Renormalization of gauge theories

Dim. Reg. provides a BRST inv. regularization.

By power counting we only generate dim 4 counterterms.

By BRST inv. (plus $\bar{c} \to \bar{c} + \eta$; $c \to e^{i\alpha} c$ $\bar{c} \to e^{-i\alpha} \bar{c}$)

we only get the terms we have.

For power counting ghosts have dim 1 and also A_μ.

At 1-loop we compute divergencies:

$$\text{~O~} \quad \frac{ig^2}{2} C_2(G) \frac{\delta^{ab}}{(4\pi)^{d/2}} \Gamma(1-\tfrac{d}{2}) (-p^2)^{d/2-1} B(\tfrac{d}{2}, \tfrac{d}{2}) (6d-4) \cdot \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

$d \to 4$ $\quad d = 4 - \varepsilon$ $\quad 1 - d/_2 = 1 - 2 + \varepsilon/_2 = -1 + \varepsilon/_2$

$$\frac{ig^2}{2} C_2(G) \frac{\delta^{ab}}{16\pi^2} \underbrace{\Gamma(-1+\tfrac{\varepsilon}{2})}_{\frac{\Gamma(\varepsilon/_2)}{-1+\varepsilon/_2} \sim -\frac{2}{\varepsilon}} (-p^2)^{+1} B(2,2)\, 20 \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$$

$$+\frac{2}{\varepsilon} \frac{ig^2}{\cancel{2}} C_2(G) \frac{\delta^{ab}}{16\pi^2} p^2 \frac{\overset{1}{\Gamma(2)} \overset{1}{\Gamma(2)}}{\underset{6}{\Gamma(4)}}\, 20 \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \qquad \frac{\overset{5}{2\cancel{0}}}{\underset{4}{\cancel{16}} \times 6}$$

$$\frac{1}{\varepsilon} ig^2 C_2(G) \delta^{ab} p^2 \left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \frac{5}{24\pi^2}$$

$$\text{~O~}\Big|_{div} = \frac{ig^2}{\varepsilon} \frac{5}{24\pi^2} C_2(G) \delta^{ab} (\eta_{\mu\nu} p^2 - p_\mu p_\nu)$$

 ghost prop

$$-g^2 \delta^{ab} \, C_2(G) \, (-p^2)^{d/2-1} \frac{i}{(4\pi)^{d/2}} \, \Gamma(1-d/2) \, (d-1) \, B\left(\tfrac{d}{2}, \tfrac{d}{2}\right)$$

$d = 4-\epsilon$

$$+g^2 \delta^{ab} \, C_2(G) \, (+p^2) \frac{i}{(4\pi)^2} \underbrace{\Gamma(-1+\epsilon/2)}_{\simeq -2/\epsilon} \, 3 \, \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}$$

$$-\frac{\cancel{2}}{\epsilon} g^2 \delta^{ab} \, C_2(G) \frac{i}{\underset{8}{\cancel{16}}\pi^2} \frac{\cancel{3}}{\underset{2}{\cancel{6}}} p^2 = -\frac{1}{\epsilon} \frac{ig^2}{16\pi^2} \delta^{ab} \, C_2(G) \, p^2$$

 $\Big)_{div} = \frac{1}{\epsilon} \frac{5}{24\pi^2} ig^2 \, C_2(G) \, \delta^{ab} \, (\eta_{\mu\nu} p^2 - p_\mu p_\nu)$

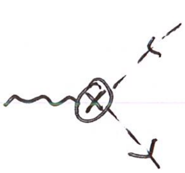
 $\Big)_{div} = -\frac{1}{\epsilon} \frac{ig^2}{16\pi^2} \delta^{ab} \, C_2(G) \, p^2$

 $\Big|_{div} = -\frac{1}{\epsilon} \frac{g^3}{16\pi^2} C_2(G) \, f^{cab} \, k^\mu$

Counter vertices

 $-\frac{1}{\epsilon} \frac{5}{24\pi^2} ig^2 \, C_2(G) \, \delta^{ab} \, (\eta_{\mu\nu} p^2 - p_\mu p_\nu)$

 $\frac{1}{\epsilon} \frac{ig^2}{16\pi^2} \delta^{ab} \, C_2(G) \, p^2$

 $\frac{1}{\epsilon} \frac{g^3}{16\pi^2} C_2(G) \, f^{cab} \, k^\mu$

Counter terms

Consider quadratic part of A Lagrangian:

$$S = -\frac{1}{4}\int (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^{a\nu} - \partial^\nu A^{a\mu})$$

$$= -\frac{1}{2}\int (\partial_\mu A^a_\nu \partial^\mu A^{a\nu} - \partial_\mu A^a_\nu \partial^\nu A^{a\mu})$$

$$= \frac{1}{2}\int A^a_\nu \partial^2 A^{a\nu} - A^a_\nu \partial_\mu \partial^\nu A^{a\mu}$$

$$e^{iS} \xrightarrow[\text{expanded}]{} \frac{i}{2}\int d^dx \left(A^a_\nu \partial^2 A^{a\nu} - A^a_\nu \partial_\mu \partial^\nu A^{a\mu}\right)$$

