Critical Phenomena

663 (1)

2^{nd} order phase transitions, QFT etc.

3d $O(N)$ model.

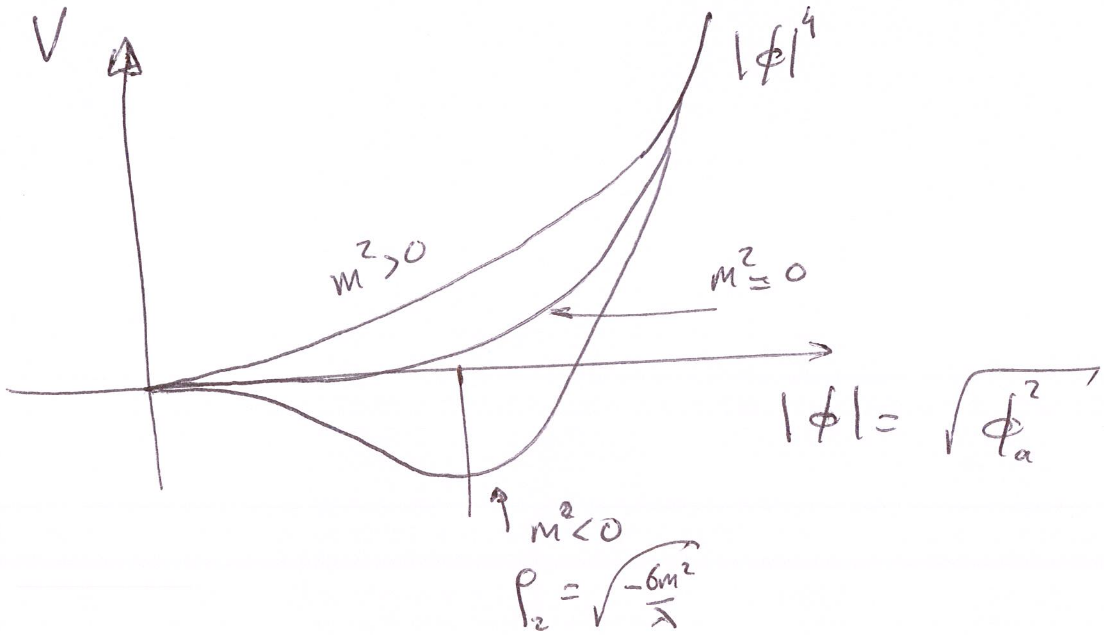
Euclidean

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a + \frac{1}{2} m^2 \phi_a^2 + \frac{\lambda}{4!}(\phi_a^2)^2$$

$a = 1 \dots N \qquad \phi_a \in \mathbb{R}.$

$$V(\phi) = \frac{1}{2} m^2 \phi_a^2 + \frac{\lambda}{4!}(\phi_a^2)^2$$

for $\nabla \phi_a = 0$ constant field.



$m^2 < 0 \qquad \rho = \phi_a^2$

$$V(\rho) = \frac{1}{2} m^2 \rho^2 + \frac{\lambda}{24}\rho^4 \qquad \frac{\partial V}{\partial \rho} = m^2 \rho + \frac{\lambda}{6}\rho^3 = 0$$

$$\rho_1 = 0 \qquad \rho_2 = \sqrt{-\frac{6m^2}{\lambda}}$$

For $m^2 > 0$ $\langle \phi_a^2 \rangle = 0$ in vacuum.

$\delta\phi_a$ small oscillations around $\phi_a = 0$

$\rightarrow$ N excitations of mass m^2

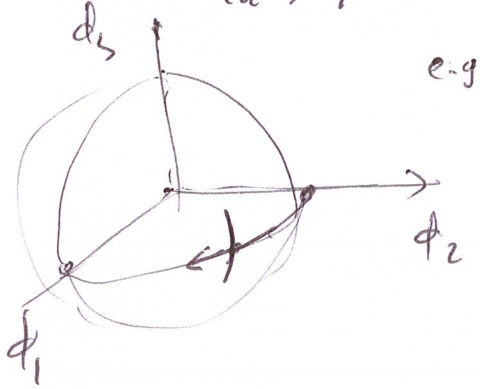
For $m^2 = 0$

N massless excitations of mass m^2

$$\mathcal{L} = \frac{1}{2} |D\phi_a|^2 + \frac{\lambda}{24} (\phi_a^2)^2$$

For $m^2 < 0$

$\langle \phi_a^2 \rangle \neq 0$ $\langle \phi_a^2 \rangle = -\frac{6m^2}{\lambda}$ in vacuum



e.g. $N=3$

$N-1$ massless excitations (non-linear sigma model on S^{N-1})

1 massive excitation (radial)

Possible parametrization:

$\phi_a = R_{ab} \hat{\phi}_b$ $\hat{\phi}_b = (0, 0 \text{---}, 0, \bar{\phi})$ $\bar{\phi} = \sqrt{-\frac{6m^2}{\lambda}}$

rotation

However $SO(N-1)$ does not do anything.

