Gauge theories, perturbative computations

$$\mathcal{L}_g = \partial_\mu \bar{c}^a D^\mu c^a - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^{a\mu})^2$$

pure YM theory.

$$F^{a\mu\nu} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + g f^{abc} A^{b\mu} A^{c\nu}$$

$$D^\mu c^a = \partial^\mu c^a + g f^{abc} A^{\mu b} c^c$$

$\boxed{\xi = 1}$

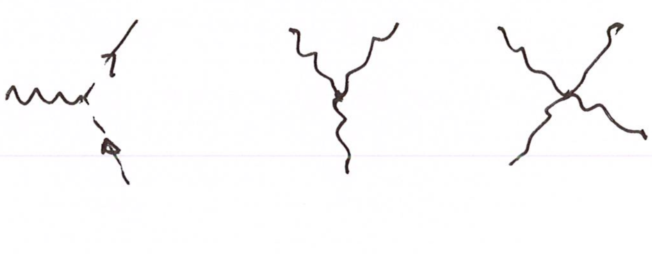
$$\Delta^{ab}_{\mu\nu} = - \frac{i}{k^2 + i\varepsilon} \eta_{\mu\nu} \delta^{ab}$$

$$\Delta^{ab}_{gh} = + \frac{i}{k^2 + i\varepsilon} \delta^{ab}$$

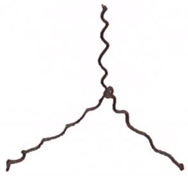
Interactions

$$\mathcal{V} = g f^{abc} A^{\mu b} \partial_\mu \bar{c}^a c^c - g \partial^\mu A^{\nu a} f^{abc} A^{b\mu} A^{c\nu}$$

$$- \frac{1}{4} g^2 f^{abc} f^{ade} A^{b\mu} A^{c\nu} A^d_\mu A^e_\nu$$



Vertices



$$-ig\, f^{abc} \int_x \partial_\mu A^a_\nu A^b_\mu A^c_\nu \, d^d x \qquad A^{a_1}_{\mu_1}(x_1)\, A^{a_2}_{\mu_2}(x_2)\, A^{a_3}_{\mu_3}(x_3)$$

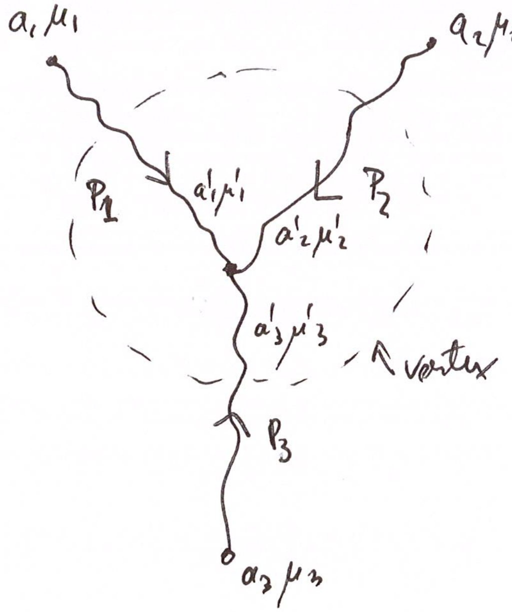
+ permutations (123).

$$-ig\, f^{abc} \int_x \partial^x_\mu \, \Delta^{aa_1}_{\nu\mu_1}(x-x_1)\, \Delta^{ba_2}_{\mu\mu_2}(x-x_2)\, \Delta^{ca_3}_{\nu\mu_3}(x-x_3)$$

$$-ig\, f^{abc} \int_x \underset{\displaystyle -ip_{1\mu}}{\underset{\downarrow}{\partial^x_\mu}} \prod_{i=1}^{3} \int \frac{d^d p_i}{(2\pi)^d}\, e^{-ip_1(x-x_1)-ip_2(x-x_2)-ip_3(x-x_3)}\, \Delta^{aa_1}_{\nu\mu_1}(p_1)\, \Delta^{ba_2}_{\mu\mu_2}(p_2)\, \Delta^{ca_3}_{\nu\mu_3}(p_3)$$

$$-g\, f^{abc} \underbrace{\prod_{j=1}^{3} \int \frac{d^d p_j}{(2\pi)^d}}_{\text{momentum integral}} \underbrace{(2\pi)^d\, \delta^{(d)}(p_1+p_2+p_3)}_{\delta\text{-function at vertex}}\, p_1^\mu \underbrace{\Delta^{aa_1}_{\nu\mu_1}(p_1)\, \Delta^{ba_2}_{\mu\mu_2}(p_2)\, \Delta^{ca_3}_{\nu\mu_3}(p_3)}_{\text{propogators}}$$

the rest is the vertex



$$\to -(2\pi)^d \delta^{(d)}(p_1+p_2+p_3) \int \prod_{j=1}^{3} \frac{d^d p_j}{(2\pi)^d} \times g f^{a'_1 a'_2 a'_3}$$

$$\times\, p_1^{\mu'_2}\, \eta^{\mu'_1 \mu'_3} \quad \Delta^{a'_1 a_1}_{\mu'_1 \mu_1}(p_1)\, \Delta^{a'_2 a_2}_{\mu'_2 \mu_2}(p_2)\, \Delta^{a'_3 a_3}_{\mu'_3 \mu_3}(p_3)$$

Vertex : $\quad -g f^{a'_1 a'_2 a'_3}\, p_1^{\mu'_2}\, \eta^{\mu'_1 \mu'_3} \quad +$ permutations.

123 132 213 231 312 321

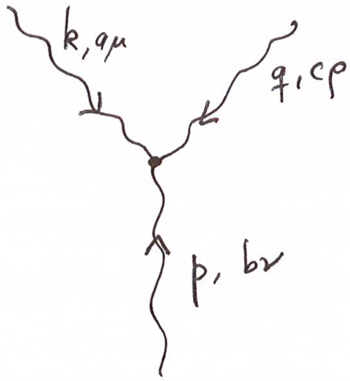
$$-g\Big(f^{a'_1 a'_2 a'_3} p_1^{\mu'_2} \eta^{\mu'_1 \mu'_3} + f^{a'_1 a'_2 a'_3} p_1^{\mu'_3} \eta^{\mu'_1 \mu'_2} + f^{a'_2 a'_1 a'_3} p_2^{\mu'_1} \eta_{\mu'_2 \mu'_3}$$

$$+ f^{a'_2 a'_3 a'_1} p_2^{\mu'_3} \eta_{\mu'_1 \mu'_2} + f^{a'_3 a'_1 a'_2} p_3^{\mu'_1} \eta^{\mu'_2 \mu'_3} + f^{a'_3 a'_2 a'_1} p_3^{\mu'_2} \eta_{\mu'_1 \mu'_3}\Big)$$

$$-g f^{a'_1 a'_2 a'_3} \Big(p_1^{\mu'_2} \eta^{\mu'_1 \mu'_3} - p_1^{\mu'_3} \eta^{\mu'_1 \mu'_2} - p_2^{\mu'_1} \eta_{\mu'_2 \mu'_3} + p_2^{\mu'_3} \eta_{\mu'_1 \mu'_2}$$

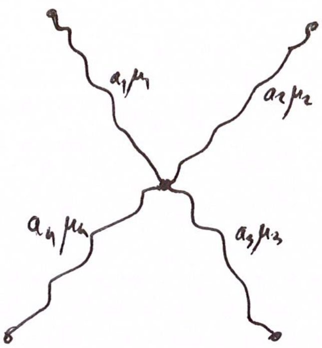
$$+ p_3^{\mu'_1} \eta^{\mu'_2 \mu'_3} - p_3^{\mu'_2} \eta_{\mu'_1 \mu'_3} \Big)$$

$$= -g f^{a'_1 a'_2 a'_3} \Big((p_1 - p_3)^{\mu'_2} \eta^{\mu'_1 \mu'_3} + (p_2 - p_1)^{\mu'_3} \eta_{\mu'_1 \mu'_2} + (p_3 - p_2)^{\mu'_1} \eta^{\mu'_2 \mu'_3} \Big)$$



$$-g f^{acb}\left(\left((k-p)^{\rho}\eta^{\mu\nu}+(q-k)^{\nu}\eta^{\mu\rho}+(p-q)^{\mu}\eta^{\nu\rho}\right)\right.$$

$$\boxed{g f^{abc}\left((k-p)^{\rho}\eta^{\mu\nu}+(q-k)^{\nu}\eta^{\mu\rho}+(p-q)^{\mu}\eta^{\nu\rho}\right)}$$



$$-\frac{1}{4} g^2 f^{abc} f^{ade}\int_x A^b_{\mu} A^c_{\nu} A^{d\mu} A^{e\nu}$$

$$\cdot A^{a_1}_{\mu_1}(x_1)\, A^{a_2}_{\mu_2}(x_2)\, A^{a_3}_{\mu_3}(x_3)\, A^{a_4}_{\mu_4}(x_4)$$

$$-\frac{i}{4} g^2 f^{abc} f^{ade}\int_x \Delta^{ba_1}_{\mu\mu_1}(x-x_1)\,\Delta^{ca_2}_{\nu\mu_2}(x-x_2)\,\Delta^{da_3\mu}{}_{\mu_3}(x-x_3)\,\Delta^{ca_4\nu}{}_{\mu_4}(x-x_4)$$

