2.

What does it mean ?

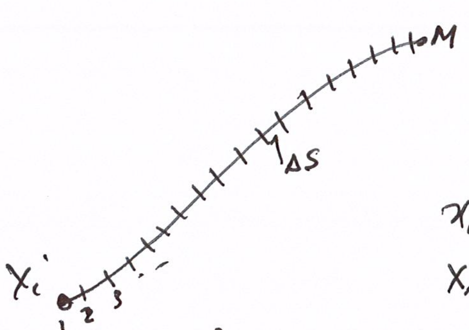
Suppose we make a gauge transf. $U(x)$; $\psi(x) \to U(x)\psi(x)$

then $\psi_1 \to U(x_1)\psi_1$

But $\hat{P} e^{ig\int_1^2 A_\mu dx^\mu}\psi_1 \to \underset{\substack{\uparrow \\ U(x_2)}}{U_2}\left(\hat{P} e^{ig\int_1^2 A_\mu dx^\mu}\psi_1\right)$

this is true if $\quad \hat{P} e^{i\int_1^2 gA_\mu dx^\mu} \to U_2 \hat{P} e^{ig\int_1^2 A_\mu dx^\mu} U_1^+$

Divide the path:



$$\hat{P} e^{i\int_{x_i}^{x_f} A_\mu dx^\mu} = e^{i\int_{M-1}^{M}} \cdots e^{i\int_2^3} e^{i\int_1^2}$$

$x_M = x_f$

$x_1 = x_i$

$$e^{gi\int_1^2 A_\mu \frac{dx^\mu}{ds} ds} \simeq e^{giA_\mu \dot{x}^\mu \Delta s} \simeq 1 + i A_\mu \dot{x}^\mu \Delta s$$

$$1 + igA_\mu \dot{x}^\mu \Delta s \longrightarrow 1 + i\left(gUA_\mu U^\dagger - \frac{i}{g}g\, \partial_\mu U U^\dagger\right)\Delta s\, \dot{x}^\mu$$

$$= \underset{UU^\dagger}{1} + \partial_\mu U U^\dagger \dot{x}^\mu ds + i g\, UA_\mu U^\dagger \Delta s\, \dot{x}^\mu$$

$$= (U + \partial_\mu U \dot{x}^\mu ds)\, U^\dagger + ig\, UA_\mu U^\dagger \Delta s\, \dot{x}^\mu$$

$$\simeq U(x+dx)\,(1 + ig A_\mu \Delta s\, \dot{x}^\mu)\, U^\dagger$$

$$\simeq U(x+dx)\; e^{ig A_\mu \Delta s \dot{x}^\mu}\, U^\dagger(x)$$

$$e^{ig\int_1^2 A_\mu \frac{dx^\mu}{ds} ds} \longrightarrow \underbrace{U(x+dx)}_{x_2}\; e^{i\int_1^2 gA_\mu \frac{dx^\mu}{ds} ds}\; U^\dagger(x)$$

$$e^{ig\int_{x_i}^{x_f} A_\mu \frac{dx^\mu}{ds} ds} \longrightarrow U(x_f)\, e^{i\int_{n-1}^{n}}\, \cancel{U^\dagger(x_{n-1})}\, \cancel{U(x_{n-1})}\, e^{i\int_{n-2}^{n-1}}\, \cancel{U^\dagger(x_{n-2})}$$

$$\cdots \cancel{U_2}\, e^{i\int_1^2}\, U^\dagger(x_i)$$

$$= U(x_f)\; e^{ig\int_{x_i}^{x_f} A_\mu \frac{dx^\mu}{ds} ds}\; U^\dagger(x_i) \quad \checkmark$$

$$W(s) = \hat{P}\, e^{ig\int_{x_i}^{x(s)} A_\mu \frac{dx^\mu}{ds} ds}$$

$$\partial_s W = \, ?$$

$$W(s+\Delta s) = e^{ig\int_{x(s)}^{x(s+\Delta s)} A_\mu \frac{dx^\mu}{ds} ds} \, W(s)$$

$$= \left(1 + ig\, A_\mu(x(s))\, \dot{x}^\mu \Delta s\right) W(s)$$

$$\Delta W = ig\, A_\mu \dot{x}^\mu W \Delta s$$

$$\boxed{\partial_s W = ig\, A_\mu(x(s))\, \dot{x}^\mu\, W}$$ dif eq. for W

$$W(0) = \mathbb{1}. \quad \text{b.c.}$$

gauge transf. $\tilde{A}_\mu = U A_\mu U^\dagger - \frac{i}{g} \partial_\mu U U^\dagger$

