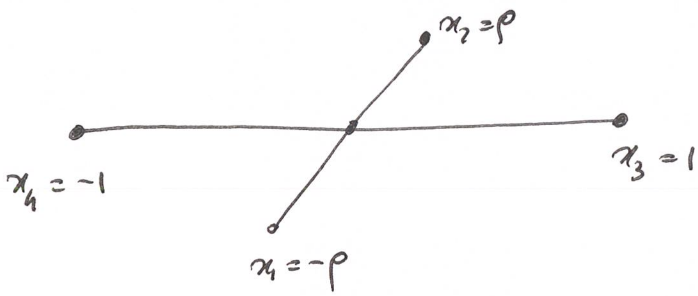
Conformal bootstrap

We want to solve

$$v^{\Delta} f(u,v) = u^{\Delta} f(v,u)$$

$$f(u,v) = \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}} \, \boxed{G_{\mathcal{O}}(u,v)}$$

Consider the configuration



$$u = \frac{x_{12}^2 \, x_{34}^2}{x_{13}^2 \, x_{24}^2} = \frac{4\rho\bar\rho \, 4}{(1+\rho)^2(1+\bar\rho)^2} = \frac{16\rho\bar\rho}{(1+\rho)^2(1+\bar\rho)^2} = z\bar z \; ; \quad \boxed{z = \frac{4\rho}{(1+\rho)^2}}$$

$$v = \frac{x_{14}^2 \, x_{23}^2}{x_{13}^2 \, x_{24}^2} = \frac{(1-\rho)^2(1-\bar\rho)^2}{(1+\rho)^2(1+\bar\rho)^2} = (1-z)(1-\bar z) \qquad 1-z = \frac{(1-\rho)^2}{(1+\rho)^2}$$

$$u = v \qquad \boxed{z_0 = \frac{1}{2}} \qquad 1+2\rho+\rho^2 = 8\rho \qquad \rho^2-6\rho+1=0$$

$$\rho = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2} \qquad \boxed{\rho_0 = 3-2\sqrt{2}}$$

Extract 1:

$$f(u,v) = 1 + \sum_{\mathcal{O}_{\Delta,\ell}} \lambda_{\mathcal{O}}^2 \, G_{\Delta,\ell}(u,v)$$

$\uparrow$ spin

$$\underbrace{v^{\Delta} - u^{\Delta}} + \sum_{\mathcal{O}_{\Delta,\ell}} \lambda_{\mathcal{O}}^2 \left(\cancel{u}\, v^{\Delta} G_{\Delta,\ell}(u,v) - u^{\Delta} G_{\Delta,\ell}(v,u) \right) = 0$$

$$\boxed{F_{0,0}(u,v) + \sum_{\mathcal{O}_{\Delta,\ell}} \lambda_{\mathcal{O}}^2 \, F_{\Delta,\ell}(u,v) = 0}$$

Bounds on $\lambda_{\mathcal{O}_0}^2$ for some $\mathcal{O}_0$

$$\lambda_{\mathcal{O}_0}^2 \, F_{\Delta_0,\ell_0}(u,v) = -F_{0,0}(u,v) - \sum_{\mathcal{O} \neq \mathcal{O}_0} \lambda_{\mathcal{O}}^2 \, F_{\Delta,\ell}(u,v)$$

Linear functional on the space of functions:

$$\alpha : F(u,v) \to \mathbb{R}$$

Usually:

$$\alpha(F(u,v)) = \sum_{m,n \leq k}{}' \, a_{m,n} \, \partial_z^m \partial_{\bar z}^n F(z,\bar z) \Big|_{z=\bar z=\frac{1}{2}}$$

We need to choose $a_{m,n}$

Normalization

$$\alpha(F_{\Delta_0,l_0}) = 1 \quad ; \quad \sum_{m,n} a_{m,n}\, \partial_z^m \partial_{\bar z}^n F_{\Delta_0,l_0}\Big|_{z=\bar z=1/2} = 1$$

$$\lambda_{\mathcal{O}_0}^2 = -\alpha(F_{0,0}) - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda_{\mathcal{O}}^2\, \alpha(F_{\Delta,l})$$

Request $\alpha(F_{\Delta,l}) \geq 0 \quad \forall\, \Delta \geq \Delta_{min,l}$
$\forall l$.

$$\Rightarrow \lambda_{\mathcal{O}_0}^2 \leq -\alpha(F_{0,0})$$

minimize $(-\alpha(F_{0,0}))$ if min is negative $\Rightarrow$

$\Rightarrow$ given Δ's not allowed.

Minimize (w/ respect to $a_{m,n}$) $-\alpha(F_{0,0})$

Such that $\alpha(F_{\Delta_0,l_0}) = 1$ and $\alpha(F_{\Delta,l}) \geq 0 \quad \forall l,\ \Delta_l > \Delta_l^{min}$

SDP

$$F = \sum_i x_i F_i - F_0$$

Minimize $\sum c_i x_i$ with constraint $F \geq 0$

positive semi-definite.

---

relation?

•) $p(x) \geq 0 \quad \forall x \geq 0$

polynomial.