when contracting we get factor of 2

$$A^{a_1\mu_1}_{x_1} \quad - \quad A^{a_2\mu_2}_{x_2}$$

$$\int d^dx \, i \left(\Delta^{a_1 a}_{\mu_1\nu}(x_1 - x)\, \partial^2_x \, \Delta^{aa_2}_{\nu\mu_2}(x-x_2) - \Delta^{a_1 a}_{\mu_1\nu}(x_1 - x)\partial_\mu \partial^\nu_x \Delta_{\mu\mu_2}(x-x_2)\right)$$

$$i\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}\underbrace{\int d^dx \, e^{-ip_1(x_1-x)}}_{(2\pi)^d\delta(p_1-p_2)}\Big(\Delta^{a_1a}_{\mu_1\nu}(p_1)(-p_2^2)\, e^{-ip_2(x-x_2)}\Delta^{aa_2}_{\nu\mu_2}(p_2)$$

$$+ \Delta^{a_1a}_{\mu_1\nu}(p_1)\, p_2^\mu p_2^\nu \, \Delta^{aa_2}_{\mu\mu_2}(p_2)\Big)$$

$$i\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}(2\pi)^d\delta(p_1-p_2)\, e^{-ip_1x_1+ip_2x_2}\, \Delta^{a_1a}_{\mu_1\nu}(p_1)\left(-p_2^2\eta^{\mu\nu} + p_2^\mu p_2^\nu\right)\Delta^{aa_2}_{\mu\mu_2}$$

Vertex

 $-i\,(p^2\eta^{\mu\nu}-p^\mu p^\nu)\,\delta^{ab}$

If we take as counterterm

$$-\frac{1}{4}\delta_3\,F^a_{\mu\nu}F^{a\mu\nu}$$

then we get

 $-i\,\delta_3\,(p^2\eta^{\mu\nu}-p^\mu p^\nu)\,\delta^{ab}$

we get $\boxed{\delta_3=\frac{1}{\varepsilon}\,\frac{5}{24\pi^2}\,g^2\,C_2(G)}$

$$S=\int\partial^\mu\bar{c}^a D_\mu c^a\ \to\ \partial^\mu\bar{c}^a\,\partial_\mu c^a$$

$$e^{iS}\to c^{a_1}_{x_1}\int_x i\,\partial^\mu\bar{c}^a\,\partial_\mu c^a\ \bar{c}^{a_2}_{x_2}$$

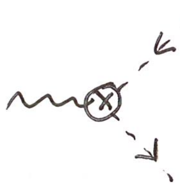
$$i\int_x\partial^\mu_x\,\Delta^{a_1a}_{gh}(x_1-x)\,\partial^x_\mu\,\Delta^{aa_2}_{gh}(x-x_2)$$

$$\int d^dx\ i\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}\ i p_1^\mu(-i p_{2\mu})\,e^{-ip_1(x_1-x)-ip_2(x-x_2)}\ \Delta^{a_1a}_{gh}(p_1)\,\Delta^{aa_2}_{gh}(p_2)$$

$$i\int\frac{d^dp_1}{(2\pi)^d}\int\frac{d^dp_2}{(2\pi)^d}\,(2\pi)^d\,\delta^{(d)}(p_1-p_2)\,e^{-ip_1x_1+ip_2x_2}\ p_1^2\ \Delta^{a_1a}_{gh}(p_1)\,\Delta^{aa_2}_{gh}(p_2)$$

$$i\delta_{2c}\, p^2\, \delta^{ab}$$

$$\delta_{2c} = \frac{1}{\varepsilon}\frac{g^2}{16\pi^2} C_2(G)$$



$$g\,\delta_{1c}\, \partial_\mu \bar{c}^a f^{abc} A^{b\mu} c^c \rightarrow -g\,\delta_{1c} f^{abc} p_\mu$$

$$\delta_{1c} = -\frac{1}{\varepsilon}\frac{g^2}{16\pi^2} C_2(G)$$

Summary: $\delta_{1c} = -\frac{1}{\varepsilon}\frac{g^2}{16\pi^2} C_2(G)$; $\delta_{2c} = \frac{1}{\varepsilon}\frac{g^2}{16\pi^2} C_2(G)$

$$\delta_3 = \frac{1}{\varepsilon}\frac{5}{24\pi^2} g^2 C_2(G)$$

$$\mathcal{L} = -\frac{1}{4}(1+\delta_3)(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a})$$

$$+ (1+\delta_{2c})\, \partial^\mu \bar{c}^a \partial_\mu c^a$$

$$+ g(1+\delta_{1c})\, \partial^\mu \bar{c}^a f^{abc} A_\mu^b c^c$$

$+$ gauge fixing, AAA, AAAA.