$SO(N)/SO(N-1)$

example

$N=1 \to \rho$

$N=2 \to \rho, \vartheta$

$N=3 \to \rho, \underbrace{\boxed{\theta, \varphi}}_{S^2}$

$$\partial_\mu \phi_a = \partial_\mu R_{ab} \hat{\phi}_b + R_{ab} \partial_\mu \hat{\phi}_b$$

$$= R_{ab} \left(\partial_\mu \hat{\phi}_b + R^{-1}_{bc} \partial_\mu R_{cd} \hat{\phi}_b \right)$$

$$R^{-1}_{ab} R_{bc} = \delta_c$$

$$R^{-1}_{ab} = R_{ba}$$

$$R^t = R^{-1}$$

$$A_{\mu\, ab} = R^{-1}_{ac} \partial_\mu R_{cb} = R_{ca} \partial_\mu R_{cb}$$

$$R_{ca} R_{cb} = \delta_{ab} \qquad \partial_\mu R_{ca} R_{cb} + R_{ca} \partial_\mu R_{cb} = 0$$

$$R_{cb} \partial_\mu R_{ca} = - R_{ca} \partial_\mu R_{cb}$$

A_μ anti-symmetric. $\boxed{A_{\mu\, ba} = -A_{\mu\, ab}}$

$$\partial_\mu \vec{\phi} = \partial_\mu \hat{\phi} + A_\mu \cdot \hat{\phi}$$

$$\partial_\mu \phi_a = \partial_\mu \hat{\phi}_a + A_{\mu\, ab} \hat{\phi}_b$$

$$\partial_\mu \phi_a \partial^\mu \phi_a = \partial_\mu \hat{\phi}_a \partial^\mu \hat{\phi}_a + 2 \partial_\mu \hat{\phi}_a A_{\mu ab} \hat{\phi}_b +$$

$$+ A_{\mu ab} A_{\mu ac} \hat{\phi}_b \hat{\phi}_c$$

$$\vec{\phi}_a = (\overset{1}{0}, \;\text{——}\; 0, \overset{N}{\rho})$$

$\rho = \rho_0 + \sigma \qquad \rho_0 = \sqrt{-\frac{6m^2}{\lambda}}$

A_μ anti-symmetric.

$$\partial_\mu \phi_a \partial^\mu \phi_a = \partial_\mu \sigma \partial_\mu \sigma + 2\, \cancel{\partial_\mu \sigma A_{\mu NN} \rho_0}^{\,0} + \cancel{2 \partial_\mu \sigma A_{\mu NN} \sigma}^{\,0}$$
$$+ A_{\mu aN} A_{\mu aN} (\rho_0 + \sigma)^2$$

$A_{\mu aN}$: first order in derivatives of the angles.

$$\mathcal{L} = \frac{1}{2} \partial_\mu \sigma \partial_\mu \sigma + \frac{1}{2} A_{\mu aN} A_{\mu aN} \rho_0^2 + \frac{1}{2} m^2 (\rho_0 + \sigma)^2 +$$
$$+ \frac{\lambda}{24} (\rho_0 + \sigma)^4 + \frac{1}{2} A_{\mu aN} A_{\mu aN} (2\rho_0 \sigma + \sigma^2)$$

No potential for the angles $\rightarrow$ $N-1$ massless fields

$$\begin{array}{c} 1 \\ 1\;2\;1 \\ 1\;3\;3\;1 \\ 1\;4\;6\;4\;1 \end{array}$$

$$\frac{1}{2} m^2 \rho_0^2 + m^2 \rho_0 \sigma + \frac{1}{2} m^2 \sigma^2 + \frac{\lambda}{24} (\rho_0^4 + 4\rho_0^3 \sigma + 6 \rho_0^2 \sigma^2 + 4 \rho_0 \sigma^3 + \sigma^4$$

$$V_0 = \frac{1}{2} m^2 \rho_0^2 + \frac{\lambda}{24} \rho_0^4$$: vacuum energy.

$$V_1 = m^2 \rho_0 \sigma + \frac{\lambda}{6} \rho_0^3 \sigma = 0 \qquad (\rho_0^2 = -6m^2/\lambda)$$

$$V_2 = \frac{1}{2} m^2 \sigma^2 + \frac{\lambda}{4} \rho_0^2 \sigma^2 = \frac{1}{2} m^2 \sigma^2 - \frac{\lambda}{4} \frac{6m^2}{\lambda} \sigma^2 = - m^2 \sigma^2$$

To quadratic order:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\,\partial_\mu\sigma + \frac{1}{2}\underbrace{(-2m^2)}_{>0}\sigma^2$$

mass $m_\sigma = \sqrt{-2m^2}$ mass of radial excitation.

$$\mathcal{L} = \frac{1}{2}\rho_0^2\, A_{\mu aN} A_{\mu aN}$$

(Renormalizable only in 2d)

kinetic term for motion on the sphere S^{N-1}

radius of the sphere

(f_π)

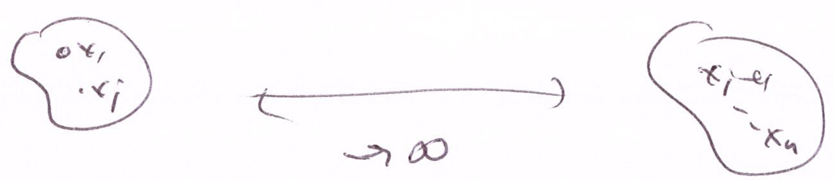
At low energy this dominates → non-linear sigma model.

Cluster decomposition property.

$$\langle 0|\,\phi_{a_1}(x_1)\,\phi_{a_2}(x_2)\cdots\phi_{a_{n-1}}(x_{n-1})\,\phi_{a_n}(x_n)\,|0\rangle \longrightarrow$$

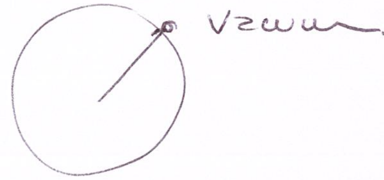
$$\to \langle 0|\,\phi_{a_1}(x_1)\cdots\phi_{a_j}(x_j)\,|0\rangle\,\langle 0|\,\phi_{a_{j+1}}(x_{j+1})\cdots\phi_{a_n}(x_n)\,|0\rangle$$

if



If vacuum is a superposition of ϕ over sphere then it is not true.