+ permutation (1234)

$$-\frac{i}{4} g^2 f^{abc} f^{ade}\int\prod_{j=1}^{4}\frac{d^d p_j}{(2\pi)^d}(2\pi)^d\,\delta^{(d)}(p_1+\dots+p_4)\,\Delta^{ba_1}_{\mu\mu_1}(p_1)\,\Delta^{ca_2}_{\nu\mu_2}(p_2)\,\Delta^{da_3\mu}{}_{\mu_3}(p_3)\,\Delta^{ca_4\nu}{}_{\mu_4}(p_4)$$

$$-\frac{i}{4} g^2\int\prod_{j=1}^{4}\frac{d^d p_j}{(2\pi)^d} f^{aa_1'a_2'} \underbrace{f^{aa_3'a_4'}}_{(2\pi)^d\delta^{(d)}(p_1+\dots+p_4)} \eta^{\mu_1'\mu_3'}\eta^{\mu_2'\mu_4'}\,\Delta^{a_1'a_1}_{\mu_1'\mu_1}(p_1)\,\Delta^{a_2'a_2}_{\mu_2'\mu_2}(p_2)\,\Delta^{a_3'a_3}_{\mu_3'\mu_3}(p_3)\,\Delta^{a_4'a_4}_{\mu_4'\mu_4}(p_4)$$

Vertex

$$-\frac{i}{4} g^2 f^{a a'_1 a'_2} f^{a a'_3 a'_4} \eta^{\mu'_1 \mu'_3} \eta^{\mu'_2 \mu'_4} \quad + \quad (1234) \text{ permutations.}$$

(1234) (1243) (1324) (1342) (1423) (1432) ← only this are relevant

(2 - -

} 24 permutations.

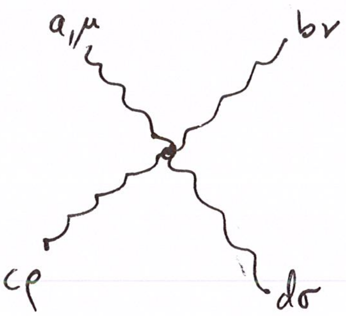
However we can fix 1 since they are equivalent. We get

$$-ig^2 f^{a a'_1 a'_2} f^{a a'_3 a'_4} \eta^{\mu'_1 \mu'_3} \eta^{\mu'_2 \mu'_4} , \; - g^2 f^{a a'_1 a'_2} f^{a a'_4 a'_3} \eta^{\mu'_1 \mu'_4} \eta^{\mu'_2 \mu'_3}$$

$$-ig^2 f^{a a'_1 a'_3} f^{a a'_2 a'_4} \eta^{\mu'_1 \mu'_2} \eta^{\mu'_3 \mu'_4} - g^2 f^{a a'_1 a'_3} f^{a a'_4 a'_2} \eta^{\mu'_1 \mu'_4} \eta^{\mu'_3 \mu'_2}$$

$$-ig^2 f^{a a'_1 a'_4} f^{a a'_2 a'_3} \eta^{\mu'_1 \mu'_2} \eta^{\mu'_4 \mu'_3} - g^2 f^{a a'_1 a'_4} f^{a a'_3 a'_2} \eta^{\mu'_1 \mu'_3} \eta^{\mu'_4 \mu'_2}$$

$$\begin{aligned} -ig^2 \Big[& f^{a a'_1 a'_2} f^{a a'_3 a'_4} \left(\eta^{\mu'_1 \mu'_3} \eta^{\mu'_2 \mu'_4} - \eta^{\mu'_1 \mu'_4} \eta^{\mu'_2 \mu'_3} \right) \\ & f^{a a'_1 a'_3} f^{a a'_2 a'_4} \left(\eta^{\mu'_1 \mu'_2} \eta^{\mu'_3 \mu'_4} - \eta^{\mu'_1 \mu'_4} \eta^{\mu'_2 \mu'_3} \right) \\ & f^{a a'_1 a'_4} f^{a a'_2 a'_3} \left(\eta^{\mu'_1 \mu'_2} \eta^{\mu'_3 \mu'_4} - \eta^{\mu'_1 \mu'_3} \eta^{\mu'_2 \mu'_4} \right) \Big] \end{aligned}$$



$$\begin{aligned} -ig^2 \Big[& f^{abe} f^{cde} \left(\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho} \right) \\ & f^{ace} f^{bde} \left(\eta^{\mu\nu} \eta^{\sigma\rho} - \eta^{\mu\sigma} \eta^{\nu\rho} \right) \\ & f^{ade} f^{bce} \left(\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma} \right) \Big] \end{aligned}$$

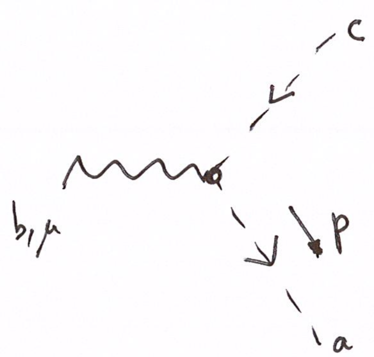
$$ig f^{abc} \int_x A^{\mu b}\, \partial_\mu \bar{c}^{a}\, c^{c} \qquad A^{a_1\mu_1}(x_1)\, \bar{c}^{a_2}(x_2)\, c^{a_3}(x_3)$$

$$-ig f^{abc} \int_x \Delta^{\mu\mu_1\, ba_1}(x-x_1)\, \partial^x_\mu \Delta_{gh}^{a a_2}(x-x_2)\, \Delta_{gh}^{a_3 c}(x_3-x)$$

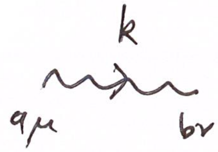
$$-ig f^{abc} \int \prod_{j=1}^{3} \frac{d^d p_j}{(2\pi)^d}\, \Delta^{ba_1}_{\mu\mu_1}(p_1)\, (-i p_{2\mu})\, \Delta_{gh}^{aa_2}(p_2)\, \Delta_{gh}^{a_3 c}(-p_3)\, \underset{(2\pi)^d}{\delta(p_1+p_2+p_3)}.$$

$$-g f^{abc} \int \prod_{j=1}^{3} \frac{d^d p_j}{(2\pi)^d}\, p_{2\mu}\, (2\pi)^d\, \delta(p_1+p_2+p_3)\, \Delta^{ba_1}_{\mu\mu_1}(p_1)\, \Delta_{gh}^{aa_2}(p_2)\, \Delta_{gh}^{a_3 c}(-p_3)$$

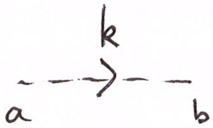
$$-g f^{a_2' a_1' a_3'} \int \prod_{j=1}^{3} \frac{d^d p_j}{(2\pi)^d}\, p_{2\mu_1'} \qquad \Delta^{a_1' a_1}_{\mu_1'\mu_1}(p_1)\, \Delta^{a_2' a_2}(p_2)\, \Delta^{a_3 a_3'}(-p_3)$$



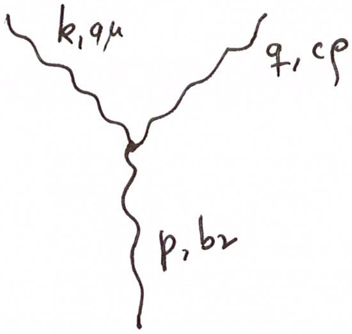
$$-g f^{abc}\, p_{2\mu}$$



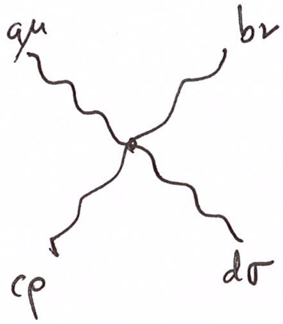
$$-\frac{i}{k^2+i\varepsilon}\,\eta_{\mu\nu}\,\delta^{ab}$$