Define bare fields

$$A_\mu^{(0)} = \sqrt{1+\delta_3}\, A_\mu^R \leftarrow \text{renormalized}$$

$$c^{(0)a} = \sqrt{1+\delta_{2c}}\; c^{aR}$$

$$\bar{c}^{(0)a} = \sqrt{1+\delta_{2c}}\; \bar{c}^{aR}$$

then

$$g(1+\delta_{1c})\, \partial^\mu \bar{c}^a f^{abc} A_\mu^b c^c = g\frac{(1+\delta_{1c})}{(1+\delta_{2c})}\frac{1}{\sqrt{1+\delta_3}}\, \partial^\mu \bar{c}^{(0)a} f^{abc} A_\mu^{(0)b} c^{(0)c}$$

$$g_0 = g\frac{(1+\delta_{1c})}{(1+\delta_{2c})}\frac{1}{\sqrt{1+\delta_3}} \underset{\substack{\uparrow \\ \text{pert. theory}}}{\hat{=}} g\left(1+\delta_{1c}-\delta_{2c}-\frac{1}{2}\delta_3\right)$$

$$g_0 = g\left(1 + \frac{1}{\varepsilon}\frac{g^2}{16\pi^2}C_2(G) - \frac{1}{\varepsilon}\frac{g^2}{16\pi^2}C_2(G) - \frac{1}{\varepsilon}\frac{5}{48\pi^2}g^2 C_2(G)\right)$$

$$= g\left(1 - \frac{g^2}{\varepsilon\, 16\pi^2}C_2(G)\underbrace{\left(1+1+\frac{5}{3}\right)}_{11/3}\right)$$

$$= g_R\left(1 - \frac{1}{\varepsilon}\frac{11}{3}\frac{g_R^2}{16\pi^2}C_2(G)\right)$$

renormalization scale

$$S^{(d)} = \int \underbrace{d^d x}_{M^{-d}} \; (\partial A \, \partial A + g \underbrace{\partial A A A}_{M^{1+\frac{3}{2}(d-2)}} + g^2 \underbrace{A A A A}_{M^{\frac{4}{2}(d-2)}} + \ldots)$$

$\uparrow$ schematically

$\partial \to M$, $A \to M^{\frac{d-2}{2}}$

$$[A] = \frac{d-2}{2} \quad \text{mass units.} \qquad \text{in } d=4 \quad [A]=1.$$

$$d = 4 - \varepsilon$$

$$[g] = d - 1 - \frac{3}{2}(d-2) = -\frac{d}{2} + 2 \qquad = \cancel{-2} + \frac{\varepsilon}{2} \cancel{+2} = \frac{\varepsilon}{2}$$

$$[g^2] = d - 2(d-2) = -d + 4 \quad \checkmark$$

Notice if $d=3$ $[g] = 1/2$ $\to$ $g_{eff} = g/\sqrt{E} \underset{E \to 0}{\to} \infty$

E $\uparrow$ energy.

$d=5$ $[g] = -1/2$ $\to$ $g_{eff} = g\sqrt{E} \underset{E \to 0}{\to} 0$

in $d<4$ gauge theories are strongly coupled in the IR $(E \to 0)$

in $d>4$ " " " " " " UV $(E \to \infty)$

in $d=4$ we have to see.

We define $g = \mu^{\varepsilon/2} \tilde{g}$

^ no units.

g_0 is ind $d=4 \Rightarrow$ no units.

we put this in finite piece.

$1 + \varepsilon \log \mu + \ldots$

$$g_0 = \mu^{\varepsilon/2} \tilde{g}_R \left(1 - \frac{1}{\varepsilon} \frac{11}{3} \frac{\tilde{g}_R^2 \mu^\varepsilon}{16\pi^2} C_2(G)\right)$$

rename $\tilde{g} \to g$ (notation)

then

$$g_0 = \mu^{\varepsilon/2} g_R \left(1 - \frac{1}{\varepsilon} \frac{11}{3} \frac{g_R^2}{16\pi^2} C_2(G)\right) \quad (1)$$

This is a <u>definition</u> of g_R. If we compute Green functions with the bare lagrangian, rescale the fields and replace g_0 by g_R according to (1) then we get something finite as $\varepsilon \to 0$.

In general

$$g_0 = \mu^{\varepsilon/2}\left(g_R + \sum_{r=1}^{\infty} \frac{a_r(g_R)}{\varepsilon^r} \right)$$

(no $\varepsilon^{n>0}$ terms only pole part)

For given g_0 (this defines the theory) how does g_R change if we change μ? $\to$ this gives renormalization group

$$\mu \to (1+\delta)\mu = \tilde{\mu}$$

$(1+\delta)^{\varepsilon/2} \simeq 1 + \frac{\varepsilon}{2}\delta$ (for small δ, no assumption on ε)

$$g_0 = \tilde{\mu}^{\varepsilon/2}\left(\tilde{g}_R + \sum_{r=1}^{\infty} \frac{a_r(\tilde{g}_R)}{\varepsilon^r}\right) \quad ; \quad \tilde{g}_R = g_R + \delta g_R = ?$$

$$= \left(1+\frac{\varepsilon}{2}\delta\right)\mu^{\varepsilon}\left(g_R + \delta g_R + \sum_{r=1}^{\infty}\frac{a_r(g_R)}{\varepsilon^r} + \sum_{r=1}^{\infty}\frac{\partial_g a_r\,\delta g_R}{\varepsilon^r}\right)$$