Example



$|\theta\rangle$

$$|0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \; |\theta\rangle$$

$$\langle 0|\phi_1(x)\,\phi_1(y)|0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \; \langle\theta|\rho_0 c\theta \, \rho_0 c\theta\,|\theta\rangle =$$

$$= \frac{1}{2\pi}\rho_0^2 \int_{-\pi}^{\pi} d\theta \; c^2\theta \; = \frac{1}{2}\rho_0^2$$

$$\langle 0|\phi_1(x)|0\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \; \langle\theta|\rho_0 c\theta|\theta\rangle = 0.$$

$$\langle 0|\phi_1(x)\,\phi_1(y)|0\rangle \neq \langle\phi_1(x)\rangle \; \langle\phi_1(y)\rangle$$

for any distance $|x-y| \to \infty$

for $|\theta\rangle$ it is <u>OK</u>

$$\langle\theta|\phi_1(x)\;\phi_1(y)|\theta\rangle = \rho_0^2 c^2\theta$$

$$\langle\theta|\phi_1(x)\,|\theta\rangle = \rho c\theta.$$

External source

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_a \partial_\mu \phi_a + \frac{1}{2} m^2 \phi_a^2 + \frac{\lambda}{24}(\phi_a^2)^2 + j_a \phi_a .$$

↑ fixed.

Take $j_N \neq 0$ $\quad j_{a \neq N} = 0$ (We choose N axis like this)

energy minimizes for $\phi \parallel j$

$$\mathcal{L} = \frac{1}{2}\partial_\mu \rho \partial_\mu \rho + \frac{1}{2} A_{\mu a \alpha} A_{\mu a \alpha} \rho^2 + \frac{1}{2} m^2 \rho^2 +$$

$$+ \frac{\lambda}{24}\rho^4 + j_N \rho$$

$m^2 > 0$ phase. $\quad V(\rho) = \frac{1}{2} m^2 \rho^2 + \frac{\lambda}{24}\rho^4 - j_N \rho$

$$\frac{\partial V}{\partial \rho} = m^2 \rho + \frac{\lambda}{6}\rho^3 - j_N = 0$$

$\rho \simeq + j_N / m^2$ $\quad$ small j_N

$\underline{m^2 = 0}$ phase $\quad \rho = \left(+ \frac{6 j_N}{\lambda} \right)^{1/3}$

$\underline{m^2 < 0}$ phase $\quad \rho = \rho_0 + \sigma$ ← small

$$m^2(\cancel{\rho_0} + \sigma) + \frac{\lambda}{6}(\cancel{\rho_0^3} + 3\rho_0^2 \sigma + \underbrace{3\rho_0 \sigma^2 + \sigma^3}_{\simeq 0}) = + j_N$$

$$\left[m^2 + \frac{3\cancel{\lambda}}{\cancel{6}}\left(-\frac{6m^2}{\cancel{\lambda}}\right)\right]\sigma = +j_N$$

$$-2m^2\sigma = +j_N$$

$$\boxed{\sigma = -\frac{1}{2m^2}\, j_N}$$

For a magnetic system $\quad \rho \leftrightarrow M \quad j_N \leftrightarrow B$

$$m^2 = f(T) \simeq \alpha \frac{(T-T_c)}{T_c} = \alpha t \quad ; \quad t = \frac{T-T_c}{T_c}$$

$T < T_c \qquad m^2 < 0$

$T > T_c \qquad m^2 > 0$

Critical exponents

$m \sim t^\nu \qquad \boxed{\nu = 1/2} \qquad (m^2 \sim t) \quad ; \quad \boxed{\eta = 0}$

$\frac{M}{V} \sim |t|^\beta \qquad M = \langle\rho\rangle = \sqrt{\frac{-6m^2}{\lambda}} \sim |t|^{1/2} \qquad \boxed{\beta = 1/2}$

$\chi \sim |t|^{-\gamma} \qquad \rho \sim -\frac{j_N}{m^2} = -\frac{j_N}{t} \qquad \boxed{\gamma = 1}$

$\frac{M}{V} \sim H^{1/\delta} \quad t = 0 \qquad \rho = \left(-\frac{6 j_N}{\lambda}\right)^{1/3} \qquad \boxed{\delta = 3}$

$$\langle \sigma(r)\sigma(0)\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{e^{i\vec{p}\vec{x}}}{p^2+m^2} \sim \frac{e^{-mr}}{r}$$

$$\frac{1}{r^p} = \frac{1}{r^{d-2+\eta}} \qquad \boxed{\eta = 0} \qquad \boxed{p=1} \qquad \boxed{d=3}$$

$\eta = 0,\ \nu = 1/2$

Relations	$d=3$	$d=4$
$\gamma = \nu(2-\eta)$	1	1
$\beta = \frac{\nu}{2}(d-2+\eta)$	$1/4$	$1/2$
$\delta = \dfrac{d+2-\eta}{d-2+\eta}$	5	3
	wrong.	<u>ok</u>
	requires fluctuations	mean field <u>ok</u>