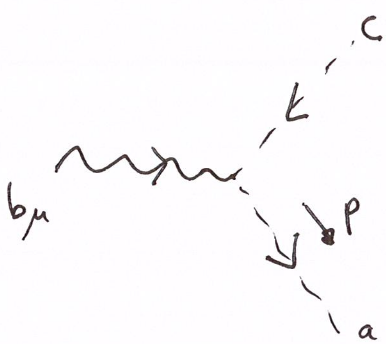
$$+\frac{i}{k^2+i\varepsilon}\,\delta^{ab}$$



$$g f^{abc}\left((k-p)^\rho \eta^{\mu\nu} + (q-k)^\nu \eta^{\mu\rho} + (p-q)^\mu \eta^{\nu\rho}\right)$$

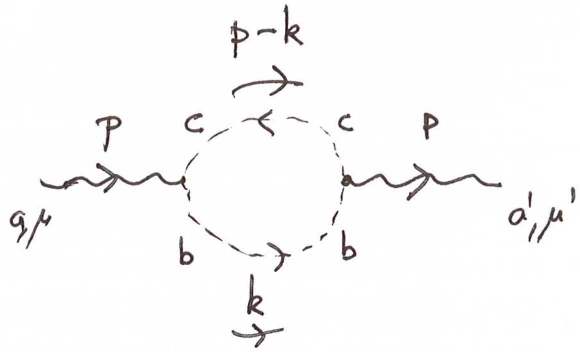


$$\begin{aligned}-ig^2\Big[& f^{abe}f^{cde}\left(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}\right)\\ & + f^{ace}f^{bde}\left(\eta^{\mu\nu}\eta^{\sigma\rho}-\eta^{\mu\sigma}\eta^{\nu\rho}\right)\\ & + f^{ade}f^{bce}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}\right)\Big]\end{aligned}$$



$$-g f^{abc}\, p_\mu$$

gluon propagator at 1-loop



$$-\int_k \left(-g f^{bac} k_\mu\right)\left(-g f^{ca'b} (k-p)_{\mu'}\right) \frac{+i}{k^2+i\epsilon}\,\frac{+i}{(p-k)^2+i\epsilon}$$

fermion loop

$$+g^2 f^{acb} f^{a'bc} \int \frac{d^dk}{(2\pi)^d} \frac{k_\mu (k-p)_{\mu'}}{k^2+i\epsilon \quad (p-k)^2+i\epsilon}$$

$$-g^2 f^{abc} f^{a'bc} \int \frac{d^dk}{(2\pi)^d} \int_0^1 d\alpha \frac{k_\mu (k-p)_{\mu'}}{\left(\alpha(p^2-2pk+k^2)+(1-\alpha)k^2+i\epsilon\right)^2}$$

$$-g^2 C_2(G)\, \delta^{aa'} \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{k_\mu (k-p)_{\mu'}}{\left(k^2-2\alpha pk+\alpha p^2+i\epsilon\right)^2}$$

$$(k-\alpha p)^2+\alpha(1-\alpha)p^2+i\epsilon$$

$$-g^2 C_2(G)\, \delta^{aa'} \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{(k_\mu+\alpha p_\mu)(k_{\mu'}-(1-\alpha)p_{\mu'})}{\left(k^2+\alpha(1-\alpha)p^2+i\epsilon\right)^2}$$

$$-g^2 C_2(G)\,\delta^{aa'} \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{k_\mu k_{\mu'} - \alpha(1-\alpha) p_\mu p_{\mu'}}{(k^2 + \alpha(1-\alpha)p^2 + i\varepsilon)^2}$$

$$-g^2 C_2(G)\,\delta^{aa'} \int_0^1 d\alpha \int \frac{d^dk}{(2\pi)^d} \frac{k^2 \frac{\eta_{\mu\mu'}}{d} - \alpha(1-\alpha) p_\mu p_{\mu'}}{(k^2 + \alpha(1-\alpha)p^2 + i\varepsilon)^2}$$

$$-g^2 C_2(G)\,\delta^{aa'} \int_0^1 d\alpha \left[\frac{1}{\cancel{d}} \eta_{\mu\mu'} \frac{(-i)}{(4\pi)^{d/2}} \frac{\cancel{d}}{2} \frac{\Gamma(1-d/2)}{\underset{1}{\Gamma(2)}} \frac{1}{\Delta^{1-d/2}} - \right.$$

$(\Delta = -\alpha(1-\alpha)p^2 - i\varepsilon)$

$$\left. - \alpha(1-\alpha) p_\mu p_{\mu'} \frac{i\,\Gamma(2-d/2)}{(4\pi)^{d/2}\,\underset{1}{\cancel{\Gamma(2)}}} \frac{1}{\Delta^{2-d/2}} \right]$$

$$i g^2 \frac{C_2(G)\,\delta^{aa'}}{(4\pi)^{d/2}} \int_0^1 d\alpha \left[\frac{\Gamma(1-d/2)\,\Delta}{2\,\Delta^{2-d/2}} \eta_{\mu\mu'} + \frac{\alpha(1-\alpha)}{\Delta^{2-d/2}} p_\mu p_{\mu'} \underset{(1-d/2)\Gamma(1-d/2)}{\Gamma(2-d/2)} \right]$$

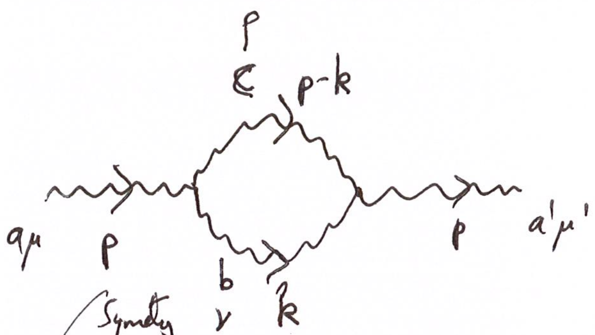
$$i g^2 \frac{C_2(G)\,\delta^{aa'}\,\Gamma(1-d/2)}{(4\pi)^{d/2}} \int_0^1 d\alpha \frac{1}{\Delta^{2-d/2}} \left[\frac{-\alpha(1-\alpha)p^2}{2} \eta_{\mu\mu'} + \alpha(1-\alpha) p_\mu p_{\mu'} (1-d/2) \right]$$

$$i g^2 C_2(G)\,\delta^{aa'} \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} \left(-\frac{p^2}{2}\eta_{\mu\mu'} + \left(1-\frac{d}{2}\right) p_\mu p_{\mu'} \right) \int_0^1 d\alpha \frac{\alpha(1-\alpha)}{(-p^2)^{2-d/2}\,\alpha^{2-d/2}\,(1-\alpha)^{2-d/2}}$$

$$\alpha^{d/2-1} \quad (1-\alpha)^{d/2-1}$$

$$ig^2 C_2(G)\,\delta^{aa'}\,\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\,(-p^2)^{d/2-2}\left(-\frac{p^2}{2}\eta_{\mu\mu'} + (1-d/2)\,p_\mu p_{\mu'}\right) B\left(\frac{d}{2},\frac{d}{2}\right)$$

$$\boxed{\frac{ig^2}{2}\, C_2(G)\,\delta^{aa'}\,\frac{\Gamma(1-d/2)}{(4\pi)^{d/2}}\,(-p^2)^{d/2-1}\left(\eta_{\mu\mu'} - (2-d)\frac{p_\mu p_{\mu'}}{p^2}\right) B(d/2,d/2)}$$



$-k-(-p+k) = p-2k$

$-p+k-p$

Symmetry factor

$$\frac{1}{2}\int g f^{abc}\left((p+k)^\rho \eta_{\mu\nu} + (p-2k)^\mu \eta_{\rho\nu} + (k-2p)^\nu \eta_{\mu\rho}\right)$$

symmetry

$$g f^{a'cb}\left((-2p+k)^\nu \eta_{\mu'\rho} + (p-2k)^{\mu'}\eta_{\rho\nu} + (k+p)^\rho \eta_{\nu\mu'}\right)$$

$$\left(\frac{i}{k^2+i\epsilon}\right)\left(\frac{i}{(p-k)^2+i\epsilon}\right)\frac{d^dk}{(2\pi)^d}$$

$$+\frac{g^2}{2} f^{abc} f^{a'bc}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha \cancel{\int\frac{d^dk}{(2\pi)^d}}\,\frac{N}{\left[k^2+\alpha(1-\alpha)p^2+i\epsilon\right]^2}$$

$$N = \left[((1+\alpha)p+k)^\rho \eta_{\mu\nu} + ((1-2\alpha)p-2k)^\mu \eta_{\rho\nu} + (k+(\alpha-2)p)^\nu \eta_{\mu\rho}\right]\cdot$$
$$\cdot\left[((\alpha-2)p+k)^\nu \eta_{\mu'\rho} + ((1-2\alpha)p-2k)^{\mu'}\eta_{\rho\nu} + (k+(1+\alpha)p)^\rho \eta_{\nu\mu'}\right]$$