$$= \mu^{\varepsilon/2}\Bigg(\overbrace{g_R + \sum_{r=1}^{\infty}\frac{a_r(g_R)}{\varepsilon^r}}^{g_0} + \overbrace{\frac{\varepsilon}{2}\delta \times g_R}^{g_R \times \delta} + \sum_{r=1}^{\infty}\frac{a_r(g_R)\cdot\delta}{2\varepsilon^{r-1}} +$$

$$+ \delta g_R + \sum_{r=1}^{\infty}\frac{\partial_g a_r\,\delta g_R}{\varepsilon^r}\Bigg)$$

$$\Rightarrow \quad \frac{\varepsilon}{2} g_R \delta + \sum_{r=1}^{\infty}\frac{a_r(g_R)\cdot\delta}{2\varepsilon^{r-1}} + \delta g_R + \sum_{r=1}^{\infty}\frac{\partial_g a_r\,\delta g_R}{\varepsilon^r} = 0 .$$

$\delta g_R = \delta g_R^{(0)} + \varepsilon\,\delta g_R^{(1)}$; the "source" term is $\varepsilon g_R \frac{\delta}{2} + \sum \frac{a_r\delta}{2\varepsilon^{r-1}}$ of order ε (or less). We do not get higher powers of ε.

$$\frac{\varepsilon}{2} g_R \delta + \sum_{r=1}^{\infty}\frac{a_r(g_R)\,\delta}{2\varepsilon^{r-1}} + \delta g_R^{(0)} + \varepsilon\,\delta g_R^{(1)} + \sum_{r=1}^{\infty}\frac{\partial_g a_R\,\delta g_R^{(0)}}{\varepsilon^r} +$$

$$+ \sum_{r=1}^{\infty}\partial_g a_r\,\frac{\delta g_R^{(1)}}{\varepsilon^{r-1}} = 0$$

$\varepsilon^1)\quad g_R\frac{\delta}{2} + \delta g_R^{(1)} = 0$

$\varepsilon^0)\quad a_1\frac{\delta}{2} + \delta g_R^{(0)} + \partial_g a_1\,\delta g_R^{(1)} = 0$

$\varepsilon^{(j\lneq 0)})\quad \frac{1}{2}\delta a_{j+1} + \partial_g a_j\,\delta g_R^{(0)} + \partial_g a_{j+1}\,\delta g_R^{(1)} = 0$

$$\delta g_R^{(1)} = -\frac{\delta}{2} \times g_R \qquad \delta = \frac{\delta\mu}{\mu}$$

$$\delta g_R^{(0)} = -a_1 \cdot \frac{\delta}{2} - \delta g_R^{(1)} \partial_g a_1 = (-a_1 + g_R \partial_{g_R} a_1) \cdot \frac{\delta}{2}$$

$$\delta g_R = \frac{\delta\mu}{2\mu}\left(-\varepsilon g_R + (-a_1 + g_R \frac{\partial}{\partial g_R} a_1)\right)$$

Now we take $\varepsilon \to 0$ (for d $\phi 4$ we could do ε expansion setting e.g. $\varepsilon = 1$ .. as in ϕ^4 model)

$$\mu \frac{\delta g_R}{\delta\mu} = \frac{1}{2}\left(-a_1 + g_R \frac{\partial a_1}{\partial g_R}\right) = \beta(g_R).$$

$$a_1 = -\frac{11}{3} \frac{g_R^3}{16\pi^2} C_2(G)$$

$$\beta(g_R) = -\frac{11}{3} \frac{C_2(G)}{2 \times 16\pi^2}\left(-g_R^3 + 3 g_R g_R^2\right) = -\frac{11}{3} \frac{C_2(G)}{16\pi^2} g_R^3$$

$$\boxed{\beta(g_R) = -\frac{11}{3} \frac{C_2(G)}{16\pi^2} g_R^3}$$

Notice for $SU(N)$ $\quad C_2(G) = N$

$$\mu \frac{\partial g_R}{\partial \mu} = -\frac{11}{3} \frac{N}{16\pi^2} g_R^3$$

Define $\lambda = g_R^2 N \Rightarrow$

$$\mu \frac{\partial \lambda}{\partial \mu} = \mu\, 2\, g_R \frac{\partial g_R}{\partial \mu} N$$

$$= -\cancel{2}\, g_R N \frac{11}{3} \frac{N}{16\pi^2} g_R^3$$

$$= - \frac{11}{3} \frac{1}{8\pi^2} \lambda^2$$

$$\mu \frac{\partial \lambda}{\partial \mu} = -\frac{11}{24\pi^2} \lambda^2 \Rightarrow \beta(\lambda) = -\frac{11}{24\pi^2}\lambda^2 \quad \text{(N goes away)}$$

$\lambda \equiv$ 't Hooft coupling.

relevant in $N \to \infty$
λ fixed limit.