$$\partial_s \tilde{W} = \left(ig\, U A_\mu U^\dagger + \partial_\mu U U^\dagger\right) \dot{x}^\mu\, \tilde{W} \qquad ; \quad \tilde{W}(0) = \mathbb{1}$$

$$= ig\, U A_\mu U^\dagger \dot{x}^\mu \tilde{W} + \partial_s U U^\dagger \tilde{W}$$

$$\hat{W} = U^\dagger \tilde{W}$$

~~$\partial_s \hat{W} = \partial_s U^\dagger \tilde{W} +$~~

$$\partial_s \hat{W} = \partial_s U^\dagger \tilde{W} + U^\dagger\left(ig\, U A_\mu \hat{W} + \partial_s U \hat{W}\right)$$

$$= \cancel{\partial_s U^\dagger U \hat{W}} + ig\, A_\mu \hat{W} + \cancel{U^\dagger \partial_s U \hat{W}}$$

$$\partial_s \hat{W} = ig\, A_\mu \hat{W} \qquad \hat{W}(0) = U^\dagger$$

$$W = \hat{W}\, U(0)$$

$$\partial_s W = ig A_\mu W$$

$$\hat{W}(0) = 0$$

$$\hat{\tilde{W}} = U\hat{W} = U W U^\dagger(0)$$

$$\boxed{\hat{\tilde{W}}(s) = U(s)\, W(s)\, U^\dagger(0)}$$

same gauge transf.

more formal derivation.

Wilson loop.

$$\hat{P}\, e^{ig\int_{x_i}^{x_f} A_\mu dx^\mu}$$ depends on the path

closed path

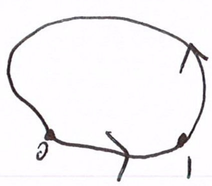
$$W = \mathrm{Tr}\left\{ e^{ig\oint A_\mu dx^\mu} \right\}$$

Wilson loop.

$U(x) \text{ - — } U^\dagger(x)$ gauge transf.

cancels in the trace.

Does not depend on the initial point



$$W_0 = \mathrm{Tr}(W_{10} W_{01}) =$$

$$= \mathrm{Tr}(W_{01} W_{10}) = W_1$$

$$W_{xx} = \hat{P}\, e^{ig\oint A_\mu dx^\mu} \neq \mathbb{1}$$

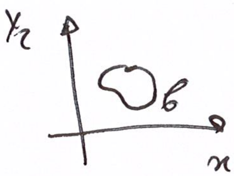
ψ_x Does not come back to ψ_x after a loop.

$$\psi_x \longrightarrow W_{xx}\,\psi_x$$

holonomy. Should be an element of the gauge group.

Measure "curvature" of the bundle.

when size $\to 0$.



$x_1(s)$ $x_2(s)$

$$W_\sigma = \hat{P}\left(\mathbb{1} + ig\oint A_\mu dx^\mu - \frac{g^2}{2}\oint A_\mu(x)\,\dot{x}^\mu ds_1 \oint A_\nu(x)\,\dot{x}^\nu ds_2 \dots\right)$$

$$= \mathbb{1} + ig\int_0^s ds\, A_\mu \dot{x}^\mu ds - g^2 \int_0^s ds_1 \int_0^{s_1} ds_2\, A_\mu(x(s_1)) A_\nu(x(s_2)) \cdot \dot{x}^\mu(s_1)\,\dot{x}^\nu(s_2)$$

further approximation

at center, for example.