$$\mathcal{N} = ((1+\alpha)p + k)_{\mu'} \, ((\alpha-2)p + k)_{\mu} + ((1+\alpha)p + k)_{\mu} \, ((1-2\alpha)p - 2k)_{\mu'}$$

$$((1+\alpha)p + k)(k + (1+\alpha)p)\,\eta_{\mu\mu'} + ((1-2\alpha)p - 2k)_{\mu} \, ((\alpha-2)p + k)_{\mu'}$$

$$d\,((1-2\alpha)p - 2k)_{\mu} \, ((1-2\alpha)p - 2k)_{\mu'} + ((1-2\alpha)p - 2k)_{\mu} \, (k + (1+\alpha)p)_{\mu'}$$

$$+ (k + (\alpha-2)p)((\alpha-2)p + k)\,\eta_{\mu\mu'} + (k + (\alpha-2)p)_{\mu} \, ((1-2\alpha)p - 2k)_{\mu'}$$

$$+ (k + (\alpha-2)p)_{\mu'} \, (k + (1+\alpha)p)_{\mu}$$

$$\rightarrow \; p_{\mu} p_{\mu'} \left(2(1+\alpha)(\alpha-2) + 2(1+\alpha)(1-2\alpha) + 2(1-2\alpha)(\alpha-2) + d\,(1-2\alpha)^2\right)$$

$$k_{\mu} k_{\mu'} \left(\cancel{1} - 2 - 2 + 4d - 2 - \cancel{2} + \cancel{1}\right)$$

$$\eta_{\mu\mu'} \left((1+\alpha)^2 p^2 + k^2 + k^2 + (\alpha-2)^2 p^2 \right)$$

$$\rightarrow \; p_{\mu} p_{\mu'} \left(2 \left(\alpha - 2 + \alpha^2 - 2\alpha + 1 - 2\alpha + \alpha - 2\alpha^2 + \alpha - 2 - 2\alpha^2 + 4\alpha\right) + d\,(1-2\alpha)^2\right)$$

$$\frac{k^2 \eta_{\mu\mu'}}{d} (4d - 6) + \eta_{\mu\mu'} \left(2k^2 + (\alpha^2 - 4\alpha + 4 + 1 + 2\alpha + \alpha^2)\, p^2\right)$$

$$\rightarrow \; p_{\mu} p_{\mu'} \left(2(-3\alpha^2 + 3\alpha - 3) + d\,(1 - 4\alpha + 4\alpha^2)\right) + \eta_{\mu\mu'} k^2 \left(4 - \frac{6}{d} + 2\right)$$

$$+ \eta_{\mu\mu'} p^2 \left(2\alpha^2 - 2\alpha + 5\right)$$

$$\rightarrow \; p_{\mu} p_{\mu'} \left(\alpha^2(4d-6) + \alpha(-4d+6) + (d-6)\right) + \eta_{\mu\mu'} k^2 \left(6 - 6/d\right) +$$

$$+ \eta_{\mu\mu'} p^2 \left(2\alpha^2 - 2\alpha + 5\right)$$

$$\frac{1}{2} g^2 f^{abc} f^{a'bc} \int_0^1 d\alpha \Big\{ \Big[p_\mu p_{\mu'} \left(\alpha^2(4d-6) + \alpha(-4d+6) + (d-6) \right) +$$

$$+ \eta_{\mu\mu'} p^2 (2\alpha^2 - 2\alpha + 5) \Big] \cdot \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\cancel{\Gamma(2)}} \frac{1}{\Delta^{2-d/2}} +$$

$$+ \eta_{\mu\mu'} (6 - 6/d) \frac{(-)\, i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(1-d/2)}{\cancel{\Gamma(2)}} \frac{1}{\Delta^{1-d/2}} \Big\}$$

$$\frac{1}{2} g^2 \frac{C_2(G)\,\delta^{aa'}}{(4\pi)^{d/2}}\, i\, \Gamma(1-d/2) \int_0^1 d\alpha \frac{1}{\Delta^{2-d/2}} \Big\{ (1-d/2) \Big(p_\mu p_{\mu'} \left(\alpha^2(4d-6) + \alpha(-4d+6) + (d-6) \right)$$

$$+ \eta_{\mu\mu'} p^2 (2\alpha^2 - 2\alpha + 5) \Big) - \eta_{\mu\mu'} \left(6 - \frac{6}{d}\right) \frac{d}{2}\, \Delta \Big\}$$

$$\frac{i g^2}{2} \frac{C_2(G)\,\delta^{aa'}}{(4\pi)^{d/2}} \Gamma\left(1 - \frac{d}{2}\right) (-p^2)^{d/2-2} \int_0^1 d\alpha\, \alpha^{d/2-2} (1-\alpha)^{d/2-2} \times$$

$$\times \Big\{ (1-d/2)\, p_\mu p_{\mu'} \left(\alpha^2(4d-6) + \alpha(-4d+6) + (d-6) \right) + \overset{(1-d/2)}{\eta_{\mu\mu'}}\, p^2 (2\alpha^2 - 2\alpha + 5) \Big)$$

$$+ 3\eta_{\mu\mu'}\, p^2 (d-1)\, \alpha(1-\alpha) \Big\}$$

$$\frac{i}{2} g^2 C_2(G) \frac{\delta^{aa'}}{(4\pi)^{d/2}} \Gamma(1-d/2) (-p^2)^{d/2-2} \times$$

$$\times \Big\{ (1-d/2)\, p_\mu p_{\mu'} \Big((4d-6)\, B\left(\tfrac{d}{2}+1, \tfrac{d}{2}-1\right) + (-4d+6) B\left(\tfrac{d}{2}, \tfrac{d}{2}-1\right) +$$

$$+ (d-6)\, B(d/2-1, d/2-1) \Big) + (1-d/2)\, \eta_{\mu\mu'} p^2 \Big(2B\left(\tfrac{d}{2}+1, \tfrac{d}{2}-1\right) - 2d\, B\left(\tfrac{d}{2}, \tfrac{d}{2}-1\right) + 5B\left(\tfrac{d}{2}-1, \tfrac{d}{2}-1\right) \Big)$$

$$+ 3\eta_{\mu\mu'} p^2 (d-1)\, B\left(\tfrac{d}{2}, \tfrac{d}{2}\right) \Big\}$$

$$B\left(\tfrac{d}{2}+1, \tfrac{d}{2}-1\right) = \frac{\Gamma\left(\frac{d}{2}+1\right)\Gamma\left(\frac{d}{2}-1\right)}{\Gamma(d)} = \frac{\frac{d}{2}\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)} \frac{\Gamma(d/2)}{\frac{d}{2}-1} = \frac{d}{d-2} B\left(\tfrac{d}{2}, \tfrac{d}{2}\right)$$

$$B\left(\tfrac{d}{2}, \tfrac{d}{2}-1\right) = \frac{\Gamma(d/2)\Gamma(d/2-1)}{\Gamma(d-1)} = \frac{\Gamma(d/2)}{\frac{\Gamma(d)}{d-1}} \frac{\Gamma(d/2)}{(d/2-1)} = \frac{2(d-1)}{d-2} B\left(\tfrac{d}{2}, \tfrac{d}{2}\right)$$

$$B\left(\tfrac{d}{2}-1, \tfrac{d}{2}-1\right) = \frac{\Gamma(d/2-1)\Gamma(d/2-1)}{\Gamma(d-2)} = \frac{\Gamma(d/2)\Gamma(d/2)}{\left(\frac{d}{2}-1\right)\left(\frac{d}{2}-1\right) \frac{\Gamma(d)}{(d-2)(d-1)}}$$

$$= \frac{4\cancel{(d-2)}(d-1)}{(d-2)^{\cancel{2}}} B\left(\tfrac{d}{2}, \tfrac{d}{2}\right)$$

$$\{\} = \Big\{ p_\mu p_{\mu'} \left(\frac{2-d}{2}\right)\Big((4d-6)\frac{d}{d-2} + (-4d+6)\frac{2(d-1)}{d-2} +$$

$$+ (d-6)\frac{4(d-1)}{d-2}\Big) + \frac{2-d}{2}\eta_{\mu\mu'} p^2 \left(\frac{2d}{d-2} - \frac{4(d-1)}{d-2} + \frac{20(d-1)}{d-2} \right) +$$

$$+ 3\eta_{\mu\mu'} p^2 (d-1) \Big\} B\left(\frac{d}{2}, \frac{d}{2}\right)$$

$$= B\left(\frac{d}{2},\frac{d}{2}\right) \Big\{ -\frac{1}{2} p_\mu p_{\mu'} \Big(4d^2 - 6d + (6-4d)(2d-2) + (4d-4)(d-6)\Big)$$