Remember sliding scale:

we can define a $g_R(\mu)$ by solving.

$$\mu \frac{\partial \lambda}{\partial \mu} = -\frac{11}{24\pi^2} \lambda^2$$

$$\int_{\lambda_0}^{\lambda} \frac{d\lambda}{\lambda^2} = -\frac{11}{24\pi^2} \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \rightarrow -\frac{1}{\lambda}\Big|_{\lambda_0}^{\lambda} = -\frac{11}{24\pi^2} \ln(\mu/\mu_0)$$

$$-\frac{1}{\lambda} + \frac{1}{\lambda_0} = -\frac{11}{24\pi^2} \ln(\mu/\mu_0)$$

$$\frac{1}{\lambda} = \frac{1}{\lambda_0} + \frac{11}{24\pi^2} \ln(\mu/\mu_0)$$

$$\lambda = \frac{1}{\frac{1}{\lambda_0} + \frac{11}{24\pi^2} \ln(\mu/\mu_0)}$$

$\lambda \to 0$
as $\mu \to \infty$

for $\frac{11}{24\pi^2} \ln(\tilde{\mu}/\mu_0) = -1/\lambda_0$ $\quad \lambda \to \infty$

$$\tilde{\mu}/\mu_0 = e^{-\frac{24\pi^2}{11}\frac{1}{\lambda_0}}$$

but as λ grows
pert. theory is not reliable
any more.

However define $\Lambda = \tilde{\mu}$ then

$$\lambda = \frac{1}{\frac{11}{24\pi^2} \ln(\mu/\mu_0) - \frac{11}{24\pi^2} \ln(\Lambda/\mu_0)} = \frac{1}{\frac{11}{24\pi^2} \ln(\mu/\Lambda)}$$

$\lambda(\mu \to \infty) = 0$ Asymptotic freedom. At large energies gluons behave as free particles up to logarithmic corrections

$\lambda(\mu \to \Lambda^{+}) \to \infty$

Λ is the only parameter! It is a scale (units of energy)

Dimensional transmutation: $g_R \to \Lambda$.

RG equation.

Notice we also have to rescale A_μ to compensate for $\mu \to \mu + \delta\mu$

$$A_\mu^{(c)} = \left(1 + \sum_{r=1}^{\infty} \frac{c_r(g)}{\varepsilon^r}\right) A_\mu^{(R)}$$

$$\delta A_\mu^{(c)} = 0 = \sum_{r=1}^{\infty} \frac{\partial_g c_r}{\varepsilon^r} \delta g_R \, A_\mu^{(R)} + \left(1 + \sum_{r=1}^{\infty} \frac{c_r}{\varepsilon^r}\right) \delta A_\mu^{(R)}$$

$$\delta g_R = \delta g^{(0)} + \varepsilon \delta g^{(1)}$$

There are only two terms $\mathcal{O}(1)$ (no term $\mathcal{O}(\varepsilon)$)

$$\partial_g c_1 \, \delta g^{(1)} A_\mu^{(R)} + \delta A_\mu^{(R)} = 0$$

$$\delta A_\mu^{(R)} = - \underbrace{\delta g^{(1)}}_{-\frac{1}{2} g \cdot \delta = -\frac{1}{2} g \frac{\delta\mu}{\mu}} \cdot \partial_g c_1 \cdot A_\mu^{(R)} \qquad (\text{as } \varepsilon \to 0).$$

$$\delta A_\mu^{(R)} = \frac{1}{2} g \, \partial_g c_1 \frac{\delta\mu}{\mu} A_\mu^{(R)}$$

We can define $A_\mu^{(R)} = \chi(\mu) \bar{A}_\mu^{(R)}$

then $\delta \chi = \frac{1}{2} g \, \partial_g c_1 \frac{\delta\mu}{\mu} \chi \qquad \frac{\mu}{\chi} \frac{\delta\chi}{\delta\mu} = \frac{1}{2} g \, \partial_g c_1 \overset{!}{=} \gamma$ definition given that

$$A_\mu^{(0)} = \sqrt{1+\delta_3}\, A_\mu^{(R)} \overset{\text{pert. theory}}{\simeq} \left(1+\frac{\delta_3}{2}\right) A_\mu^{(R)} \quad ; \quad \delta_3 = \frac{1}{\varepsilon}\frac{5}{24\pi^2} g^2 C_2(G)$$

then

$$C_1 = \frac{5}{48\pi^2} g^2 C_2(G)$$

$$\gamma = \frac{1}{2} g \partial_g C_1 = \frac{5}{48\pi^2} g^2 C_2(G) \overset{SU(N)}{=} \frac{5}{48\pi^2} g^2 N = \frac{5}{48\pi^2}\lambda$$

we have.