Hilbert $\quad p(x) = \sum_i (q_i(x))^2 + x \sum_j (r_j(x))^2$

$$q_i(x) = \sum_j C_{ij} x^j$$

$$(q_i(x))^2 = \sum_{j,j'} C_{ij} x^j C_{ij'} x^{j'} \qquad (r_i(x))^2 = \sum_{l,l'} \tilde{C}_{jl} \tilde{C}_{jl'} x^l x^{l'}$$

$$p(x) = \sum_{i,j,j'} C_{ij} C_{ij'} x^j x^{j'} + x \sum_{j,l,l'} \tilde{C}_{jl} \tilde{C}_{jl'} x^l x^{l'}$$

Define $X = (1, x, x^2, \ldots x^M)$

$$p(x) = X^t C^t C X + x\, X^t \tilde{C}^t \tilde{C} X$$

$$C^t C \geq 0 \qquad v^t C^t C v = \|Cv\| \geq 0 \quad \forall v.$$

$$p(x) = X^t A X + x\, X^t B X$$

$$A \geq 0 \qquad B \geq 0$$

Find approximation

indep. of m,n

$$F^{m,n}_{\Delta,\ell} = \chi_\ell(\Delta)\, \underbrace{P^{m,n}_\ell(\Delta)}_{\text{polynomials.}}$$

$$\alpha(F_{\Delta,\ell}) = \chi_\ell(\Delta) \sum_{m,n} a_{m,n} P^{m,n}_\ell\big((1+x)\Delta_{min}\big)$$

We need $\alpha(F_{\Delta,\ell}) \geq 0 \quad \forall x \geq 0.$

We compute $A(a_{m,n})$, $B(a_{m,n})$ linear dependence in $a_{m,n}$

minimize $(-\alpha(F_{0,0})) \;/\; A(a_{m,n}) \geq 0,\ B(a_{m,n}) \geq 0,$

$\alpha(F_{\Delta_0,\ell_0}) = 1$

Conf. group $SO(d+1,1)$

generators L_{AB}

$\underbrace{x_1 \, - - \, x_d}_{\mathbb{R}^d} \; x_{d+1} \; x_0$

$$[L_{AB}, L_{CD}] = -i\left(\eta_{AC} L_{BD} - \eta_{AD} L_{BC} - \eta_{BC} L_{AD} + \eta_{BD} L_{AC}\right)$$

$$L^2 = L_{AB} L_{CD} \eta^{AC} \eta^{BD}$$

$$x_\pm = x_0 \pm x_{d+1} \qquad ds^2 = dx_a^2 - dx_+ dx_-$$

$$L^2 = L_{aB} L_{aD} \eta^{BD} \doteq L_{+B} L_{-D} \eta^{BD}$$

$$= L_{ab} L_{ab} - 2 L_{a+} L_{a-} + \frac{1}{2} L_{+-} L_{-+}$$

classical

$$L^2 = L_{ab} L_{ab} - L_{a+} L_{a-} - L_{a-} L_{a+} + \frac{1}{2} L_{+-}^2$$

$$L_{+-} = (L_{0-} + L_{d+1\,-}) = \cancel{L_{00}} - L_{0\,d+1} + L_{d+1\,0} - \cancel{L_{d+1\,d+1}}$$

$$= -2 L_{0\,d+1}$$

②

$$\frac{1}{2} L^2 = \sum_{a<b} L_{ab} L_{ab} - \frac{1}{2} L_{a+} L_{a-} - \frac{1}{2} L_{a-} L_{a+} - \frac{1}{4} L_{+-}^2$$

$$[L_{+-}, L_{a+}] = -2 [L_{0\,d+1}, L_{a0} + L_{a\,d+1}] =$$

$$= 2i (L_{d+1\,a} + L_{0a}) = 2i L_{+a} = -2i L_{a+}$$

$$[L_{+-}, L_{a-}] = -2 [L_{0\,d+1}, L_{a0} - L_{a\,d+1}] =$$

$$= +2i (L_{d+1\,a} - L_{0a}) = -2i L_{-a} = 2i L_{a-}$$

$L_{+-} |\psi\rangle = \alpha |\psi\rangle$ consider highest weight.

$$\cancel{L_{a+} L_{+-} |\psi\rangle = L_{+-} L_{a+} |\psi\rangle + 2i L_{a+} |\psi\rangle}$$

$$\cancel{= (\alpha + 2i) L_{a+} |\psi\rangle}$$

$$L_{+-} L_{a+} |\psi\rangle = L_{a+} L_{+-} |\psi\rangle - 2i L_{a+} |\psi\rangle$$

$$= (\alpha - 2i) |\psi\rangle$$

$$L_{+-} L_{a-} |\psi\rangle = (\alpha + 2i) |\psi\rangle$$

$$L_{+-} |\psi\rangle = 2i \Delta |\psi\rangle$$

$$[L_{a+}, L_{a-}] = [L_{a0} + L_{ad+1}, L_{a0} - L_{ad+1}] =$$

$$= -i\left(-d\, L_{0d+1} + d\, L_{d+1\,0}\right)$$

$$= 2id\, L_{0d+1} = -id\, L_{+-}$$

$$\frac{1}{2}L^2 = \sum_{a<b} L_{ab}L_{ab} - L_{a-}L_{a+} + \frac{id}{2}L_{+-} - \frac{1}{4}L_{+-}^2$$

$$L_{a-}L_{a+} = L_{a+}L_{a-} + [L_{a+}, L_{a-}] = L_{a-}L_{a+} + id\, L_{+-}$$

$$\frac{1}{2}L^2 = J^2 - L_{a+}L_{a-} + \frac{id}{2}(2i\Delta) - \frac{1}{4}(-4\Delta^2)$$

$$= J^2 - \Delta d + \Delta^2 - L_{a-}L_{a+}$$

$L_{a-}|\psi\rangle = 0$ cannot decrease Δ.