$$-\frac{1}{2}\eta_{\mu\mu'} p^2 \left(2d - 4d + 4 + 20d - 20\right) + 3\eta_{\mu\mu'} p^2 (d-1) \Big\}$$

$$= B\left(\frac{d}{2},\frac{d}{2}\right) \Big\{ -\frac{1}{2} p_\mu p_{\mu'} \Big(\cancel{4d^2} \underbrace{-6d} + \underbrace{12d} - 12 - \cancel{8d^2} + \underbrace{8d} + \cancel{4d^2} \underbrace{-24d - 4d} + 24\Big)$$

$$-\frac{1}{2}\eta_{\mu\mu'} p^2 \left(18d - 16\right) + 3\eta_{\mu\mu'} p^2 (d-1) \Big\}$$

$$= B\left(\frac{d}{2},\frac{d}{2}\right) \Big\{ -\frac{1}{2} p_\mu p_{\mu'} \left(-14d + 12\right) - \frac{1}{2}\eta_{\mu\mu'} p^2 (18d-16) + 3\eta_{\mu\mu'} p^2 (d-1)\Big)$$

$$= B\left(\frac{d}{2},\frac{d}{2}\right) \Big\{ p_\mu p_{\mu'} (7d-6) + \eta_{\mu\mu'} p^2 (3d - 3 - 9d + 8) \Big\}$$

$$= B\left(\frac{d}{2},\frac{d}{2}\right) \Big\{ p_\mu p_{\mu'} (7d-6) + \eta_{\mu\mu'} p^2 (-6d + 5) \Big\}$$

 $$= \frac{ig^2}{2} C_2(G) \frac{\delta^{aa'}}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) (-p^2)^{d/2-1} B\left(\frac{d}{2},\frac{d}{2}\right)$$

$$\times \left\{ \eta_{\mu\mu'} (6d-5) + \frac{p_\mu p_{\mu'}}{p^2} (6-7d) \right\}$$

 $= 0$ in dim. reg.

 $$= \frac{ig^2}{2} C_2(G) \frac{\delta^{aa'}}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) (-p^2)^{d/2-1} B\left(\frac{d}{2},\frac{d}{2}\right) \times$$

$$\times \left\{ \eta_{\mu\mu'} \underbrace{(1+6d-5)}_{6d-4} + \frac{p_\mu p_{\mu'}}{p^2} \underbrace{(-2+d+6-7d)}_{-6d+4} \right\}$$

 $$\overset{\text{1-loop}}{=} \frac{ig^2}{2} C_2(G) \frac{\delta^{aa'}}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) (-p^2)^{d/2-1} B\left(\frac{d}{2},\frac{d}{2}\right) (6d-4) \times$$

$$\times \left(\eta_{\mu\mu'} - \frac{p_\mu p_{\mu'}}{p^2} \right)$$

transverse.

Why transverse.? At lowest order:

$$i\Delta^{ab}_{\mu\nu} = -\frac{i}{(k^2+i\varepsilon)}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\nu k_\mu}{k^2}\right)\delta^{ab}$$

$$i\Delta^{ab}_{\mu\nu}k^\mu = -\frac{i}{k^2+i\varepsilon}\left(\cancel{k_\nu} - (\cancel{1}-\xi)k_\nu\right)\delta^{ab} = -\frac{i\xi k_\nu}{k^2+i\varepsilon}\delta^{ab}$$

Not transverse but $i\Delta^{ab}_{\mu}k^\mu$ given by gauge fixing parameter ξ

We can also write:

$$i k^\nu \Delta^{ab}_{\nu\mu} k^\mu = -i\xi\frac{\cancel{k^2}}{\cancel{k^2+i\varepsilon}}\delta^{ab} = -i\xi\delta^{ab}$$

this is true at all loops.

<u>BRST identities</u> (Ward id, Slavnov Taylor id, Ward Takahashi id. BRS id. ---)

Basic identities (any QFT indep. of symmetries)

$$\frac{\delta S}{\delta \phi_A} = 0$$ e.o.m. is satisfied in QM by the Heisenberg op. $\partial_t \phi_A = -i[H, \phi_A]$

ϕ_A ← any field

this is true in QFT (but (also) up to operator ordering, divergences etc)

Consider

$$\langle 0|\hat{T}\left\{\underbrace{\frac{\delta S}{\delta\phi_A}}_{x} f(\phi_A)\right\}|0\rangle = 0$$

any function (not functional, i.e. no derivatives)

Now $\dfrac{\delta S}{\delta \phi_A} = \partial_\mu \dfrac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} - \dfrac{\delta \mathcal{L}}{\delta \phi_A} = 0$

$$\langle 0 | \hat{T} \left\{ \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} f(\phi_A) - \frac{\delta \mathcal{L}}{\delta \phi_A} f(\phi_A) \right\} | 0 \rangle = 0$$

We want to get ∂_μ out. OK except $\mu = 0$ because of $\hat{T}$

$$\partial_0 \hat{T} \Big\{ \underbrace{\overbrace{\frac{\delta \mathcal{L}}{\delta \partial_0 \phi_A}}^{x}}_{\Pi_A} \overbrace{f(\phi_A)}^{y} \Big\} = \partial_0 \Big(\Theta(t_x - t_y)\, \Pi_A(x)\, f(\phi_A(y)) + $$

$$+ \Theta(t_y - t_x)\, f(\phi_A(y))\, \Pi_A(x) \Big)$$

$$= \hat{T} \left\{ \partial_0 \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_A} f(\phi_A) \right\} + \delta(t_x - t_y) \left[\Pi_A(x), f(\phi_A(y)) \right]$$

$$\left(\left[\Pi_A(x), \phi_A(y) \right]_{E.T.} = -i\, \delta^{(d-1)}(x-y) \right)$$

$$= \hat{T} \left\{ \partial_0 \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_A} f(\phi_A) \right\} - i\, \delta^{(d)}(x-y) \frac{\delta f}{\delta \phi_A}(x)$$

$$\langle 0 | \hat{T} \left\{ \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} f(\phi_A) - \frac{\delta \mathcal{L}}{\delta \phi_A} f(\phi_A) \right\} | 0 \rangle = 0 =$$

$$= \partial_\mu \langle 0 | \hat{T} \left\{ \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_A} f(\phi_A) \right\} | 0 \rangle - \langle 0 | \hat{T} \frac{\delta \mathcal{L}}{\delta \phi_A} f(\phi_A) | 0 \rangle +$$

$$+ i\, \delta^{(d)}(x-y) \frac{\delta f}{\delta \phi_A}$$

the equivalent of e.o.m. in QFT is

$$\boxed{\partial_\mu \langle 0|\hat{T}\Big\{\underbrace{\frac{\delta h}{\delta \partial_\mu \phi_A}}_{x}\underbrace{f(\phi_A)}_{y}\Big\}|0\rangle - \langle 0|\hat{T}\underbrace{\frac{\delta h}{\delta \phi_A}}_{x}\underbrace{f(\phi_A)}_{y}|0\rangle = -i\,\delta^{(d)}(x-y)\,\frac{\delta f}{\delta \phi_A}(x)}$$

(needs to be checked after regularization)

For global symmetries (including BRST) we have:

$$\delta \langle 0|\hat{T}\,\phi_{A_1}\cdots\phi_{A_n}|0\rangle = 0$$

if they are symmetries of the vacuum.

the symmetry is generated by a charge Q.

$$\delta\phi_A = -i\,[Q_A, \phi_A]$$

$$Q|0\rangle = 0.$$

$$\langle 0|\,[Q, \hat{T}\phi_{A_1}\cdots\phi_{A_n}]\,|0\rangle = 0 \qquad (Q|0\rangle = 0 \quad \langle 0|Q = 0)$$

$$\langle 0|\hat{T}\{[Q,\phi_{A_1}]\phi_{A_2}\cdots\phi_{A_n} + \phi_{A_1}[Q,\phi_{A_2}]\cdots\phi_{A_n} + \phi_{A_1}\cdots[Q,\phi_{A_n}]|0\rangle$$

$$= i\langle 0|\hat{T}\{\delta\phi_{A_1}\phi_{A_2}\cdots\phi_{A_n} + \phi_{A_1}\delta\phi_{A_2}\cdots + \phi_{A}\cdots\delta\phi_{A_n}\}|0\rangle$$

$$= i\langle 0|\hat{T}\{\delta(\phi_{A_1}\cdots\phi_{A_n})\}|0\rangle = 0.$$

E.o.m example.