$$\underbrace{\chi^n(\mu)\, \Gamma^{(n)}(p_i, \lambda, \mu)}_{\Gamma^{(n)}(p_i, \lambda, \mu)} = \underbrace{\chi^n(\mu+\delta\mu)}_{(\chi+\delta\chi)^n}\, \Gamma^{(n)}(p_i, \lambda+\delta\lambda, \mu+\delta\mu)$$

$$= \left(\chi^n + n\chi^{n-1}\delta\chi\right)\left(\Gamma^{(n)} + \frac{\partial\Gamma}{\partial\lambda}\delta\lambda + \delta\mu\frac{\partial\Gamma}{\partial\mu}\right)$$

then $\displaystyle n\frac{\delta\chi}{\chi}\Gamma^{(n)} + \delta\lambda\frac{\partial\Gamma}{\partial\lambda} + \delta\mu\frac{\partial\Gamma}{\partial\mu} = 0$

we can define a $\Gamma^{(n)}(p_i, \lambda(\mu), \mu)$ such that

$$\boxed{\mu\frac{\partial\Gamma^{(n)}}{\partial\mu} + \beta(\lambda)\frac{\partial\Gamma^{(n)}}{\partial\lambda} + n\gamma\Gamma^{(n)} = 0}$$ RG equation.

RG equation only states that the physics is independent of the arbitrary RG scale μ.

We can get some physical result if we take $\mu \sim$ energy to avoid large logs in pert. theory $\left(\ln\left(p^2/\mu^2\right)\right)$.

In that sense $g_R(\mu)$ is the effective coupling at energy μ.

We can also use dimensional analysis to re-write the RG eqn. as an eqn. that determines what happens if we rescale the momenta p_i.

<u>Units</u>

$$S = \int \underset{M^{-d}}{d^d x}\ \underset{M}{\partial_\nu} \underset{M}{A_\mu}\, \partial^\nu A^\mu + \cdots \quad \rightarrow [A_\mu] = M^{\frac{d-2}{2}}$$

$$G^{(n)}_{\mu_1 \cdots \mu_n}(x_1 \cdots x_n) = \langle 0 | \hat{T}\{A_{\mu_1}(x_1) \cdots A_{\mu_n}(x_n)\} | 0 \rangle \sim M^{n\frac{d-2}{2}}$$

$$\underset{M^{-nd}}{\int d^d x_1 \cdots d^d x_n}\ e^{i(p_1 x_1 + \cdots + p_n x_n)} \underset{M^{n\frac{d-2}{2}}}{G^{(n)}_{\mu_1 \cdots \mu_n}(x_1 \cdots x_n)} = (2\pi)^d \underset{M^{-d}}{\delta^{(d)}(p_1 + \cdots + p_n)} \cdot \underset{M^{d-nd+n\frac{d-2}{2}}}{\tilde{G}^{(n)}_{\mu_1 \cdots \mu_n}(p_1 \cdots p_n)}$$

$$\tilde{G}^{(n)}_{\mu_1 \cdots \mu_n} \sim M^{d - \frac{nd}{2} - n} \qquad \underline{n=2} \quad \tilde{G}^{(2)} \sim M^{-2}$$

$$\Gamma^{(n)}_{\mu_1\ldots\mu_n}(p_1\ldots p_n) = \underbrace{\underset{\substack{\uparrow\\ \text{remove external legs}}}{(G^{(2)})^{-1}_{\mu_1\mu_1'}} \cdots (G^{(2)})^{-1}_{\mu_n\mu_n'}}_{M^{2n}} \underbrace{G^{(n)\mu_1'\ldots\mu_n'}}_{M^{d-\frac{nd}{2}-n}}$$

$$\Gamma^{(n)}_{\mu_1\ldots\mu_n}(p_1,\ldots p_n) \sim M^{d-\frac{nd}{2}+n} \sim M^{4-n} \quad \text{for } d=4$$

$$\Gamma^{(n)R}_{\mu_1\ldots\mu_n}(p_j;\mu;\lambda) = \mu^{n+d-\frac{nd}{2}} \underset{\substack{\uparrow\\ \text{no units.}}}{\overline{\Gamma}^{(n)}_{\mu_1\ldots\mu_n}}(p_j/\mu,\lambda)$$

Consider some fixed moment $\hat{p}_i$ and compute

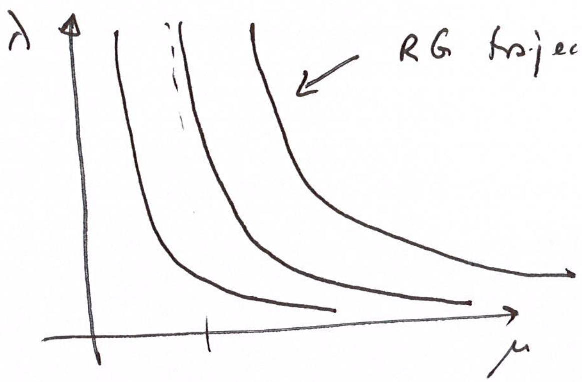
$$\begin{aligned}
\Gamma^{(n)R}_{\mu_1\ldots\mu_n}(\sigma\hat{p}_j,\mu,\lambda) &= \mu^{n+d-\frac{nd}{2}} \overline{\Gamma}^{(n)}_{\mu_1\ldots\mu_n}\left(\frac{\sigma\hat{p}_j}{\mu},\lambda\right) \\
&= \sigma^{n+d-\frac{nd}{2}} \left(\frac{\mu}{\sigma}\right)^{n+d-\frac{nd}{2}} \overline{\Gamma}^{(n)}_{\mu_1\ldots\mu_n}\left(\frac{\hat{p}_j}{(\mu/\sigma)},\lambda\right) \\
&= \sigma^{n+d-\frac{nd}{2}} \Gamma^{(n)R}_{\mu_1\ldots\mu_n}\left(\hat{p}_j,\frac{\mu}{\sigma},\lambda\right)
\end{aligned}$$

Interesting, but we want the same ren. scale for comparison.