$$\frac{1}{2}L^2 = J^2 + \Delta(\Delta - d)$$

$J^2 \to l(l+d-2)$

$\underline{(l, 0 \text{ —— } 0)}$

highest weight

$\boxed{3d}$

$$\boxed{\frac{1}{2}L^2 = l(l+1) + \Delta(\Delta-3)}$$

①

$$(X_{1A}\partial_{1B} - X_{1B}\partial_{1A}) + (X_{2A}\partial_{2B} - X_{2B}\partial_{2A}) = L^{12}_{AB}$$

$$U = \frac{(X_1X_2)(X_3X_4)}{(X_1X_3)(X_2X_4)} \qquad V = \frac{(X_1X_4)(X_2X_3)}{(X_1X_3)(X_2X_4)} \qquad ; X_1 \cdot X_2 = -\frac{1}{2}x_{12}^2$$

$(X_1X_2) \to$ invariant.

(X_1X_3) , (X_1X_4) , (X_2X_3) , (X_2X_4)

$$F(X_{13}, X_{14}, X_{23}, X_{24})$$

$$L^{12}_{AB}(X_1 \cdot X_3) = (X_{1A}X_{3B} - X_{1B}X_{3A}) = X_{13AB}$$

$$L^{12}_{AB} F = X_{13AB}\partial_1 F + X_{14AB}\partial_2 F + X_{23AB}\partial_3 F + X_{24AB}\partial_4 F$$

$$(L^{12}_{AB})^2 F = L^{12}_{AB}(X_{13AB})\partial_1 F + \partial^{12}_{AB}(X_{14AB})\partial_2 F + L^{12}_{AB}(X_{23AB})\partial_3 F + L^{12}_{AB}(X_{24AB})\partial_4 F$$

$$+ X_{13AB}X_{13AB}\partial_1^2 F + 2X_{13AB}X_{14AB}\partial_{12}F + 2X_{13AB}X_{23AB}\partial_{13}F + 2X_{13AB}X_{24AB}\partial_{14}F$$

$$+ 2X_{14AB}X_{23AB}\partial_{23}F + 2X_{14AB}X_{24AB}\partial_{24}F + X_{23AB}X_{24AB}\partial_{34}F$$

$$+ X_{14AB}X_{14AB}\partial_2^2 f + X_{23AB}X_{23AB}\partial_3^2 f + X_{24AB}X_{24AB}\partial_4^2 F$$

$$L^{12}_{AB} X_{13AB} = (X_{1A}\partial_{1B} - X_{1B}\partial_{1A})(X_{1A}X_{3B} - X_{1B}X_{3A}) = (X_1X_3) - (X_1X_3)(d+2) -$$

$$- (d+2)(X_1X_3) + (X_1X_3) = (X_1X_3)(2 - 2d - 4) = -2(d+1)(X_1X_3)$$

$$-2d-2$$

$$L^{12}_{AB} X_{14AB} = -2(d+1)X_{14} \quad ; \quad L^{12}_{AB} X_{23AB} = -2(d+1)X_{23} \quad ; \quad L^{12}_{AB} X_{24AB} = -2(d+1)X_{24}$$

②

$$X_{13AB}\, X_{13AB} = (X_{1A} X_{3B} - X_{1B} X_{3A})(X_{1A} X_{3B} - X_{1B} X_{3A})$$

$$= -2 (X_1 X_3)^2$$

$$X_{13AB}\, X_{14AB} = (X_{1A} X_{3B} - X_{1B} X_{3A})(X_{1A} X_{4B} - X_{1B} X_{4A})$$

$$= -2 (X_1 X_4)(X_1 X_3)$$

$$X_{13AB}\, X_{24AB} = (X_{1A} X_{3B} - X_{1B} X_{3A})(X_{2A} X_{4B} - X_{2B} X_{4A})$$

$$= 2 (X_1 X_2)(X_3 X_4) - 2 (X_1 X_4)(X_3 X_2)$$

$$(L^{12}_{AB})^2 F = -2(d+1)\left(x_{13}\, \partial_1 F + x_{14}\, \partial_2 F + x_{23}\, \partial_3 F + x_{24}\, \partial_4 F \right) -$$

$$+ 2 \left(x_{13}^2\, \partial_1^2 F + x_{14}^2\, \partial_2^2 F + x_{23}^2\, \partial_3^2 F + x_{24}^2\, \partial_4^2 F \right)$$

$$-4\, x_{14}\, x_{13}\, \partial_{12} F - 4\, x_{13}\, x_{23}\, \partial_{13} F + 4 (x_{12}\, x_{34} - x_{14}\, x_{23}) \partial_{14} F$$

$$+4\, (x_{12}\, x_{34} - x_{13}\, x_{24})\, \partial_{23} F - 4\, x_{14}\, x_{24}\, \partial_{24} F - 4\, x_{24}\, x_{23}\, \partial_{34} F$$