$$A_\mu(x) = \bar{A}_\mu + \overline{\partial_\alpha A_\mu}\, x^\alpha + \dots$$

$$W_6 = \mathbb{1} + ig \int_0^{s_f} ds \, \bar{A}_\mu \underbrace{\dot{x}^\mu ds}_{\cancel{0}} +$$

$$+ ig \int_0^{s_f} ds \, \partial_\alpha \overrightarrow{A}_\mu \, x^\alpha \dot{x}^\mu ds + \ldots$$

$$- g^2 \int_0^{s_f} ds_1 \int_0^{s_1} ds_2 \, \bar{A}_\mu \bar{A}_\nu \, \dot{x}^\mu(s_1) \dot{x}^\nu(s_2)$$

$$\oint_0^{s_f} \dot{x}^\mu ds = x_f - x_i = 0. \quad \text{(closed loop).}$$

$$\oint_0^{s_f} ds \; \overbrace{x^\alpha \dot{x}^\mu}^{\text{antisymmetric.}} \cancel{ds} = \underbrace{\int_0^{s_f} ds \; \partial_s (x^\alpha x^\mu)}_{0} - \oint ds \; \underline{\dot{x}^\alpha x^\mu}$$

$$\hat{f}_{\alpha\mu} = \frac{1}{2} \oint_0^{s_f} ds \left(x^\alpha \dot{x}^\mu - x^\mu \dot{x}^\alpha \right)$$

$$\oint_0^{s_f} ds_1 \int_0^{s_1} ds_2 \; \dot{x}^\mu(s_1) \dot{x}^\nu(s_2) = \int_0^{s} ds_1 \; \underbrace{\dot{x}^\mu(s_1)(x^\nu(s_1) - x^\nu(s_0)}_{\text{total derivative}})$$

$$= \oint ds \; \dot{x}^\mu x^\nu = - \hat{f}_{\mu\nu}$$

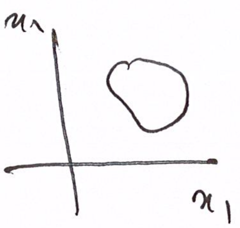
$$W_C = \mathbb{1} + \frac{ig}{2}\left(\underset{\mu\,\nu}{\partial_\alpha \mathbb{A}_\mu} - \underset{\nu}{\partial_\mu}\,\underset{\mu}{\mathbb{A}_\alpha}\right)\underset{\mu\nu}{\hat{f}_{\alpha\mu}} -$$

$$+\, g^2 \frac{1}{2}\left[\mathbb{A}_\mu , \mathbb{A}_\nu\right]\hat{f}_{\mu\nu}$$

$$= \mathbb{1} + \frac{ig}{2}\left(\partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu - ig\left[\mathbb{A}_\mu, \mathbb{A}_\nu\right]\right)\hat{f}_{\mu\nu}$$

$$= \mathbb{1} + \frac{ig}{2}\,\mathbb{F}_{\mu\nu}\,\hat{f}^{\mu\nu} + \text{---}$$

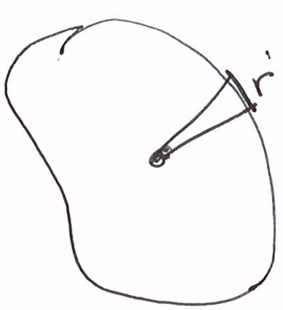
$$\hat{f}^{\mu\nu} = \frac{1}{2}\oint ds\,\left(x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu\right)$$



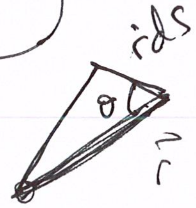
suppose loop is projected on x_1, x_2

$$\hat{f}^{12} = \frac{1}{2}\oint ds\,\left(x^1 \dot{x}^2 - x^2 \dot{x}^1\right) = \frac{1}{2}\oint ds\,\left|\vec{r}\wedge\dot{\vec{r}}\right|$$

$$\begin{array}{ccc} 1 & 2 & 3 \\ (x_1, & x_2, & 0) \\ \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \end{array}$$



$$= \oint ds\,\frac{1}{2}\,|\vec{r}|.|\dot{\vec{r}}|.s\theta = \oint d\,\underset{\text{loop}}{\text{area}} = \underline{\text{area}}_{x_1 x_2}$$



$\hat{f}^{12}$ area of projection on $\underline{12}$.

$$\underbrace{\mathbb{F}_{\mu\nu} = -\frac{i}{g}\,\frac{(W-\mathbb{1})}{\text{area}_{\mu\nu}}}_{\text{as area} \to 0.}$$

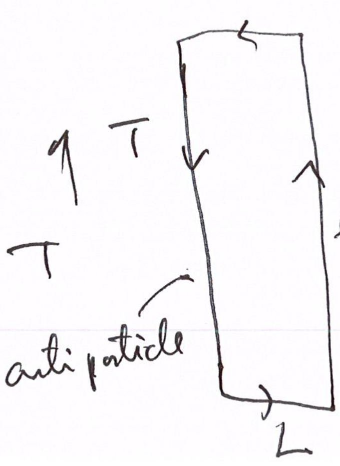
However

$$\mathrm{Tr}\,W = \underbrace{N}_{SU(N)} + \frac{ig}{2}\,\underbrace{\mathrm{Tr}\,F_{\mu\nu}}_{0}\, f^{\mu\nu} + \ldots$$

Expanding at higher orders we get gauge invariant operators.