$$\partial^\mu D_\mu c = 0 \qquad \text{e.o.m. for } \bar{c} \; ; \quad \frac{\delta \mathcal{L}}{\delta \partial_\mu \bar{c}} = D_\mu c$$

$$\partial_\mu \langle 0 | \hat{T} \left\{ \frac{\delta \mathcal{L}}{\delta \bar{c}^a} \bar{c}^b \right\} | 0 \rangle - \langle 0 | \hat{T} \underbrace{\frac{\delta \mathcal{L}}{\delta \bar{c}^a}}_{\substack{\shortparallel \\ 0}} \bar{c}^b | 0 \rangle = -i \delta^{(d)}(x-y) \underbrace{\delta^{ab}}_{\frac{\delta \bar{c}^b}{\delta \bar{c}^a}}$$

$$\boxed{\partial_\mu \langle 0 | \hat{T} \{ \underbrace{D^\mu c^a}_{x} \underbrace{\bar{c}^b}_{y} \} | 0 \rangle = -i \delta^{(d)}(x-y) \delta^{ab}}$$

$$\partial^\mu_x \delta_{BRST} \langle 0 | \hat{T} \underbrace{A^a_\mu}_{x} \underbrace{\bar{c}^b}_{y} | 0 \rangle = \partial^\mu_x \left(\langle 0 | \hat{T} D_\mu c^a \bar{c}^b | 0 \rangle + \langle 0 | \hat{T} A^a_\mu B^b | 0 \rangle \right)$$

~~since A commutes with B~~

$$= -i \delta^{(d)}(x-y) \delta^{ab} + \partial^\mu_x \langle 0 | \hat{T} A^a_\mu B^b | 0 \rangle$$

e.o.m. for B.

$$= -i \delta^{(d)}(x-y) \delta^{ab} + \partial^\mu_x \langle 0 | \hat{T} A^a_\mu \left(-\frac{1}{\xi} \partial^\nu A^b_\nu \right) | 0 \rangle$$

by BRST invariance.

$$= -i \delta^{(d)}(x-y) \delta^{ab} - \frac{1}{\xi} \partial^\mu_x \partial^\nu_y \langle 0 | \hat{T} \underbrace{A^a_\mu}_{x} \underbrace{A^b_\nu}_{y} | 0 \rangle = 0$$

we can take it out since $[A^a_\mu, A^b_\nu]_{E.T.} = 0$.

$$\Rightarrow \boxed{\partial^\mu_x \partial^\nu_y \langle 0 | \hat{T} A^a_\mu(x) A^b_\nu(y) | 0 \rangle = -i\xi \delta^{(d)}(x-y) \delta^{ab}}$$

$-ik^\mu \qquad ik^\nu$

$$\int \frac{d^dk}{(2\pi)^d} e^{-ik(x-y)} \Delta^{ab}_{\mu\nu}(k)$$

$$k^\mu k^\nu \Delta^{ab}_{\mu\nu} = -i\xi \delta^{ab}$$

Using Lorentz symmetry

$$\Delta^{ab}_{\mu\nu} = \left(A(k^2)\, \eta_{\mu\nu} + B(k^2)\, k_\mu k_\nu\right) \delta^{ab}$$

BRST identity

$$k^\mu k^\nu \Delta^{ab}_{\mu\nu} = (A k^2 + B k^4)\, \delta^{ab} = -i\xi\, \delta^{ab}$$

$$\Rightarrow \quad B k^4 = -i\xi - A k^2 \qquad B = -\frac{A}{k^2} - \frac{i\xi}{k^4}$$

$$\Delta^{ab}_{\mu\nu} = \Big[\underbrace{A(k^2)}_{\substack{\text{loop corrections}\\ \text{go here}\\ \text{(coupling constant } g)}} \underbrace{\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}_{\text{transverse}} - \underbrace{i\xi \frac{k_\mu k_\nu}{k^4}}_{\substack{\text{fixed by BRST}\\ \text{symmetry and}\\ \text{given by free propagator.}}}\Big] \delta^{ab}$$

We can do a resummation of diagrams. similar to scalar field.

Define $\Pi(k^2)$ by. $\quad A(k^2) = -\dfrac{i}{k^2 + i\varepsilon}\, \dfrac{1}{1 + \Pi(k^2)}$

$$\Delta^{ab}_{\mu\nu} = \left[-\frac{i}{k^2 + i\varepsilon}\, \frac{\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}{1 + \Pi(k^2)} - i\xi \frac{k_\mu k_\nu}{k^4}\right]^{ab}$$

for the free propagator $\Pi = 0 \Rightarrow \Pi$ is loop corrections ignore δ^{ab} indices for the moment.

$$(\Delta_{\mu\nu})^{-1} = (\Delta^{(0)}_{\mu\nu})^{-1} - \Pi_{\mu\nu}$$

self-energy ← 1 PI diagrams.

$$\left.\begin{aligned} \Delta_{\mu\nu} &= A\eta_{\mu\nu} + B k_\mu k_\nu \\ \Delta^{-1}_{\mu\nu} &= C\eta_{\mu\nu} + D k_\mu k_\nu \end{aligned}\right\} \rightarrow \Delta^{\mu\nu}\Delta^{-1}_{\nu\alpha} = AC\,\delta^\mu{}_\alpha + \underline{AD}\, k^\mu k_\alpha + BC\, k^\mu k_\alpha + \underline{BD\, k^2}\, k^\mu k_\alpha$$

$$C = 1/A \qquad AD + BDk^2 = -BC \qquad D = -\frac{BC}{A+Bk^2}$$

$$D = +\frac{1}{A}\,\frac{-A/k^2 - i\xi/k^4}{\cancel{A} - \cancel{A} + i\xi/k^2} = +\frac{ik^2}{\xi A}\left(+\frac{A}{k^2} + \frac{i\xi}{k^4}\right)$$

$$= -\frac{1}{Ak^2} + \frac{i}{\xi}$$

$$\Delta^{-1}_{\mu\nu} = +\frac{1}{A}\eta_{\mu\nu} - \frac{1}{Ak^2}k_\mu k_\nu + \frac{i}{\xi}k_\mu k_\nu$$

$$= \frac{1}{A}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{i}{\xi}k_\mu k_\nu = ik^2(1+\Pi)\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{ik_\mu k_\nu}{\xi}$$

$$= (\Delta^{(0)}_{\mu\nu})^{-1} + ik^2\Pi\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$\boxed{\Pi_{\mu\nu} = -ik^2\,\Pi(k^2)\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)}$$

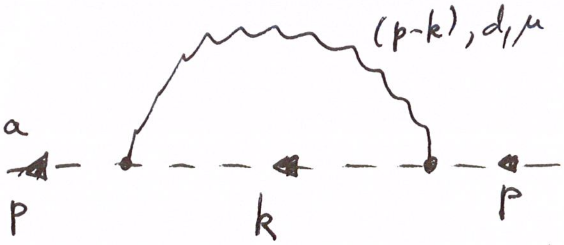
self-energy is transverse.

at 1-loop.

$$\Pi_{\mu\nu} = \frac{ig^2}{2} C_2(G) \frac{1}{(4\pi)^{d/2}} \Gamma\left(1-\frac{d}{2}\right) (-p^2)^{d/2-1} \; B\left(\frac{d}{2},\frac{d}{2}\right)(6d-4)\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

$$\boxed{\Pi(k^2) = \frac{g^2}{2} C_2(G) \frac{\Gamma(1-d/2)}{(4\pi)^{d/2}} B(d/2, d/2)\,(6d-4)\,(-p^2)^{d/2-2}}$$

ghost propagator



c   bcd   (-)(acd)

$$\int \frac{d^dk}{(2\pi)^d}\,(-g)^2\, f^{cdb}\, f^{adc}\; p_\mu\, k^\mu \left(\frac{+i}{k^2+i\epsilon}\right)\frac{(-i)}{((p-k)^2+i\epsilon)}$$

$$= \overset{(-i)^2 i}{\cancel{+}}\, g^2\, \delta^{ab}\, C_2(G)\int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\,\frac{(\cancel{k_\mu}+\alpha p_\mu)\,p^\mu}{(k^2-\Delta)^2}$$

$$\Delta = -\alpha(1-\alpha)p^2 - i\epsilon$$

$$= -g^2\,\delta^{ab}\,C_2(G)\,p^2\int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\,\alpha\,\frac{1}{(k^2-\Delta)^2}$$

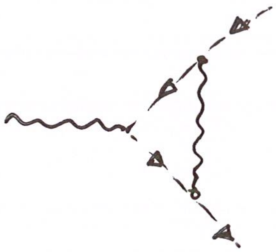
$$= -g^2\,\delta^{ab}\,C_2(G)\,p^2\int_0^1 d\alpha\,\alpha\,\frac{i}{(4\pi)^{d/2}}\,\frac{\Gamma(2-d/2)}{\underset{1}{\cancel{\Gamma(2)}}}\,\frac{1}{\Delta^{2-d/2}}$$

$$= +g^2\,\delta^{ab}\,C_2(G)\,(-p^2)^{d/2-1}\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\;\underbrace{\int_0^1 d\alpha\,\alpha^{d/2-1}(1-\alpha)^{d/2-2}}_{B\left(\frac{d}{2},\,\frac{d}{2}-1\right)\,=\,\frac{2(d-1)}{d-2}\,B\left(\frac{d}{2},\,\frac{d}{2}\right)}$$