$$\partial_1 F(u,v) = -\frac{1}{x_{13}} (u\, \partial_u + v\, \partial_v) F = -\frac{1}{x_{13}} (F_u + F_v)$$

$$\partial_2 F(u,v) = \frac{1}{x_{14}} v\, \partial_v F = \frac{1}{x_{14}} F_v$$

$$\partial_3 F(u,v) = \frac{1}{x_{23}} v\, \partial_v F = \frac{1}{x_{23}} F_v$$

$$\partial_4 F = -\frac{1}{x_{24}} (F_u + F_v)$$

$$\partial_1^2 F = \frac{1}{x_{13}^2}\left(F_u + F_v\right) + \frac{1}{x_{13}^2}\left(F_{uu} + 2F_{uv} + F_{vv}\right)$$

$$\partial_2^2 F = -\frac{1}{x_{14}^2} F_v + \frac{1}{x_{14}^2} F_{vv}$$

$$\partial_3^2 F = -\frac{1}{x_{23}^2} F_v + \frac{1}{x_{23}^2} F_{vv}$$

$$\partial_4^2 F = \frac{1}{x_{24}^2}\left(F_u + F_v\right) + \frac{1}{x_{24}^2}\left(F_{uu} + 2F_{uv} + F_{vv}\right)$$

$$\partial_{12} F = -\frac{1}{x_{14}x_{13}}\left(F_{uv} + F_{vv}\right)$$

$$\partial_{13} F = -\frac{1}{x_{13}x_{23}}\left(F_{uv} + F_{vv}\right)$$

$$\partial_{14} F = \frac{1}{x_{13}x_{24}}\left(F_{uu} + 2F_{uv} + F_{vv}\right)$$

$$\partial_{23} F = \frac{1}{x_{14}x_{23}} F_{vv}$$

$$\partial_{24} F = -\frac{1}{x_{14}x_{24}}\left(F_{uv} + F_{vv}\right)$$

$$\partial_{34} F = -\frac{1}{x_{23}x_{24}}\left(F_{uv} + F_{vv}\right)$$

(4)

$$\left(L^{12}_{AB}\right)^2 F = -2(d+1)\left(-F_u - \cancel{F_v} + \cancel{F_v} + \cancel{F_v} - F_u - \cancel{F_v}\right)$$

$$-2\Big(F_u + \cancel{F_v} + F_{uu} + 2F_{uv} + F_{vv} - \cancel{F_v} + F_{vv} - \cancel{F_v} + F_{vv} +$$

$$+F_u + \cancel{F_v} + F_{uu} + 2F_{uv} + F_{vv}\Big)$$

$$+4\left(F_{uv} + F_{vv}\right) + 4\left(F_{uv} + F_{vv}\right) + 4\,\frac{(x_{12}x_{34} - x_{14}x_{23})}{x_{13}x_{24}}\begin{pmatrix} F_{uu}+ \\ +2F_{uv}+F_{vv}\end{pmatrix}$$

$$+4\,\frac{(x_{12}x_{34} - x_{13}x_{24})}{x_{14}x_{23}}F_{vv} + 4\left(F_{uv} + F_{vv}\right) +$$

$$+4\left(F_{uv} + F_{vv}\right)$$

$$= 4(d\cancel{+1})F_u \cancel{+4F_u} - 2\left(2F_{uu} + 4F_{uv} + 4F_{vv}\right) + \overset{16}{\cancel{8}}\,F_{uv} +$$

$$+\overset{16}{\cancel{8}}\,F_{vv} + 4(u-v)\left(F_{uu} + 2F_{uv} + F_{vv}\right) +$$

$$+4\left(\frac{u}{v} - \frac{1}{v}\right)F_{vv}$$

$$= 4dF_u - 4F_{uu} - 8F_{uv} - 8F_{vv} + 16F_{uv} + 16F_{vv} + 4(u-v)\left(F_{uu} + 2F_{uv} + F_{vv}\right) +$$

$$+4\,\frac{u-1}{v}F_{vv}$$

$$= 4dF_U + 8F_{VV} + 8F_{UV} - 4F_{UU} + 4(U-V)F_{UU} +$$

$$+ 8(U-V)F_{UV} + 4\left((U-V) + \frac{U-1}{V}\right)F_{VV}$$

$$= 4dF_U + 4(U-V-1)F_{UU} + 8(U-V+1)F_{UV} +$$

$$+ 4\left(UV - V^2 + U - 1 + 2V\right)\frac{1}{V}F_{VV}$$

$$= 4\left(dF_U + (U-V-1)F_{UU} + 2(U-V+1)F_{UV} + \left(U(V+1) - (V-1)^2\right)\frac{F_{VV}}{V}\right)$$

$$= 4\Big(-\left[(1-V)^2 - U(1+V)\right]\partial_V(V\partial_V F) - (1-U+V)\,u\partial_u(u\partial_u F) +$$

$$+ 2(1+U-V)\,UV\,\partial_{uv}F + d\,u\,\partial_u F\Big)$$

$L \to iL$

$$\frac{1}{2} L^2 = 2 \Big[\left[(1-v)^2 - u(1+v)\right] \partial_v (v \partial_v F) + (1-u+v)\, u\, \partial_u (u \partial_u F)$$