To relate the value of $\Gamma^{(n)}$ at two different values of μ we can use RG eqn:

$$\mu \frac{\partial \Gamma^{(n)}}{\partial \mu} + \beta(\lambda) \frac{\partial \Gamma^{(n)}}{\partial \lambda} + n \gamma \Gamma^{(n)} = 0$$

That tells us that a change in μ is equivalent to a change in λ (up to a rescaling).



solve $\mu \frac{d\lambda(\mu)}{d\mu} = \beta(\lambda(\mu))$

Along these, the physics is equivalent (it is just arbitrary choice of ren. scale μ).

For pure YM different trajectories only differ by a choice of scale Λ.

Along the RG trajectory we can compute $\Gamma^{(n)}$:

$$\mu \frac{d}{d\mu} \Gamma^{(n)}(p_j, \lambda(\mu), \mu) = \mu \frac{\partial \Gamma^{(n)}}{\partial \mu} + \mu \frac{\partial \lambda}{\partial \mu} \frac{\partial \Gamma^{(n)}}{\partial \lambda} = \mu \frac{\partial \Gamma^{(n)}}{\partial \mu} + \beta(\lambda) \frac{\partial \Gamma^{(n)}}{\partial \lambda} =$$

$$= -n \gamma \Gamma^{(n)}(p_i, \lambda(\mu), \mu)$$

Solved by $\Gamma^{(n)}(p_i, \lambda(\mu), \mu) = \chi^n(\mu) \, \Gamma^{(n)}(p_i, \lambda(\mu_0), \mu_0)$

$$\mu \frac{d\Gamma^{(n)}}{d\mu} = \mu \, n \chi^{n-1} \frac{d\chi}{d\mu} \Gamma^{(n)}(\mu_0) = -n \gamma \chi^{(n)}(\mu) \Gamma^{(n)}(\mu_0)$$

$$\Rightarrow \frac{1}{\chi}\frac{d\chi}{d\mu} = -\frac{\gamma(\lambda(\mu))}{\mu} \Rightarrow \int_{\mu_0}^{\cancel{\chi}(\mu)} d\ln\chi = -\int^{\mu} \underbrace{\gamma(\lambda(\mu))}_{\text{known function}} \frac{d\mu}{\mu}$$

$\underline{\chi(\mu_0) = 1}$

$$\chi(\mu) = e^{-\int_{\lambda(\mu_0)}^{\lambda(\mu)} \gamma(\lambda)\frac{d\mu}{d\lambda}\frac{d\lambda}{\mu}}$$

$$\chi(\lambda(\mu)) = e^{-\int_{\lambda(\mu_0)}^{\lambda(\mu)} \frac{\gamma(\lambda)}{\beta(\lambda)} d\lambda}$$

Define $\chi(\lambda) = e^{-\int_{\lambda_0}^{\lambda} \frac{\gamma(\lambda')}{\beta(\lambda')} d\lambda'}$

then

$$\Gamma^{(n)}(p_i, \lambda(\mu), \mu) = e^{-n\int_{\lambda_0}^{\lambda} \frac{\gamma(\lambda')}{\beta(\lambda')} d\lambda'} \Gamma^{(n)}(p_i, \lambda(\mu_0), \mu_0)$$

$\lambda_0 = \lambda(\mu_0)$ here.

relates $\Gamma^{(n)}$ at different values of μ along RG trajectory.

We had $\Gamma(\hat{p}_i, \mu/\sigma, \lambda)$; Take $\mu_0 = \mu/\sigma$; $\lambda(\mu_0) = \lambda$

Solve $\mu \frac{\partial\lambda}{\partial\mu} = \beta(\lambda)$ w/ $\lambda(\mu_0) = \lambda$ and get $\lambda(\mu)$

then $\Gamma^{(n)}(\hat{p}_i, \mu, \lambda(\mu)) = e^{-n\int_{\lambda}^{\lambda(\mu)} \frac{\gamma(\lambda')}{\beta(\lambda')} d\lambda'} \Gamma^{(n)}(\hat{p}_i, \lambda, \mu/\sigma)$