Fluctuations

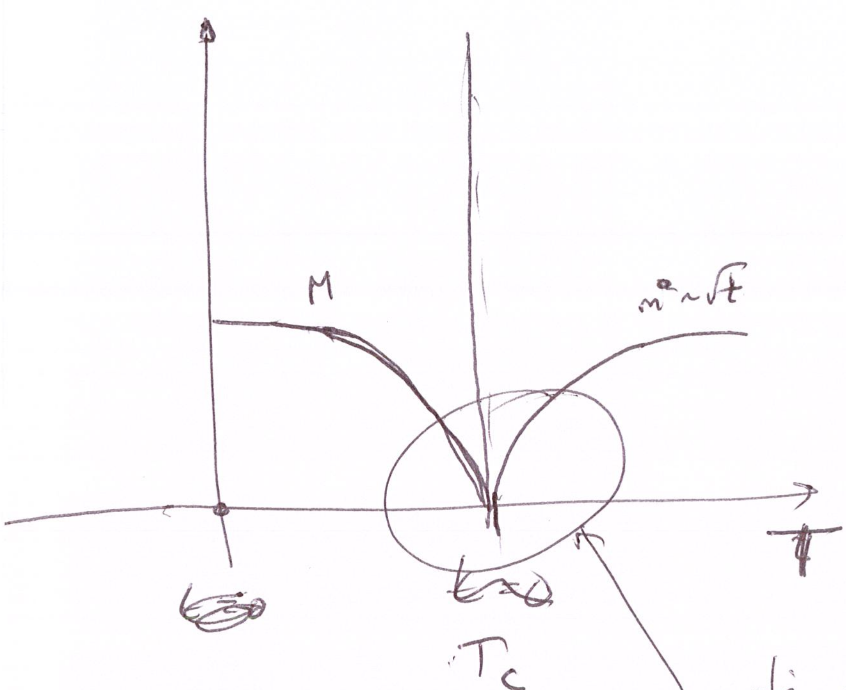
Partition function

$$Z = \int \mathcal{D}\phi_a \; e^{-\int d^3x\, \mathcal{L}(\phi)}$$

classical partition function or EQFT. Same calculation
(Euclidean)

Different interpretation.

Naively we should put $m^2=0$ but there are corrections from fluctuations (thermal or quantum).
(to get scale invariance)



critical region dominated by scale invariant theory.

Any QFT has extra dependence on the cut-off. Λ.

At the critical point

$$\langle \phi_a(x)\, \phi_b(y) \rangle = \delta_{ab} \frac{C\, \Lambda^{-\eta}}{|x-y|^{d-2+\eta}} = G_{ab}(x,y)$$

$$\int d^d x \; |\nabla \phi|^2$$

$$M^{-d} \quad M^{+1} M^{-1+d/2} \qquad [\phi] \sim \underset{\uparrow \text{mass}}{M}^{d/2-1}$$

$$\langle \phi\, \phi \rangle \sim M^{d-2} \sim L^{2-d} = \frac{1}{L^{d-2}}.$$

If we can compute the dependence of $\langle \phi_a(x) \phi_b(y) \rangle$ on Λ we can get η.

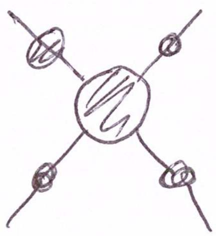
$$\Lambda \frac{\partial}{\partial \Lambda} G_{ab} = \cancel{\Lambda} (-\eta) \frac{1}{\cancel{\Lambda}} G_{ab} = -\eta\, G_{ab}(x,y)$$

Dependence on cut-off follows from RG.

$$\phi_R = Z_\phi^{-1/2} \phi_B$$

$$\lambda_B = \lambda_B(\lambda_R, \Lambda)$$

If we write everything in terms of renormalized quantities then the cut-off dependence goes away when $\Lambda \to \infty$



$$G^{(4)} = G^{(2)} G^{(2)} G^{(2)} G^{(2)} \Gamma^{(4)}$$

$$\Gamma_R^{(n)} = \frac{\overbrace{\langle \phi_R \cdots \phi_R \rangle}^{n}}{\langle \phi\phi \rangle_R \cdots \langle \phi\phi \rangle_R} = \underbrace{Z_\phi^{n/2} \Gamma_B(g_B(g_R, \Lambda), p_i, \Lambda)}_{\Lambda \to \infty}$$

is finite (and indep. of Λ)

$$\Lambda \frac{\partial \Gamma_R}{\partial \Lambda} = 0 = \frac{n}{2} \Lambda \frac{\partial Z_\phi}{\partial \Lambda} \frac{1}{Z_\phi} Z_\phi^{n/2} \Gamma_B + Z_\phi^{n/2} \Lambda \frac{\partial \lambda_B}{\partial \Lambda} \frac{\partial \Gamma_B}{\partial \lambda_B} +$$

$$+ Z_\phi^{n/2} \Lambda \frac{\partial \Gamma_B}{\partial \Lambda} = 0$$

Ren. conditions

$$\Gamma_R^{(4)}(\hat{p}_i = \hat{p}_i \overset{\text{scale}}{\mu}) = \lambda_R \mu^{2\varepsilon}$$

$$\Gamma_R^{(2)}(0) = m_R^2 \qquad \frac{\partial}{\partial p^2} \Gamma_R^{(2)}(p)\Big|_{p^2=\mu^2} = 1$$

$$\frac{n}{2}\underbrace{\Lambda\frac{\partial \ln Z_\phi}{\partial\Lambda}}_{-\eta}\Gamma_B + \underbrace{\Lambda\frac{\partial\lambda_B}{\partial\Lambda}}_{\beta}\frac{\partial\Gamma_B}{\partial\lambda_B} + \Lambda\frac{\partial\Gamma_B}{\partial\Lambda} = 0$$

$$\boxed{\Lambda\frac{\partial\Gamma_B^{(n)}}{\partial\Lambda} + \beta\frac{\partial\Gamma_B^{(n)}}{\partial\lambda_B} - \frac{n}{2}\eta\,\Gamma_B^{(n)} = 0}$$

$\beta(\lambda_B)$ $\quad\eta(\lambda_B)$ $\quad$ ($\beta(\lambda_B, \frac{\Lambda}{\mu})$, but μ does not appear in $\Gamma^{(3)}$)

no units

Suppose $\beta(\hat{\lambda}_B) = 0$

$\leftarrow$ some given value of λ_B.