$$+ 2 (v-u-1)\, uv\, \partial_{uv} F - ud\, \partial_u F \Big]$$

$u = z\bar{z} \qquad v = (1-z)(1-\bar{z})$

$\partial_z = \bar{z} \partial_u - (1-\bar{z}) \partial_v$

$\partial_{\bar{z}} = z \partial_u - (1-z) \partial_v$

$v = (1 - u/\bar{z})(1-\bar{z})$

$v\bar{z} = (\bar{z}-u)(1-\bar{z}) = -\bar{z}^2 - u + u\bar{z} + \bar{z}$

$\bar{z}^2 + (v-1-u)\bar{z} + u = 0$

$\bar{z} = (1+u-v) \pm \sqrt{}$

$z\partial_z - \bar{z}\partial_{\bar{z}} = \left[- z(1-\bar{z}) + \bar{z}(1-z)\right] \partial_v = -(z-\bar{z})\partial_v$

$-z + \cancel{z\bar{z}} + \bar{z} - \cancel{z\bar{z}}$

$(1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}} = \left[\bar{z}(1-z) - z(1-\bar{z})\right]\partial_u = -(z-\bar{z})\partial_u$

$\bar{z} - z$

$\partial_v = -\frac{1}{z-\bar{z}} (z\partial - \bar{z}\bar{\partial}) \qquad \partial_u = -\frac{1}{z-\bar{z}} \left((1-z)\partial - (1-\bar{z})\bar{\partial}\right)$

$$v\partial_v = -\frac{(1-z)(1-\bar{z})}{(z-\bar{z})} (z\partial - \bar{z}\bar{\partial})$$

$$u\partial_u = -\frac{z\bar{z}}{z-\bar{z}} (\partial - \bar{\partial} - z\partial + \bar{z}\bar{\partial}) =$$

maple.

$$\frac{1}{2} L^2 = 2 \left[\frac{1}{z-\bar{z}}\right] \Big(-\bar{z}^2 (\bar{z}-1)(z-\bar{z}) \bar{\partial}^2 F - z^2 (z-1)(z-\bar{z}) \partial^2 F + \bar{z}\left(\bar{z}^2 + z\bar{z}(d-3) - z(d-2)\right)\bar{\partial} F$$
$$- z\left((z(d-3) - d + 2)\bar{z} + z^2\right) \partial F \Big)$$

$$\frac{1}{4}L^2 = z^2(1-z)\partial^2 F + \bar z^2(1-\bar z)\bar\partial^2 F - \frac{z}{z-\bar z}\Big(\overbrace{z^2 - z\bar z}^{z(z-\bar z)} + (d-2)\underbrace{(z\bar z - \bar z)}_{\bar z(z-1)}\Big)\partial F$$

$$+ \frac{\bar z}{z-\bar z}\left(\bar z^2 - z\bar z + (d-2)(z\bar z - z)\right)\bar\partial F$$

$$= z^2(1-z)\partial^2 F + \bar z^2(1-\bar z)\bar\partial^2 F - z^2\partial F - \bar z^2\bar\partial F +$$

$$+ (d-2)\frac{z\bar z}{z-\bar z}\Big((1-z)\partial F - (1-\bar z)\bar\partial F\Big)$$

$$z^2(1-z)\partial^2 g_{\Delta,\ell} + \bar z^2(1-\bar z)\bar\partial^2 g_{\Delta,\ell} - z^2\partial g_{\Delta,\ell} - \bar z^2\bar\partial g_{\Delta,\ell} +$$

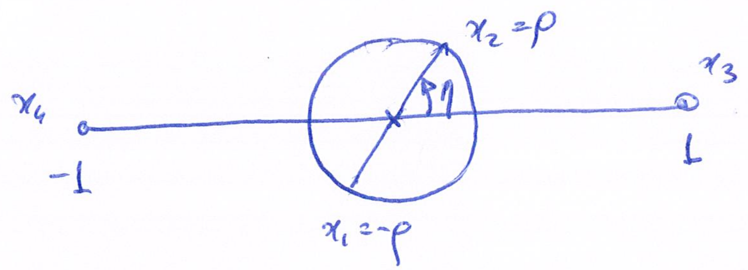
$$+ (d-2)\frac{z\bar z}{z-\bar z}\Big((1-z)\partial g_{\Delta,\ell} - (1-\bar z)\bar\partial g_{\Delta,\ell}\Big) = \frac{1}{2}\left[\ell(\ell+1) + \Delta(\Delta-3)\right] g_{\Delta,\ell}$$

$\ell \leftrightarrow -\ell-1$ symmetry.

$0 \leftrightarrow -1$

$$g_{\Delta,0} = g_{\Delta,-1}$$

↑ formally.



x_3, x_4.
inv. under rot.
$l_z = 0$.