We get

$$\Gamma^{(n)}_{\mu_1 \cdots \mu_n}(\sigma \hat{p}_j, \mu, \lambda) = \sigma^{n+d-\frac{nd}{2}} \, e^{n \int_{\lambda}^{\lambda(\mu)} \frac{\gamma(\lambda')}{\beta(\lambda')} d\lambda'} \, \Gamma^{(n)}(\hat{p}_i, \lambda(\mu), \mu)$$

$$\mu \frac{\partial \lambda(\mu)}{\mu} = \beta(\lambda) \quad \Rightarrow \quad \int_{\lambda_0 = \lambda}^{\lambda(\mu)} \frac{d\lambda}{\beta(\lambda)} = \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} = \ln \mu/\mu_0 = \ln \sigma$$

$\mu_0 = \mu/\sigma$ here.

we define $\lambda(\sigma)$ / $\boxed{\int_{\lambda}^{\lambda(\sigma)} \frac{d\lambda}{\beta(\lambda)} = \ln \sigma}$

then

$$\Gamma^{(n)}_{\mu_1 \cdots \mu_n}(\sigma \hat{p}_j, \lambda, \mu) = \sigma^{n+d-\frac{nd}{2}} \, e^{n \int_{\lambda}^{\lambda(\sigma)} \frac{\gamma(\lambda')}{\beta(\lambda')} d\lambda'} \, \Gamma^{(n)}(\hat{p}_i, \lambda(\sigma), \mu)$$

For fixed μ it determines how $\Gamma^{(n)}$ depends on the overall scale of p_j.

For a pure YM. at one loop. $\beta(\lambda) = -\frac{11}{24\pi^2} \lambda^2$

$$\int_{\lambda}^{\lambda(\sigma)} \frac{d\lambda}{\lambda^2} = -\frac{11}{24\pi^2} \ln \sigma \quad \Rightarrow \quad -\frac{1}{\lambda}\Big|_{\lambda}^{\lambda(\sigma)} = -\frac{11}{24\pi^2} \ln \sigma \Rightarrow -\frac{1}{\lambda(\sigma)} + \frac{1}{\lambda} = -\frac{11}{24\pi^2} \ln \sigma$$

$$\frac{1}{\lambda(\sigma)} = \frac{1}{\lambda} + \frac{11}{24\pi^2}\ln\sigma$$

we can define σ_0 /

$$\frac{1}{\lambda} = -\frac{11}{24\pi^2}\ln\sigma_0 \qquad (\sigma_0 < 1)$$

$$\lambda(\sigma) = \frac{1}{\frac{1}{\lambda} + \frac{11}{24\pi^2}\ln\sigma} \quad ; \quad \text{then} \quad \boxed{\lambda(\sigma) = \frac{1}{\frac{11}{24\pi^2}\ln(\sigma/\sigma_0)}}$$

Also: $$\int_\lambda^{\lambda(\sigma)} \frac{\gamma(\lambda')}{\beta(\lambda')}d\lambda' = \int_\lambda^{\lambda(\sigma)} \frac{5}{48\pi^2}\frac{\cancel{\lambda'}}{(-)\frac{11}{24\pi^2}\lambda'^{\cancel{2}}}d\lambda' = -\frac{5}{22}\ln\frac{\lambda(\sigma)}{\lambda}$$

then

$$\Gamma^{(n)}_{\mu_1\cdots\mu_n}(\sigma\hat{p}_j, \lambda, \mu) = \sigma^{\overset{d=4}{\downarrow}4-n} e^{-\frac{5}{22}n\ln\frac{\lambda(\sigma)}{\lambda}}\Gamma^{(n)}(\hat{p}_j, \lambda(\sigma), \mu)$$

$$= \sigma^{4-n}\left(\frac{\lambda}{\lambda(\sigma)}\right)^{\frac{5}{22}n}\Gamma^{(n)}(\hat{p}_j, \lambda(\sigma), \mu)$$

as $\sigma \to \infty$ (very large momenta) $\lambda(\sigma) \to 0$

then $\Gamma^{(n)}(\hat{p}_j, \lambda(\sigma), \mu) \longrightarrow \Gamma_{free}$ + pert. corrections.

$\approx$ free theory and well given by pert. theory.

asymptotic freedom

this region is good to test the theory perturbatively

At low energies $\lambda(\sigma) \to \infty$ then pert. theory is not reliable using $\frac{1}{\lambda} = -\frac{11}{24\pi^2} \ln \sigma_0$ (definition of σ_0)

$$\frac{\lambda}{\lambda(\sigma)} = \frac{\cancel{\frac{11}{24\pi^2}} \ln(\sigma/\sigma_0)}{-\cancel{\frac{11}{24\pi^2}} \ln(\sigma_0)} = \frac{\ln(\sigma/\sigma_0)}{\ln(1/\sigma_0)}$$

$$\boxed{\Gamma^{(n)}_{\mu_1 \dots \mu_n}(\sigma \vec{p}_j, \lambda, \mu) = \underbrace{\sigma^{4-n}}_{\text{dimensional analysis scaling}} \underbrace{\left(\frac{\ln(\sigma/\sigma_0)}{\ln(1/\sigma_0)}\right)^{\frac{5}{22}n}}_{\text{logarithmic corrections}} \Gamma^{(n)}_{\mu_1 \dots \mu_n}(\vec{p}_j, \lambda(\sigma), \mu)}$$