$\eta(\hat{\lambda}_B) = \hat{\eta}$

$$\Lambda\frac{\partial\Gamma_B^{(n)}}{\partial\Lambda} = \frac{n}{2}\hat{\eta}\,\Gamma_B^{(n)} \Rightarrow \Gamma_B^{(n)} = \Lambda^{n/2\,\hat{\eta}}\underbrace{\tilde{\Gamma}_B^{(n)}}_{\text{indep. of }\Lambda}$$

we got cut-off dependence!

$$G^{(2)} = (\Gamma^{(2)})^{-1}$$

$$G_B^{(2)} = \Lambda^{-\hat{\eta}}\,\tilde{G}^{(2)}$$

this is η exponent. critical exp.!!

Example of $O(N)$

by units so λ no units

$$\mathcal{L}_E = \frac{1}{2}\partial_\mu\phi_a\partial_\mu\phi_a + \frac{1}{2}m^2\phi_a^2 + \frac{\lambda}{4!}\Lambda^{2\varepsilon}(\phi_a^2)^2$$

$[\phi] = M^{-1+d/2}$

$2\varepsilon + 4(-1+d/2) = d$

$2\varepsilon - 4 + 2(4-2\varepsilon) = 4-2\varepsilon$

✓

-) β-function ($\beta(\lambda^*) = 0$) $\Lambda \frac{\partial\lambda_B}{\partial\Lambda}\Big|_{\lambda_R}$

-) critical exponent from correlation of operators.

1-loop diagrams. ($4-2\varepsilon$ dim)

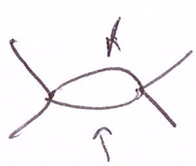
$\hookrightarrow \int d^4k \frac{1}{k^2} \sim \underline{\Lambda^2} \rightarrow$ $C_0 p^2 + C_1 +$ finite piece.

actually zero

divergent

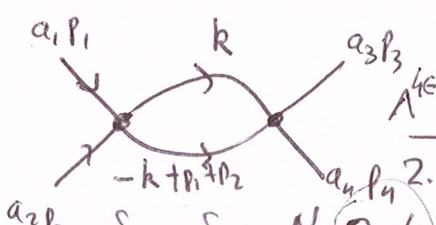
C_0 : wave-function ren.

C_1 : mass renorm.



$$\int \frac{d^4k}{k^2k^2} \sim C_0 \ln\Lambda + \text{finite}$$

coupling constant ren.



$$\frac{\Lambda^{4\varepsilon}\lambda^2}{2.(4!)^2}\phi_{a_1}\phi_{a_2}\phi_{a_3}\phi_{a_4}\int d^4x\,\phi_{b_1}\phi_{b_1}\phi_{c_1}\phi_{c_1}\int d^4x\,\phi_{b_2}\phi_{b_2}\phi_{c_2}\phi_{c_2}$$

$\delta_{a_1a_2}\delta_{a_3a_4}\,N$ $2\times4\times1\times4\times2$

$\delta_{a_1a_2}\delta_{a_3a_4}$ $2\times4\times1\times4\times2\times2$

$\delta_{a_1a_2}\delta_{a_3a_4}$ $2\times4\times2\times4\times2$

$\delta_{a_1a_3}\delta_{a_2a_4}$ $2\times4\times2\times4\times2$

$\delta_{a_1a_4}\delta_{a_2a_3}$ $2\times4\times2\times4\times2$

$$\delta_{a_1a_2}\,\delta_{a_3a_4}\;\frac{\lambda^2}{2\cdot(4!)^2}\;2\times 4\times 2\times 4\times (N+4) = \delta_{a_1a_2}\,\delta_{a_3a_4}\;\lambda^2\frac{(N+4)}{18}$$

$\cancel{1234}\ \cancel{1234}$

$$\delta_{a_1a_3}\,\delta_{a_2a_4}\;\frac{\lambda^2}{2(1234)(1234)}\;\cancel{2\times 4\times 2\times 4\times 2} = \delta_{a_1a_3}\,\delta_{a_2a_4}\;\frac{\lambda^2}{9}$$

$$\delta_{a_1a_4}\,\delta_{a_2a_3}\;\frac{\lambda^2}{9}$$

All these $\displaystyle B(p_1+p_2) = \int \frac{d^4k}{(2\pi)^4}\;\frac{\Lambda^{2\varepsilon}}{k^2\,(p_1+p_2-k)^2}$

we add

$$\frac{\lambda^2}{18}\,B(p_1+p_2)\left((N+4)\,\delta_{a_1a_2}\delta_{a_3a_4} + 2\,\delta_{a_1a_3}\delta_{a_2a_4} + 2\,\delta_{a_1a_4}\delta_{a_2a_3}\right)$$

$$\frac{\lambda^2}{18}\,B(p_1+p_4)\left((N+4)\,\delta_{a_1a_4}\delta_{a_2a_3} + 2\,\delta_{a_1a_3}\delta_{a_2a_4} + 2\,\delta_{a_1a_2}\delta_{a_3a_4}\right)$$

$$\frac{\lambda^2}{18}\,B(p_1+p_3)\left((N+4)\,\delta_{a_1a_3}\delta_{a_2a_4} + 2\,\delta_{a_1a_2}\delta_{a_3a_4} + 2\,\delta_{a_1a_4}\delta_{a_2a_3}\right)$$

Bubble in dim. reg.