In a pure gauge theory (no fermions), Wilson loops are all observables. (theory of Wilson loops?)

Physical properties $e^{ig\oint A_\mu \underbrace{dx^\mu}_{\text{like a current}}}$



← particle $\langle W \rangle_{\text{vacuum}} \sim e^{-E(L)T}$ $(T \to \infty)$

$E(L)$: energy of two charges separated by distance L.

We can already write the QCD Lagrangian

$$\mathcal{L} = -\frac{1}{2} \mathrm{Tr} \mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu} + \sum_{k=1}^{N_f} \bar{q}_k \, (i \not{D} - m_k) \, q_k$$

$$\mathbb{F}_{\mu\nu} = \partial_\mu \mathbb{A}_\nu - \partial_\nu \mathbb{A}_\mu - ig \, [\mathbb{A}_\mu, \mathbb{A}_\nu]$$

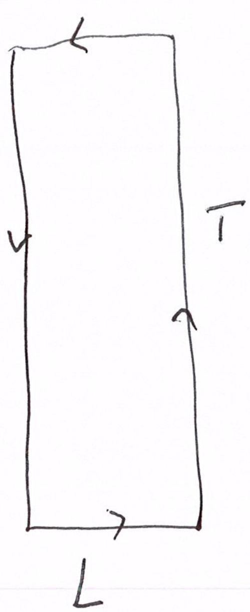
$$D_\mu q_k = (\partial_\mu - ig \mathbb{A}_\mu) q_k$$

$$\mathbb{A}_\mu = \sum_{a=1}^{8} A_\mu^a \, T^a \qquad \mathrm{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$$

$$[T^a, T^b] = i f^{abc} T^c$$

There appears to be 8 massless vector bosons. However they do not exist as asymptotic states due to confinement. Only non-charged (under SU(3)) states have finite energy.

Wilson loop.



$E \sim \alpha L$ potential goes as L

$W \sim e^{\underbrace{-\alpha TL}_{\text{area law}}}$ force as L?

∞ energy to pull probe charges apart

with quarks, $q\bar{q}$ pairs appear in the middle.

In the case of the weak interactions there are massive vector bosons.

$W^{\pm}_{\mu}$; Z_{μ} : How is it possible?

Spontaneous symmetry breaking (Higgs phenomenon).

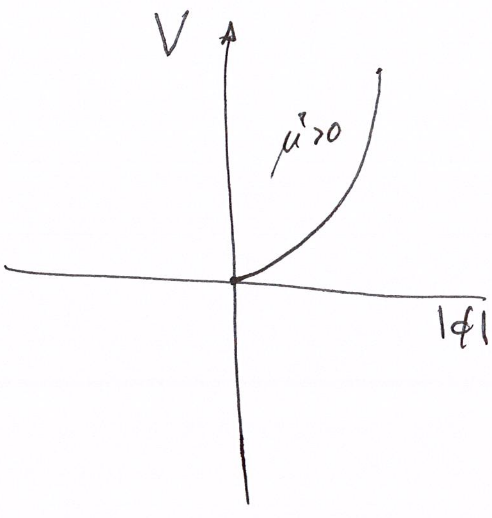
·) $U(1)$; Abelian Higgs model.