$\rho = r e^{i\eta}$

$$g_{\Delta,\ell}(u,v) \rightleftharpoons g_{\Delta,\ell}(z,\bar z) \rightleftharpoons g_{\Delta,\ell}(\rho,\eta)$$

$$\rho = \frac{z}{(1+\sqrt{1-z})^2}$$

legendre polynomial.

$$g_{\Delta,\ell}(r,\eta) = \sum_{n=0}^{\infty} \sum_{j} B_{n,j}\, r^{\Delta+n}\, P_j(\cos\eta)$$

j: $l-n$ .. $l+n$
steps 2.

$$z = \frac{4\rho}{(1+\rho)^2} \quad ; \quad (1-z) = \frac{(1-\rho)^2}{(1+\rho)^2}$$

$$\partial_z = \frac{\partial\rho}{\partial z}\partial_\rho = \frac{1}{\partial z/\partial\rho}\partial_\rho = \frac{(1+\rho)^3}{4(1-\rho)}\partial_\rho$$

$$\frac{\partial z}{\partial\rho} = \frac{4(1+\rho)^{2} - 2\cancel{(1+\rho)}4\rho}{(1+\rho)^{\cancel{4}\,3}} = \frac{4(1-\rho)}{(1+\rho)^3}$$

$$z^2(1-z)\,\partial_z^2 g$$

$$\partial_z g = \frac{(1+\rho)^3}{4(1-\rho)}\,\partial_\rho g$$

$$\partial_z^2 g = \frac{(1+\rho)^3}{4(1-\rho)}\,\partial_\rho\left[\frac{(1+\rho)^3}{4(1-\rho)}\,\partial_\rho g\right]$$

$$= \frac{(1+\rho)^3}{4(1-\rho)}\left(\frac{3(1+\rho)^2\,4(1-\rho)+4(1+\rho)^3}{16\;(1-\rho)^2}\,\partial_\rho g + \frac{(1+\rho)^3}{4(1-\rho)}\,\partial_\rho^2 g\right)$$

$$= \frac{(1+\rho)^6}{16(1-\rho)^2}\,\partial_\rho^2 g + \frac{(1+\rho)^3}{16(1-\rho)^3}\,(1+\rho)^2\,\underbrace{(3-3\rho+1+\rho)}_{4-2\rho=2(2-\rho)}\,\partial_\rho g$$

$$= \frac{(1+\rho)^6}{16(1-\rho)^2}\,\partial_\rho^2 g + \frac{1}{8}\,\frac{(1+\rho)^5}{(1-\rho)^3}\,(2-\rho)\,\partial_\rho g$$

$$z^2(1-z) = \frac{16\rho^2}{(1+\rho)^4}\,\frac{(1-\rho)^2}{(1+\rho)^2} = \frac{16\rho^2(1-\rho)^2}{(1+\rho)^6}$$

$$z^2(1-z)\,\partial_z^2 g = \rho^2\,\partial_\rho^2 g + \frac{2\rho^2(2-\rho)}{(1-\rho)(1+\rho)}\,\partial_\rho g$$

$$z^2\,\partial g = \frac{16\rho^2}{(1+\rho)^{\cancel{4}}}\,\frac{\cancel{(1+\rho)^3}}{4(1-\rho)}\,\partial_\rho g = \frac{4\rho^2}{(1-\rho^2)}\,\partial_\rho g$$

$$z^2(1-z)\,\partial_z^2 g - z^2\,\partial g = \rho^2\,\partial_\rho^2 g + \frac{2\rho^2}{(1-\rho^2)}\,\partial g\,(\cancel{2}-\rho\cancel{-2}) = \rho^2\,\partial_\rho^2 g - \frac{2\rho^3}{1-\rho^2}\,\partial_\rho g$$

$$\frac{z\bar z}{z-\bar z}(1-z)\,\partial g = \frac{16\rho\bar\rho}{(1+\rho)^2(1+\bar\rho)^2}\;\frac{(1-\rho)^{\cancel{2}}}{\cancel{(1+\rho)^2}}\;\frac{1}{\frac{4\rho}{(1+\rho)^2}-\frac{4\bar\rho}{(1+\bar\rho)^2}}\;\frac{(1+\rho)^{\cancel{2}}}{4\cancel{(1-\rho)}}\,\partial_\rho g$$

$$= \frac{\cancel{4}\rho\bar\rho}{\cancel{(1+\rho)^2(1+\bar\rho)^2}}\;\frac{(1-\rho^2)\,\cancel{(1+\rho)^2(1+\bar\rho)^2}}{\left[\rho(1+\bar\rho)^2-\bar\rho(1+\rho)^2\right]}\,\partial_\rho g$$

$$= \frac{\rho\bar\rho\,(1-\rho^2)\,\partial_\rho g}{(\rho+\cancel{2\rho\bar\rho}+\rho\bar\rho^2-\bar\rho-\cancel{2\rho\bar\rho}-\bar\rho\rho^2)} = \frac{\rho\bar\rho\,(1-\rho^2)}{(\rho-\bar\rho)+(\bar\rho-\rho)\rho\bar\rho}\,\partial_\rho g$$

$$= \frac{\rho\bar\rho\,(1-\rho^2)\,\partial_\rho g}{(1-\rho\bar\rho)(\rho-\bar\rho)}$$

<u>We get:</u>

$$\rho^2\partial_\rho^2 g - \frac{2\rho^3}{1-\rho^2}\partial_\rho g + \bar\rho^2\partial_{\bar\rho}^2 g - \frac{2\bar\rho^3}{1-\bar\rho^2}\partial_{\bar\rho} g +$$

$$+(d-2)\frac{\rho\bar\rho}{(1-\rho\bar\rho)(\rho-\bar\rho)}\left((1-\rho^2)\partial_\rho g-(1-\bar\rho^2)\partial_{\bar\rho} g\right) = \frac{1}{2}\left[\ell(\ell+1)+\Delta(\Delta-3)\right]g$$