$$B(p) = \int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \quad \frac{\Lambda^{2\varepsilon}}{k^2\,(p-k)^2} =$$

$$= \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2\, \lambda^{2\varepsilon} \int \frac{d^dk}{(2\pi)^d}\, e^{-\alpha_1 k^2}\, e^{-\alpha_2(p^2-2pk+k^2)}$$

$$= \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2 \left(\frac{\pi}{\alpha_1+\alpha_2}\right)^{d/2} \frac{1}{(2\pi)^d}\, e^{\frac{4\alpha_2^2 p^2}{4(\alpha_1+\alpha_2)}}\, e^{-\alpha_2 p^2}\, \lambda^{2\varepsilon}$$

$$e^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2}$$

$$= \int_0^\infty d\lambda \int_0^\infty d\alpha_1 \int_0^\infty d\alpha_2\, \lambda^{2\varepsilon} \frac{\delta(\alpha_1+\alpha_2-\lambda)}{(2\pi)^d} \frac{\pi^{d/2}}{(\alpha_1+\alpha_2)^{d/2}}\, e^{-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p^2}$$

$\alpha_1 = \lambda\beta_1 \quad \alpha_2 = \lambda\beta_2$

$$= \lambda^{2\varepsilon} \int_0^\infty \lambda^2 d\lambda \int_0^\infty d\beta_1 \int_0^\infty d\beta_2 \frac{\delta(\beta_1+\beta_2-1)}{(2\pi)^d\,\lambda} \frac{\pi^{d/2}}{\lambda^{d/2}}\, e^{-\lambda\beta_1\beta_2 p^2}$$

$$= \lambda^{2\varepsilon} \int_0^\infty d\beta_1 \int_0^\infty d\beta_2\, \delta(\beta_1+\beta_2-1) \frac{\pi^{d/2}}{(2\pi)^d} \int_0^\infty d\lambda\, \lambda^{-d/2+1}\, e^{-\lambda\beta_1\beta_2 p^2}$$

$$= \int_0^1 d\beta_1 \int_0^1 d\beta_2\, \delta(\beta_1+\beta_2-1) \frac{\pi^{d/2}}{(2\pi)^d}\, \Gamma(2-d/2)\, (\beta_1\beta_2 p^2)^{-2+d/2}\, \lambda^{2\varepsilon}$$

$$= \frac{\pi^{d/2}}{(2\pi)^d}\, p^{d-4} \int_0^1 d\beta_1\, \beta_1^{d/2-2}\, (1-\beta_1)^{d/2-2}\, \Gamma(2-d/2)\, \lambda^{2\varepsilon}$$

$$B(p) = \frac{\pi^{d/2}}{(2\pi)^d}\, p^{d-4}\, \Gamma(2-d/2)\, B(d/2-1, d/2-1)\, \Lambda^{2\varepsilon}$$

$$= \frac{\pi^{d/2}}{(2\pi)^d}\, p^{d-4}\, \Gamma(2-d/2)\, \frac{\Gamma(d/2-1)\,\Gamma(d/2-1)}{\Gamma(d-2)}\, \Lambda^{2\varepsilon}$$

$$= \frac{1}{(4\pi)^{d/2}}\, p^{d-4}\, \frac{\Gamma\left(\frac{4-d}{2}\right)}{\Gamma(d-2)} \left(\Gamma(d/2-1)\right)^2 \Lambda^{2\varepsilon}$$

$d = 4-2\varepsilon$

$$B(p) = \frac{\Lambda^{2\varepsilon}}{(4\pi)^{2-\varepsilon}}\, p^{-2\varepsilon}\, \frac{\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)} \left(\Gamma(1-\varepsilon)\right)^2$$

$\varepsilon\Gamma(\varepsilon) = \Gamma(1+\varepsilon) \simeq 1 + \Gamma'(1)\,\varepsilon$

$\varepsilon \to 0$

$\Gamma(\varepsilon) \simeq 1/\varepsilon$

$$B(p) \underset{\uparrow}{=} \frac{1}{\varepsilon}\, \frac{1}{(4\pi)^2}$$

pole

Λ dependence

$$e^{2\varepsilon \ln \Lambda/p}$$

$$B(p) = \frac{1}{\varepsilon}\, \frac{1}{(4\pi)^2} \left(\frac{\Lambda}{p}\right)^{2\varepsilon} = \frac{1}{16\pi^2\varepsilon} + \frac{1}{8\pi^2} \ln \Lambda/p + \ldots$$

Cut-off dependence.

$$\Gamma^{(4)}_{a_1a_2a_3a_4}(p_1,p_2,p_3,p_4) = \frac{\lambda^2}{18}(N+8)\frac{1}{8\pi^2}\ln\Lambda\,(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3}) + \text{finite piece}$$

as $\Lambda\to\infty$

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order λ.

$$\frac{\lambda\Lambda^{2\varepsilon}}{4!}\phi_{a_1}\phi_{a_2}\phi_{a_3}\phi_{a_4}\int d^4x\,\phi_{b_1}\phi_{b_1}\phi_{c_1}\phi_{c_1}$$

$$\frac{\cancel{8}}{1\cancel{2}3\cancel{4}}\lambda\Lambda^{2\varepsilon}\left(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3}\right)$$

$$\Gamma^{(4)} = \frac{\lambda\Lambda^{2\varepsilon}}{3}(\delta_{a_1a_2}\delta_{a_3a_4}+\delta_{a_1a_3}\delta_{a_2a_4}+\delta_{a_1a_4}\delta_{a_2a_3}).$$

$$\cdot\left[1+\frac{\lambda}{6}(N+8)\frac{1}{8\pi^2}\ln\Lambda+\dots\right] + \text{finite piece.}$$

$$\mu^{2\varepsilon}\lambda_R = \Lambda^{2\varepsilon}\left(\lambda_B - \frac{\lambda_B^2}{48\pi^2}(N+8)\ln\frac{\Lambda}{\mu} + \text{finite}\right)$$