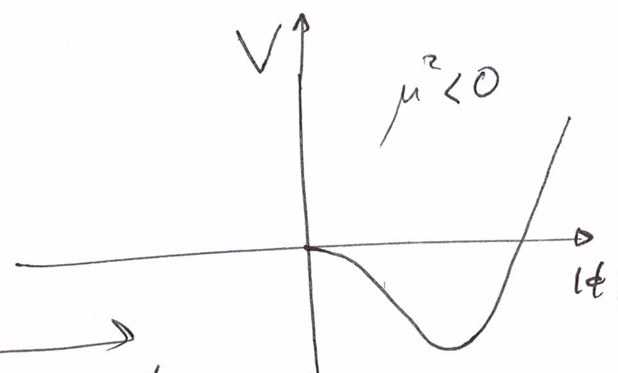
$$\mathcal{L} = (D_\mu \phi)^\dagger (D_\mu \phi) + \mu^2 |\phi|^2 - \lambda |\phi|^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

charged scalar. Same model as we used for phase transitions but now it is charged.

$$\phi \to e^{-i\alpha} \phi \qquad A_\mu \to A_\mu - \frac{1}{g} \partial_\mu \alpha . \qquad \boxed{\lambda > 0}$$

for stability





phase transition

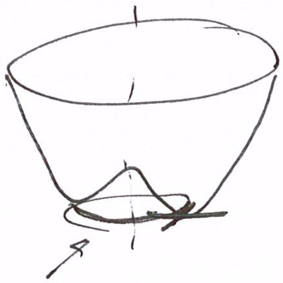
But different physics now.

$$V = +\mu^2 |\phi|^2 + \lambda |\phi|^4 \qquad \frac{\partial V}{\partial |\phi|} = 2\mu^2 |\phi| + 4\lambda |\phi|^3 = 0$$

·) $|\phi| = 0$ ··) $|\phi|^2 = -\frac{\mu^2}{2\lambda}$ if $\mu^2 < 0 \Rightarrow |\phi| = \sqrt{\frac{-\mu^2}{2\lambda}}$

$v = \sqrt{\frac{-\mu^2}{\lambda}}$ $= v/\sqrt{2}$

If $\underline{\mu^2<0}$ then $|\phi| = \frac{v}{\sqrt{2}}$ has the lowest energy.



all these are vacua

Without the gauge field we would have a massive scalar ("radial" excitation) and a massless scalar (Goldston boson, "angular" excit.)

However, let's see here:

Unitary parameterization (gauge)

$|\phi_0| = \frac{v}{\sqrt{2}}$ $\rightarrow$ choose vacuum $\phi_0 = \frac{v}{\sqrt{2}}$ (we are choosing a phase)

Now we introduce two scalar fields $\eta(x)$, $\xi(x)$:

$$\phi = \frac{(v+\eta(x))}{\sqrt{2}} e^{i\frac{\xi(x)}{v}} \rightarrow |\phi|^2 = \frac{(v+\eta)^2}{2}$$

$$D_\mu\phi = \partial_\mu\phi - igA_\mu\phi = \frac{1}{\sqrt{2}}\partial_\mu\eta\, e^{i\xi/v} + \frac{i\eta}{\sqrt{2}}\frac{\partial_\mu\xi}{v} e^{i\xi/v} +$$

$$+ \frac{\cancel{v}}{\sqrt{2}} i \frac{\partial_\mu\xi}{\cancel{v}} e^{i\xi/v} - igA_\mu\frac{v}{\sqrt{2}} e^{i\xi/v} - igA_\mu\frac{\eta}{\sqrt{2}} e^{i\xi/v}$$

$$= \left(\frac{1}{\sqrt{2}}\partial_\mu\eta + i\left(\frac{\eta}{\sqrt{2}}\frac{\partial_\mu\xi}{v} + \frac{\partial_\mu\xi}{\sqrt{2}} - gA_\mu\frac{v}{\sqrt{2}} - gA_\mu\frac{\eta}{\sqrt{2}}\right)\right) e^{i\xi/v}$$

$$|D_\mu\phi|^2 = \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{2}\left(\partial_\mu\xi + \frac{\eta}{v}\partial_\mu\xi - gA_\mu v - gA_\mu\eta\right)^2$$

$\partial_\mu \xi - g v A_\mu$ → will give $A_\mu \partial^\mu \xi$ mixed terms in the quadratic part.

rewrite $\quad -gv\left(A_\mu - \frac{1}{gv}\partial_\mu \xi\right)$

Define $\quad B_\mu = A_\mu - \frac{1}{gv}\partial_\mu \xi$

$$|D_\mu \phi|^2 = \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta + \frac{1}{2}\left(-gv\,B_\mu - g\eta\,B_\mu\right)^2$$

$$= \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta + \frac{g^2v^2}{2}B_\mu B^\mu + g^2 v\eta\,B_\mu B^\mu + \frac{1}{2}g^2\eta^2 B_\mu B^\mu$$

massive Vector boson.