Using.

$$g_{\Delta,\ell} = \sum_{n=0}^{\infty} \sum_{\substack{j=\ell-n \\ \text{steps of } 2}}^{\ell+n} B_{n,j}\, r^{\Delta+n} P_j(c\eta)$$

$\rho = re^{i\eta}$

Assume $B_{0,\ell} = 1$

Then

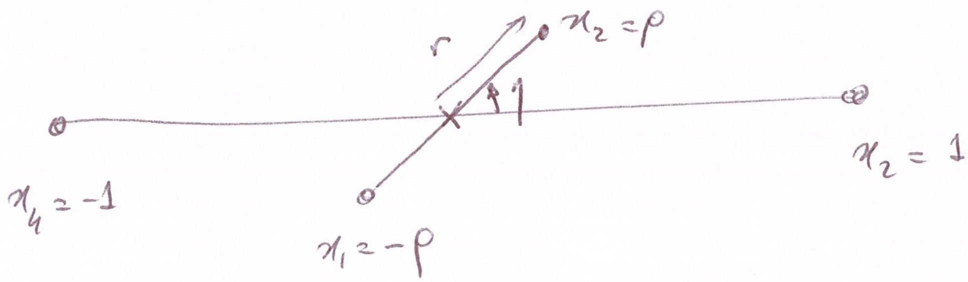
$$B_{2,\ell-2} = \frac{2\ell(\ell-1)(\Delta-\ell-1)}{(2\ell-1)(2\ell+1)(\Delta-\ell)}$$

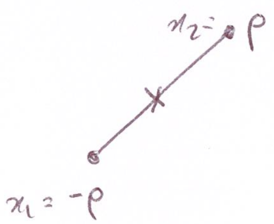
$$B_{2,\ell} = \frac{(-\Delta + 4(\Delta-1)\ell(\ell+1))}{(2\Delta-1)(2\ell+3)(2\ell-1)}$$

$$B_{2,\ell+2} = \frac{2(\Delta+\ell)(\ell+1)(\ell+2)}{(\Delta+\ell+1)(2\ell+1)(2\ell+3)}$$

$$v^{\Delta\phi} f(u,v) - u^{\Delta\phi} f(v,u) = 0$$

$$f(u,v) = 1 + \sum_{\mathcal{O}} C^2_{\phi\phi\mathcal{O}}\, g_{\Delta,\ell}(u,v)$$





$$\phi(x_1)\,\phi(x_2) = \frac{1}{x_{12}^{2\Delta}}\left(1 + \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}\, C(x_{12}, \partial_y)\, \mathcal{O}_{\Delta,\ell}(y)\Big|_{y=0}\right)$$

$$\underbrace{\partial^{\mu_1}\cdots\partial^{\mu_n}\mathcal{O}_{\Delta,\ell}}_{\Delta_n = \Delta + n} \quad ; \qquad \partial^{\mu} \to j = 1, \quad \to \Delta = 1$$

$$j = \ell - n \;\dots\; \ell + n$$

parity even $\rho \leftrightarrow -\rho$ $\to j$ even.

$$\partial^{\mu_1} \cdots \partial^{\mu_n} \mathcal{O}_{\Delta,\ell} \to \sum_{j=\ell-n}^{\ell+n} \tilde{\mathcal{O}}_{j,n,j_z}$$

$$-j \cdots j$$

$\phi(x_1)\,\phi(x_2)$ scalar

$$R\,\phi(x_1)\,\phi(x_2)\,R^{-1} = \phi(Rx_1)\,\phi(Rx_2)$$

$$R\,\phi(R^{-1}x_1)\,\phi(R^{-1}x_2)\,R^{-1} = \phi(x_1)\,\phi(x_2)$$

↑ rotating operator and coordinate leaves $\phi(x_1)\cdot\phi(x_2)$ invariant.

$$R\,\tilde{\mathcal{O}}_{j j_z n}\,R^{-1} \quad \text{rotates as} \quad |j\, j_z\rangle$$

We need dependence on x_{12} also to rotate as

$$\sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}} \sum_n Y^*_{j j_z}(\theta,\varphi)\,\tilde{\mathcal{O}}_{j j_z, n}(0)\; r^{\Delta+n}$$

$$\langle \tilde{\mathcal{O}}_{j j_z}(0)\,\phi(x_3)\,\phi(x_4)\rangle \quad \text{inv. rot. z-axis.}$$

$$\underline{j_z = 0}$$

$$Y_{j\, j_z=0} = P_j(\cos\theta)$$

$$\phi(x_1)\,\phi(x_2) \to \frac{1}{x_{12}^{2\Delta}}\Big(1 + \sum_{\mathcal{O}} C_{\phi\phi\mathcal{O}}\, r^{\Delta+n}\, Y^*_{j j_z}\, \tilde{\mathcal{O}}_{j j_z, n}\Big)$$

$$g_{\Delta,\ell} = \sum_{n=0}^{\infty} \sum_{j=\ell-n}^{\ell+n} r^{\Delta+n} P_j(c\eta) \cdot \underbrace{B_{n,j}}_{\text{coefficients}}$$

$$\underset{\uparrow \text{even}}{r^{\Delta} P_\ell(c\eta)} + \cdots \qquad (\text{Take } B_{0,\ell} = 1)$$

j even $\Rightarrow$ n even ; $\left.\begin{array}{l}\ell \text{ odd} \\ n \text{ odd}\end{array}\right\}$ allowed?