$1+2\varepsilon\ln\Lambda/\mu$

$$\lambda_R = \cancel{\lambda_B 2+}\ \lambda_B\left(\frac{\Lambda}{\mu}\right)^{2\varepsilon} - \frac{\lambda_B^2}{48\pi^2}(N+8)\ln\frac{\Lambda}{\mu} + \dots$$

1-loop

$$= \lambda_B + \ln\frac{\Lambda}{\mu}\left(2\varepsilon\lambda_B - \frac{\lambda_B^2}{48\pi^2}(N+8)\right) + \text{finite}$$

$$\lambda_B = \lambda_R - \ln\frac{\Lambda}{\mu}\left(2\varepsilon\lambda_R - \frac{\lambda_R^2}{48\pi^2}(N+8)\right) + \dots$$

$$\Lambda\frac{\partial\lambda_B}{\partial\Lambda} = -2\varepsilon\lambda_R + \frac{\lambda_R^2}{48\pi^2}(N+8) = \beta$$

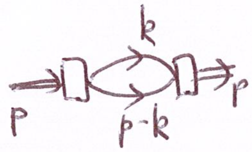
$$\beta(\lambda_B) = -2\varepsilon\lambda_B + \frac{\lambda_B^2}{48\pi^2}(N+8)$$

$\boxed{\lambda_B = 0}$ scale inv.

$$\lambda_B^* = \frac{96\varepsilon\pi^2}{(N+8)} \qquad g = \frac{\lambda}{16\pi^2}$$

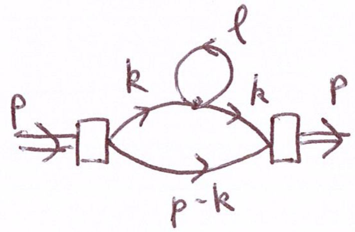
$$\boxed{g_B^* = \frac{6\varepsilon}{N+8}}$$ zero of β-function

$\langle \phi_a^2 \, \phi_b^2 \rangle$

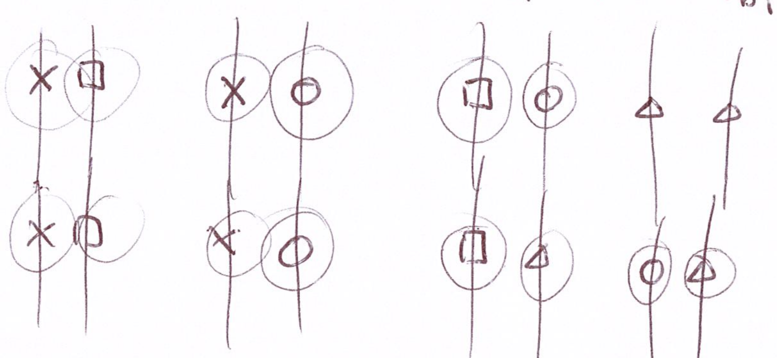


$\phi_a \; \phi_a \qquad \phi_b \; \phi_b$

 $2N \, B(p)$



$-\frac{\lambda}{4!} \quad \phi_a \; \phi_a \qquad \phi_b \; \phi_b \qquad \phi_{a_1} \; \phi_{a_1} \; \phi_{b_1} \; \phi_{b_1}$



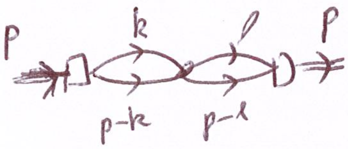
$4 \times 4 \times 1 \; N^2$

$4 \times 4 \times 2 \; N$

$$-\frac{\lambda}{\cancel{1234}} \overset{2}{\cancel{4}\times\cancel{4}} \times N(N+2) \int \frac{d^dk}{(2\pi)^d} \int \frac{d^dl}{(2\pi)^d} \frac{1}{(k^2)^2} \frac{1}{(p-k)^2} \frac{1}{l^2}$$

$\int \frac{d^dl}{l^2} = 0$ dim. reg.

$$-\frac{2\lambda N}{3}(N+2) \; \Delta(0) \; \underbrace{\int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2)^2} \frac{1}{(p-k)^2}}_{\text{A div} = -3}$$



$$-\frac{\lambda}{4!} \quad \phi_a \phi_a \quad \phi_b \phi_b \qquad \phi_{a_1} \phi_{a_1} \quad \phi_{b_1} \phi_{b_1}$$

$$\times \quad \square \quad \triangle \quad \circ \qquad \times \quad \square \quad \triangle \quad \circ \qquad 4 \times 1 \times 2 \times N^2$$

$$\times \quad \square \quad \triangle \quad \circ \qquad \times \quad \triangle \quad \square \quad \circ \qquad 4 \times 2 \times 2 \times N$$

$$-\frac{\lambda}{1 \times \cancel{2} \times 3 \times \cancel{4}} \cancel{8} (N+2) N \; B^2(p)$$

$$\langle \phi_a^2 \phi_b^2 \rangle = 2N B(p) - \frac{\lambda N}{3}(N+2) \; B^2(p) + \ldots$$

$$= 2N B(p) \left(1 - \frac{\lambda}{6}(N+2) B(p) + \ldots \right)$$

not n_I for counting.

$$4 n_v = 2 n_I + 4 \qquad n_I = 2 n_v - 2$$

$$n_L = n_I + 2 - n_v + 1 \qquad n_L = 2n_v - \cancel{2} + \cancel{2} - n_v + 1 = n_v + 1$$

$$\# \text{div} = d\, n_L - 2(n_I + 2) = d n_v + d - 4 n_v + 4 - 8 = (d-4) n_v + d - 4$$

$$\# \text{div} = (n_v + 1)(d - 4)$$

$\underline{d=4}$ $\quad \# \text{div} = 0 \quad \Rightarrow \underline{\ln \Lambda} \times$ Constant.