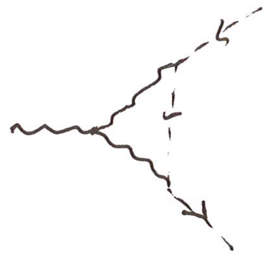
$$\Gamma(2-d/2) = (1-d/2)\,\Gamma(1-d/2) = \frac{2-d}{2}\,\Gamma(1-d/2)$$

$$\boxed{= -g^2\,\delta^{ab}\,C_2(G)\,(-p^2)^{d/2-1}\,\frac{i}{(4\pi)^{d/2}}\,\Gamma(1-d/2)\,(d-1)\,B\left(\frac{d}{2},\frac{d}{2}\right)}$$

gluon – ghost, $\overline{\text{ghost}}$ vertex

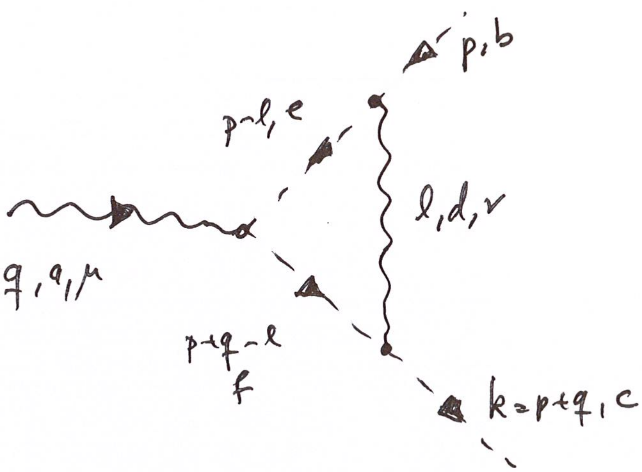


$+$



similar to
electron photon

from non-abelian cases



$$\int \frac{d^d l}{(2\pi)^d} (-g)^3 f^{fae} f^{edb} f^{cdf} (p+q-l)^\mu (p-l)^\nu (p+q)_\nu \times$$

$$\times \frac{(\cancel{-}i)}{l^2+i\varepsilon} \frac{(\cancel{-}i)}{(p+q-l)^2+i\varepsilon} \frac{(+i)}{(p-l)^2+i\varepsilon}$$

$$f^{fae} f^{edb} f^{cdf} = -\left(f^{dfe} f^{eab} + f^{ade} f^{efb} \right) f^{cdf}$$

$$\left(\begin{array}{l} \quad fae \;\; edb \qquad dfe \;\; eab \quad ade \;\; efb \\ f^{abd} f^{dce} + f^{cad} f^{dbe} + f^{bcd} f^{dae} = 0 \end{array} \right)$$

$$\qquad\qquad\qquad\qquad f \quad\; d \quad\; fd$$

$$= -C_2(G)\, \delta^{ce} f^{eab} - f^{ade} f^{efb} f^{cdf}$$

$$f^{fae} f^{edb} f^{cdf} + f^{afe} f^{edb} f^{efd} = -C_2(G)\, f^{cab}$$

$$\overset{-}{(fae)} \quad \overset{+}{(edb)} \quad \overset{-}{(cdf)}$$

same

$$f^{fae} f^{edb} f^{cdf} = -\frac{1}{2} C_2(G)\, f^{cab}.$$

$$= +ig^3 \frac{1}{\cancel{2}} C_2(G) f^{cab} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^d l}{(2\pi)^d} \frac{\cancel{2}(p+q-l)^\mu (p-l)^\nu (p+q)_\nu}{\left[(1-\alpha-\beta) l^2 + \alpha(p-l)^2 + \beta(p+q-l)^2 + i\epsilon\right]^3}$$

$$(1-\alpha-\beta) l^2 + \alpha (p-l)^2 + \beta (p+q-l)^2 = l^2 - 2\alpha p l - 2\beta (p+q) l + \alpha p^2 + \beta (p+q)^2$$

$$= (l - \alpha p - \beta \underbrace{(p+q)}_{k})^2 + \alpha(1-\alpha) p^2 + \beta(1-\beta) \underbrace{(p+q)^2}_{k})$$

$$l \to l + \alpha p + \beta k$$

no linear term in l.

$$(k - l - \alpha p - \beta k)^\mu (p - l - \alpha p - \beta k)^\nu (k)_\nu \Rightarrow (1-\beta) k^\mu ((1-\alpha) p - \beta k)^\nu k_\nu -$$

$$-\alpha p^\mu ((1-\alpha) p - \beta k)^\mu k_\nu + l^\mu l^\nu k_\nu = ((1-\beta) k^\mu - \alpha p^\mu)((1-\alpha) pk - \beta k^2) +$$

$$+ \frac{1}{d} k^\mu l^2$$

$$= i g^3 C_2(G) f^{cab} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \int \frac{d^d l}{(2\pi)^d} \left\{ \frac{((1-\beta)k^\mu - \alpha p^\mu)((1-\alpha)pk - \beta k^2) + \frac{1}{d} k^\mu l^2}{(l^2 - \Delta)^3} \right\}$$

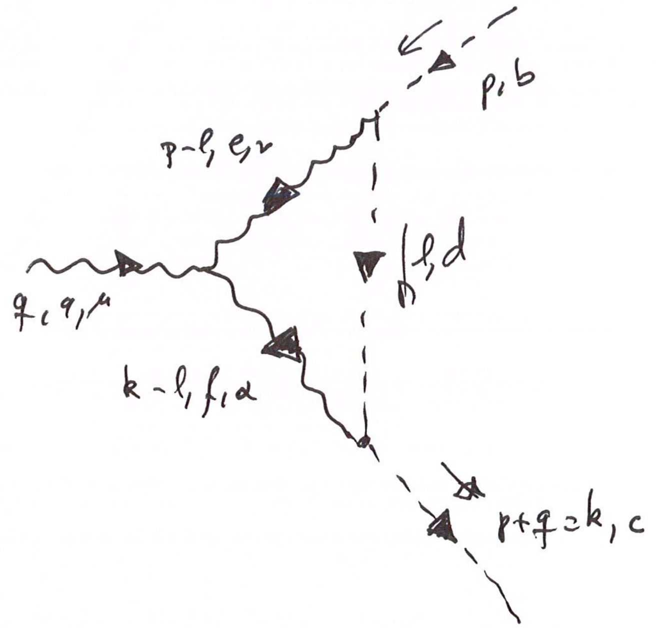
$$\Delta = -\alpha(1-\alpha)p^2 - \beta(1-\beta)k^2 - i\varepsilon$$

$$= i g^3 C_2(G) f^{cab} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \left\{ ((1-\beta)k^\mu - \alpha p^\mu)((1-\alpha)pk - \beta k^2) \frac{(-) i}{(4\pi)^{d/2}} \frac{\Gamma(3-\frac{d}{2})}{\Gamma(3)\, \Delta^{3-d/2}} + \right.$$

$$\left. + \frac{1}{\cancel{d}} k^\mu \frac{\cancel{(-)^2}}{(4\pi)^{d/2}} \frac{i\cancel{d}}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)} \frac{1}{\Delta^{2-d/2}} \right\}$$

$$= i g^3 C_2(G) f^{cab} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \frac{i}{(4\pi)^{d/2}} \frac{1}{2} \frac{1}{\Delta^{2-d/2}} \left\{ -((1-\beta)k^\mu - \alpha p^\mu)((1-\alpha)pk - \beta k^2) \cdot \frac{\Gamma(3-d/2)}{\Delta} + \frac{k^\mu}{2} \Gamma(2-\tfrac{d}{2}) \right\}$$

$$= -\frac{g^3}{2(4\pi)^{d/2}} C_2(G) f^{cab} \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \; \frac{1}{\Delta^{2-d/2}} \left\{ -((1-\beta)k^\mu - \alpha p^\mu)((1-\alpha)pk - \beta k^2) \cdot \frac{\Gamma(3-\frac{d}{2})}{\Delta} + \frac{1}{2} k^\mu \Gamma(2-\tfrac{d}{2}) \right\}$$



$$\boxed{k=p+q}$$

$p=k-q$

$q=k-p$

$$\int \frac{d^dl}{(2\pi)^d}\, g f^{afe}\left[(q+k-l)^\nu \eta^{\mu\alpha} + (2l-p-k)^\mu \eta^{\nu\alpha} + (p-q-l)^\alpha \eta^{\mu\nu}\right]$$