interaction

interaction

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu\left(B_\nu + \frac{1}{gv}\partial_\nu\xi\right) - \partial_\nu\left(B_\mu + \frac{1}{gv}\partial_\mu\xi\right)$$

$$= \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$V = \mu^2\frac{(v+\eta)^2}{2} + \lambda\frac{(v+\eta)^4}{4} = -\frac{v^2\lambda}{2}\left(v^2+2v\eta+\eta^2\right) + \frac{\lambda}{4}\left(v^4+4v^3\eta+6v^2\eta^2+4v\eta^3+\eta^4\right)$$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\eta\,\partial^\mu\eta - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2v^2}{2}B_\mu B^\mu + g^2 v\eta\,B_\mu B^\mu + \frac{1}{2}g^2\eta^2 B_\mu B^\mu$$

$$+ \frac{\lambda v^4}{2} + \cancel{v^3\lambda\eta} + \frac{v^2\lambda}{2}\eta^2 - \frac{\lambda}{4}v^4 - \cancel{\lambda v^3\eta} - \frac{3}{2}\lambda v^2\eta^2 - \lambda v\eta^3 - \frac{\lambda}{4}\eta^4$$

$$-\left(-\lambda v^4/4 + v^2\lambda\eta^2 + \lambda v\eta^3 + \frac{\lambda}{4}\eta^4\right)$$

$$\mathcal{L} = +\frac{\lambda v^4}{4} \qquad \text{(vacuum energy)}$$

$$+\frac{1}{2}\partial_\mu \eta\, \partial^\mu \eta - \frac{2v^2\lambda}{2}\eta^2 \qquad \text{massive scalar } m = \sqrt{2\lambda v^2}$$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g^2 v^2}{2} B_\mu B^\mu \qquad \text{massive vector boson } M = gv$$

$$-\lambda v \eta^3 - \frac{\lambda}{4}\eta^4$$

 scalar self-interaction.

$$+ g^2 v\, \eta\, B_\mu B^\mu$$

 ηBB interaction

$$+\frac{1}{2} g^2 \eta^2 B_\mu B^\mu$$

 $\eta\eta BB$ interaction.

(Notice $B_\mu B^\mu = B_0^2 - \vec{B}^2$)

$\uparrow$ mass term has $-$ sign

__Spectrum__

1 scalar $m = \sqrt{2\lambda v^2}$

1 vector boson $M = gv$ (3 polarizations)

$\Big\}$ 4 degrees of freedom

Instead of 2 scalars + 2 polarizations of a massless A_μ we get a massive B_μ and a massive scalar.

A_μ "eats" $\xi(x)$ to gain an extra polarization.

$\xi(x)$ dissappears in the Lagrangian.

The abelian Higgs model can be thought as a model of Superconductivity (Meissner effect $\equiv$ massive photon).
[However usual superconductors are non-relativistic]

Non-Abelian case

take $SU(2)$ $\qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - V(\phi) - \underbrace{\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu}}_{\frac{1}{2} \mathrm{Tr}\, \mathbb{F}_{\mu\nu} \mathbb{F}^{\mu\nu}}$$

$$D_\mu \phi = \left(\partial_\mu - ig \frac{\sigma^a}{2} A^a_\mu\right) \phi = \left(\partial_\mu - ig \mathbb{A}_\mu\right)\phi$$

$$V(\phi) = +\mu^2 (\phi^\dagger \phi) + \lambda (\phi^\dagger \phi)^2$$

Unitary parameterization

Vacuum $\quad \phi_0 = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad \rightarrow \quad \phi = \underbrace{e^{i\xi_a \frac{\sigma^a}{2}}}_{\substack{SU(2) \\ \text{rotation.}}} \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix}$

$3\,\xi_a + 1\,\eta \rightarrow 4$ fields ✓

It is important to notice that ϕ_0 is not invariant under <u>any</u> rotation (full $SU(2)$ is broken).

$$\phi = U(\xi) \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix}$$

It is like a gauge transformation.