$\Gamma^{(2,0)}$ has a divergence even for the free theory.
We need to subtract

$$\Gamma_R^{(2,0)}(p) = Z_2^2 Z_\phi^{-2} \left(\Gamma_B^{(2)}(p) - \underline{\Gamma_B^{(2)}(\mu)} \right)$$

choose this.

Take also $\Gamma_R^{(2,0)}(p = 2\mu) = 1$ for example.

$$\Gamma_R^{(2,0)}(p=\mu) = 0$$

$$\Gamma_B^{(2)}(p) - \Gamma_B^{(2)}(\mu) = 2N\left(B(p) - B(\mu)\right) -$$

$$- \frac{\lambda N(N+2)}{3} \left(B^2(p) - B^2(\mu) \right) =$$

$$= 2N\left(B(p) - B(\mu)\right) \left[1 - \frac{\lambda(N+2)}{6} \left(B(p) + B(\mu) \right) \right]$$

$$B(p) = \frac{1}{8\pi^2} \ln \frac{\Lambda}{p} + b$$

finite constant.

$$B(p) - B(\mu) = \frac{1}{8\pi^2} \ln \frac{\mu}{p}$$: finite ✓.

$$B(p) + B(\mu) = \frac{1}{8\pi^2}\left(\ln\frac{\Lambda}{p} + \ln\frac{\Lambda}{\mu}\right) + 2b$$

$$= \frac{1}{4\pi^2}\ln\frac{\Lambda}{\sqrt{\mu p}} + 2b$$

$$\Gamma_R^{(2,0)}(p) = Z_2^2 Z_\phi^{-2}\,\frac{2N}{8\pi^2}\ln\frac{\mu}{p}\left(1 - \frac{\lambda(N+2)}{24\pi^2}\ln\frac{\Lambda}{\sqrt{\mu p}} + \tilde{b}\right)$$

$$\Gamma_R^{(2,0)}(2\mu) = Z_2^2 Z_\phi^{-2}\,\frac{N}{4\pi^2}\ln\left(\tfrac{1}{2}\right)\left(1 - \frac{\lambda(N+2)}{24\pi^2}\ln\frac{\Lambda}{\sqrt{2}\mu} + \tilde{b}\right) = 1$$

$$Z_\phi = 1 + \mathcal{O}(\lambda^2)$$

$$Z_2 = \sqrt{\frac{4\pi^2}{N\ln 2}}\left(1 + \frac{\lambda(N+2)}{48\pi^2}\ln\frac{\Lambda}{\sqrt{2}\mu}\right)$$

$$\Lambda\frac{\partial \ln Z_2}{\partial\Lambda} = \frac{\lambda(N+2)}{48\pi^2} = \frac{g^*(N+2)}{3} = \eta_2$$

$$Z_2 \sim \left(\frac{\Lambda}{\mu}\right)^{\frac{1}{3}g^*(N+2)}$$

units

$$\langle \phi^2_B \phi^2_B \rangle = \frac{c\Lambda^{-2\Delta_{\phi^2}+2d-4}}{|x-y|^{2\Delta_{\phi^2}}}$$

$$[\phi] = M^{d/2-1}$$

$$\int e^{ip(x-y)} d^d x \frac{1}{|x-y|^{2\Delta_{\phi^2}}} = p^{2\Delta_{\phi^2}-d}$$

$$\Gamma_B^{(2,0)} = \frac{\tilde{c}\Lambda^{-2\Delta_{\phi^2}+2d-4}}{p^{d-2\Delta_{\phi^2}}} = \frac{\tilde{c}\left(\Lambda^{-\Delta_{\phi^2}+d-2}\right)^2}{p^{d-2\Delta_{\phi^2}}}$$

$$= Z_2^{-2}\,\Gamma_R^{(2,0)}(p^2)$$

$$Z_2^{-2} = \left(\frac{\Lambda}{\mu}\right)^{-\frac{2}{3}g^*(N+2)}$$

$$-\Delta_{\phi^2}+d-2 = -\frac{1}{3}g^*(N+2)$$

$$\Delta_{\phi^2} = d-2+\frac{1}{3}g^*(N+2)$$

$$S = S_{CFT} + \alpha \int d^d x \; t \, \phi^2$$

$$= S_{CFT} + \tilde{\alpha} \int d^d x \; \xi^{-1/\nu} \, \underbrace{\phi^2}_{\xi^{-\Delta_{\phi^2}}}$$

$$d - \frac{1}{\nu} - \Delta_{\phi^2} = 0$$

$$\frac{1}{\nu} = d - \Delta_{\phi^2} = \cancel{d} - \cancel{d} + 2 - \frac{g^*}{3}(N+2)$$

$$\nu = \frac{1}{2\left(1 - \frac{g^*}{6}(N+2)\right)} \simeq \frac{1}{2}\left(1 + \frac{g^*}{6}(N+2)\right)$$

$$\nu = \frac{1}{2}\left(1 + \frac{\cancel{6}\epsilon}{N+8} \; \frac{N+2}{\cancel{6}}\right)$$

$$= \frac{1}{2}\left(1 + \frac{\epsilon}{2} \; \frac{N+2}{N+8}\right)$$