$$f^{cfd} f^{deb} (-g)^2 k^\alpha l^\nu \frac{(+i)}{l^2+i\varepsilon}\frac{(-i)}{(k-l)^2+i\varepsilon}\frac{(-i)}{(p-l)^2+i\varepsilon}$$

$$= -ig^3 \overset{\bar{f}^{ae}}{f^{afe}}\; \overset{\bar{f}^{cdf}}{f^{cfd}}\; \overset{\bar{f}^{cdb}}{f^{deb}} \int\frac{d^dl}{(2\pi)^d}\int_0^1 d\alpha\int_0^{1-\alpha} d\beta\, \frac{2k^\alpha l^\nu\left[(q+k-l)^\nu\eta^{\mu\alpha}+(2l-p-k)^\mu\eta^{\nu\alpha}+(p-q-l)^\alpha\eta^{\mu\nu}\right]}{\left[(1-\alpha-\beta)l^2+\alpha(k-l)^2+\beta(p-l)^2+i\varepsilon\right]^3}$$

$$= \cancel{-}\frac{ig^3}{2} C_2(G) f^{cab}\int\frac{d^dl}{(2\pi)^d}\int_0^1 d\alpha\int_0^{1-\alpha}d\beta\, \frac{-\frac{d-1}{d}k^\mu l^2 + k^\mu k^2\, 2\alpha(1-2\alpha) - k^\mu p^2\beta(1+\beta) + k^\mu(pk)\left((\alpha+\beta)-\alpha\beta\right) + k^2 p^\mu(\alpha\beta-\alpha-\beta) + \beta(1+\beta)p^\mu(pk)}{(l^2-\Delta)^3}$$

$$l^2 - 2\alpha kl - 2\beta pl + \alpha k^2 + \beta p^2 = (l-\alpha k-\beta p)^2 + \alpha(1-\alpha)k^2 + \beta(1-\beta)p^2$$

$$l \to l+\alpha k+\beta p$$

$$\left[(l+\alpha k+\beta p)_\nu (q+k-\alpha k-\beta p-l)^\nu k^\mu + (2l+2\alpha k+2\beta p-p-k)^\mu k^\alpha (l+\alpha k+\beta p)_\alpha\right.$$

$$\left. + k\cdot(p-q-l-\alpha k-\beta p)(l+\alpha k+\beta p)^\mu\right]$$

removing linear in l.

$$\Big[\underbrace{-l^2 k^\mu} + (\alpha k+\beta p)\cdot((2-\alpha)k-(1+\beta)p)\,k^\mu + \underbrace{2\, l^\mu l^\alpha k_\alpha} + ((2\alpha-1)k+(2\beta-1)p)^\mu \cdot$$

$$\cdot k^\alpha(\alpha k+\beta p)_\alpha + \underbrace{(-k^\alpha l_\alpha)\, l^\mu} + k\cdot((2-\beta)p \cancel{-k} + (1+\alpha)k)\,(\alpha k+\beta p)^\mu \Big]$$

$$l^2\Big[-k^\mu + \frac{2}{d}k^\mu - \frac{1}{d}k^\mu\Big] + k^\mu\Big(\alpha(2-\alpha)k^2 - \alpha(1+\beta)pk + (2-\alpha)\beta pk -$$

$$-\beta(1+\beta)p^2\Big) + ((2\alpha-1)k+(2\beta-1)p)^\mu\,(\alpha k^2+\beta pk) + (\alpha k+\beta p)^\mu\big((2-\beta)pk-(1+\alpha)k^2\big)$$

$$l^2\Big[\frac{1}{d}-1\Big]k^\mu + k^\mu\Big(\alpha(2-\alpha)k^2 + ((2-\alpha)\beta - \alpha(1+\beta))pk - \beta(1+\beta)p^2 +$$

$$+(2\alpha-1)k^\mu(\alpha k^2+\beta pk) + (2\beta-1)p^\mu(\alpha k^2+\beta pk) + \alpha k^\mu\big((2-\beta)pk-(1+\alpha)k^2\big)$$

$$+\beta p^\mu\big((2-\beta)pk - (1+\alpha)k^2\big)\Big)$$

$$-\frac{d-1}{d}k^\mu l^2 + k^\mu k^2\big(\alpha(2-\alpha) - \alpha(2\alpha-1) - \alpha(1+\alpha)\big) + k^\mu p^2\big(-\beta(1+\beta)\big) +$$

$$2\alpha-\alpha^2-2\alpha^2 \cancel{+\alpha} \cancel{-\alpha} - \alpha^2 = 2\alpha - 4\alpha^2$$

$$+k^\mu(p\cdot k)\big((2-\alpha)\beta - \alpha(1+\beta) + (2\alpha-1)\beta + \alpha(2-\beta)\big) +$$

$$2\beta \cancel{-\alpha\beta} - \alpha - \alpha\beta + \cancel{2\alpha\beta} - \beta + 2\alpha \cancel{-\alpha\beta} = -\alpha\beta + \alpha + \beta$$

$$+p^\mu k^2\big(\alpha(2\beta-1) - \beta(1+\alpha)\big) + p^\mu p\cdot k\,\big(\beta(2\beta-1) + (2-\beta)\beta\big)$$

$$2\alpha\beta - \alpha - \beta - \alpha\beta \qquad\qquad 2\beta^2 - \beta + 2\beta - \beta^2 = \beta + \beta^2$$

$$-\alpha - \beta + \alpha\beta$$

$$-\frac{d-1}{d}k^\mu l^2 + k^\mu k^2\, 2\alpha(1-2\alpha) - k^\mu p^2\beta(1+\beta) + k^\mu(pk)(\alpha+\beta-\alpha\beta) +$$

$$+k^2 p^\mu(\alpha\beta-\alpha-\beta) + \beta(1+\beta)\,p^\mu\, p\cdot k$$

Divergent part (for renormalization)

$$+ \cancel{2}\frac{ig^3}{\cancel{2}}\, C_2(G)\, f^{cab}\, k^\mu \left(+\frac{d-1}{\cancel{d}}\right) \int_0^1 d\alpha \int_0^{1-\alpha} d\beta \underbrace{\int \frac{d^d l}{(2\pi)^d}\, \frac{l^2}{(l^2-\Delta)^3}}_{\frac{1}{(4\pi)^{d/2}}\, \frac{i d}{2}\, \frac{\Gamma(2-d/2)}{\underset{1/2}{\Gamma(3)}}\, \frac{1}{\Delta^{2-d/2}}}$$

$$- \frac{g^3}{2\times 2}\, \frac{C_2(G)}{(4\pi)^{d/2}}\, f^{cab}\, k^\mu\, (d-1)\, \Gamma(2-d/2) \int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \frac{1}{\Delta^{2-d/2}}$$

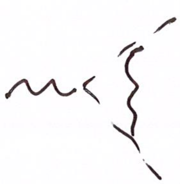
$d = 4-\varepsilon \qquad 2 - d/2 = \varepsilon/2$

div. part

$$- \frac{g^3}{4}\, \frac{C_2(G)}{16\pi^2}\, f^{cab}\, k^\mu\, 3\, \Gamma\!\left(\frac{\varepsilon}{2}\right) \underbrace{\int_0^1 d\alpha \int_0^{1-\alpha} d\beta \cdot 1}_{1/2}$$

$$- \frac{3g^3}{128\pi^2}\, C_2(G)\, f^{cab}\, k^\mu\, \frac{2}{\varepsilon}$$

div. part $= - \dfrac{3g^3}{64\pi^2}\, C_2(G)\, f^{cab}\, k^\mu\, \dfrac{1}{\varepsilon}$ 

 $\xrightarrow{\text{div. part}}$ $- \dfrac{g^3}{2(4\pi)^{d/2}}\, C_2(G)\, f^{cab}\, \dfrac{1}{2} k^\mu\, \Gamma\!\left(\dfrac{\varepsilon}{2}\right) \displaystyle\int_0^1 d\alpha \int_0^{1-\alpha} d\beta\, \underbrace{\frac{1}{\Delta^{2-d/2}}}_{\to 1 \ d\to 4}$

$$- \frac{g^3}{2\times 16\pi^2}\, C_2(G)\, f^{cab}\, \cancel{\frac{1}{2}} k^\mu\, \frac{\cancel{2}}{\varepsilon}\, \frac{1}{2} = - \frac{g^3}{64\pi^2}\, C_2(G)\, f^{cab}\, k^\mu\, \frac{1}{\varepsilon}$$

 $\Big|_{div} = - \dfrac{3+1}{64\pi^2}\, g^3\, C_2(G)\, f^{cab}\, k^\mu\, \dfrac{1}{\varepsilon} = \boxed{- g^3\, \dfrac{C_2(G)}{16\pi^2}\, f^{cab}\, k^\mu\, \dfrac{1}{\varepsilon}}$