We can do

$$\phi \rightarrow U^{-1}\phi = \tilde{\phi} = \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix}$$

$$A_\mu \longrightarrow \tilde{A}_\mu = U^{-1} A_\mu U - \frac{i}{g} \partial_\mu U^{-1} U = \mathbb{B}_\mu$$

$$\mathcal{L} = (D_\mu \tilde{\phi})^+ (D^\mu \tilde{\phi}) - V(\tilde{\phi}) - \frac{1}{2} \mathrm{Tr}\, \tilde{\mathbb{F}}_{\mu\nu} \tilde{\mathbb{F}}^{\mu\nu}$$

is equal to the original one.

$$V(\tilde{\phi}) = \mu^2 \tilde{\phi}^+ \tilde{\phi} + \lambda (\tilde{\phi}^+ \tilde{\phi})^2$$

$$= -\frac{\lambda v^4}{4} + v^2 \lambda \eta^2 + \lambda v \eta^3 + \frac{\lambda}{4} \eta^4$$

same as before

$$D_\mu \tilde{\phi} = \partial_\mu \tilde{\phi} - i g \mathbb{B}_\mu \tilde{\phi} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \partial_\mu \eta \end{pmatrix} - i g \mathbb{B}_\mu \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix}$$

$$(D_\mu \tilde{\phi})^+ (D_\mu \tilde{\phi}) = \frac{1}{2} \partial_\mu \eta \partial^\mu \eta - i g \left(0 \;\; \frac{\partial_\mu \eta}{\sqrt{2}}\right) \mathbb{B}_\mu \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix} + i g \left(0 \;\; \frac{v+\eta}{\sqrt{2}}\right) \mathbb{B}_\mu^+ \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \partial_\mu \eta \end{pmatrix} +$$

$$+ g^2 \left(0 \ \frac{v+\eta}{\sqrt{2}}\right) \mathbb{B}_\mu^+ \mathbb{B}_\mu \begin{pmatrix} 0 \\ \frac{v+\eta}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{2} \partial_\mu \eta \, \partial^\mu \eta - ig \frac{\partial_\mu \eta}{\sqrt{2}} \cancel{\frac{(v+\eta)}{\sqrt{2}} (0\,1) \mathbb{B}_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix} + ig \frac{\partial_\mu \eta}{\sqrt{2}} \frac{(v+\eta)}{\sqrt{2}} (0\,1) \mathbb{B}_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$+ g^2 \frac{(v+\eta)^2}{2} (0\,1) \mathbb{B}_\mu^+ \mathbb{B}_\mu \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathbb{B}_\mu = B_\mu^a \frac{\sigma^a}{2} \qquad \mathbb{B}_\mu \mathbb{B}^\mu = B_\mu^a B_\mu^b \frac{\sigma^a}{2} \frac{\sigma^b}{2} = \frac{1}{4} B_\mu^a B^{b\mu} \left(\delta^{ab} + \underbrace{i\varepsilon^{abc}\sigma^c}_{0}\right)$$

$$= \frac{1}{4} B_\mu^a B^{\mu a} \cdot \mathbb{1}_{2\times 2}$$

$$(0\,1) \mathbb{1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1.$$

$$(D_\mu \tilde{\phi})^+ (D^\mu \tilde{\phi}) = \frac{1}{2} \partial^\mu \eta \, \partial_\mu \eta + g^2 \frac{(v+\eta)^2}{8} B_\mu^a B^{\mu a}$$

$$\mathcal{L} = \frac{1}{2} \partial^\mu \eta \, \partial_\mu \eta + \frac{g^2 v^2}{8} B_\mu^a B^{a\mu} + \frac{g^2 v}{4} \eta B_\mu^a B^{\mu a} + \frac{g^2}{8} \eta^2 B_\mu^a B^{a\mu}$$

$$- \frac{\lambda v^4}{4} - v^2 \lambda \eta^2 - \lambda v \eta^3 - \frac{\lambda}{4} \eta^4 - \frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu}$$

$$F_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g \varepsilon^{abc} B_\mu^b B_\nu^c$$

$$B_\mu^a \rightarrow \text{mass} \sqrt{g^2 v^2 / 4} = \frac{gv}{2} \qquad \eta \text{ mass} \rightarrow \sqrt{2 v^2 \lambda}$$

3 massive vector bosons

1 massive scalar

__No__ massless particles

Comments

- For the quantum theory the problem is that massive vector bosons are not renormalizable in general.

- If the vacuum is invariant under a subgroup then the subgroup is unbroken and those gauge fields remain massless

e.g. $SU(2) \to U(1)$

one photon

2 massive vectors.