Substitute in equation

$B_{n,j}(\Delta)$ has poles in Δ.

at $\Delta = 1-\ell-2k$ $\quad k=1,2,\ldots$

$1+\ell/2-k$ $\quad k=1,2,\ldots$

$2+\ell-2k$ $\quad k=1,2\ldots[\ell/2]$

Factor poles and r^Δ

$$\frac{r^\Delta}{\prod_i (\Delta-\Delta_i)} \underbrace{\tilde{F}(u,v)}_{\substack{\text{polynomial} \\ \text{in } \Delta \\ \text{if truncated at finite order.}}} = v^{\Delta\phi} g_{\Delta,\ell}(u,v) - u^{\Delta\phi} g_{\Delta,\ell}(v,u) = F(u,v)$$

$$F_{0,0}(u,v) + \sum_{\mathcal{O}_{\Delta \ell}} C^2_{\phi\phi\mathcal{O}} F_{\Delta,\ell}(u,v) = 0$$

$$C^2_{\phi\phi\mathcal{O}_0} F_{\Delta_0,\ell_0}(u,v) = -F_{0,0}(u,v) - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}} F_{\Delta,\ell}(u,v)$$

$$\alpha\big(F(u,v)\big) = \sum_{m,n\leq K} a_{m,n}\, \partial_z^m \partial_{\bar z}^n F(z,\bar z)\Big|_{z=\bar z=1/2}$$

$$\alpha(F_{\Delta_0,\ell_0}) = 1 \quad \to \quad \sum_{m,n\leq K} a_{m,n}\, \partial_z^m \partial_{\bar z}^n F_{\Delta_0,\ell_0}\Big|_{z=\bar z=1/2} = 1.$$

$$C^2_{\phi\phi\mathcal{O}} = -\alpha(F_{0,0}) - \sum_{\mathcal{O}\neq\mathcal{O}_0} \lambda^2_{\mathcal{O}}\, \alpha(F_{\Delta,\ell})$$

$$\alpha(F_{\Delta,\ell}) \geq 0 \quad \forall\ \Delta \geq \Delta_{\min,\ell}.$$

$$\boxed{\Delta \geq \ell + 1}$$

$$C^2_{\phi\phi\mathcal{O}} \leq -\alpha(F_{0,0})$$

if $-\alpha(F_{0,0}) \leq 0 \Rightarrow$ not allowed.

minimize $\left(-\alpha\left(F_{00}\right)\right) \Big/ \quad \alpha\left(F_{\Delta,\ell}\right) \geq 0$

$$\forall \Delta \geq \ell+1$$

$$F_{\Delta,\ell} = \frac{r^{\Delta}}{\prod(\Delta-\Delta_i)} \tilde{F}_{\Delta,\ell}$$

$$\alpha\left(F_{\Delta,\ell}\right) = \alpha\left(r^{\Delta} \tilde{F}_{\Delta,\ell}\right) = r^{\Delta} \sum_{m,n} a_{m,n} \tilde{F}^{m,n}(\Delta)$$

$$\sum_{m,n} \tilde{F}^{m,n}(\Delta)\, a_{m,n} \geq 0 \qquad \forall \Delta \geq \Delta_{\ell} = \ell+1$$

$$p(x) \geq 0 \qquad \forall x \geq a$$

$$p(x) = \sum_i P_i^2(x) + \sum_j (x-a) P_j^2(x)$$

$$P_i(x) = \sum \alpha_{ni} x^n$$

$$\sum_i P_i^2(x) = \sum_{inm} x^n \alpha_{ni} \alpha_{mi} x^m = X^t A X$$

$$X = \begin{pmatrix} 1 \\ x \\ \vdots \\ x^s \end{pmatrix} \qquad A_{nm} = \alpha_{ni}\alpha_{mi} \leftarrow \text{positive definite.}$$

$$= \alpha \alpha^t$$

$$\sum x^n A_{nm} x^m = \sum x^n L_{np} L^t_{pm} x^m = \sum_p \sum x^n L_{np} L_{mp} x^m$$

$$= \sum (P_p(x))^2$$

$A = LL^t$ Cholesky decomposition

$$p(x) = X^t A X + (x-a) X^t B X$$

$$A \geq 0 \qquad B \geq 0$$

$$p(x) = \sum_{m,n} a_{m,n} \tilde{F}^{m,n}(\Delta) =$$

$$= \sum_{\substack{j,l \\ j+l-2=n}} x^{j-1} A_{jl} x^{l-1} + (x-a) \sum_{\substack{jl \\ x^{j+l-1=n}}} x^{j-1} B_{jl} x^{l-1}$$

$$= \sum_{n=1} x^n \sum_{j=1}^{n+1} \left(A_{j\, n+2-j} - a B_{j\, n+2-j}\right) + \sum_{n} x^n \sum_{j=1}^{n} B_{j\, n+1-j}$$

$$\sum_{j=1}^{s+1} \left(A^{(l)}_{j\, s+2-j} - a B^{(l)}_{j\, s+2-j}\right) + \sum_{j=1}^{n} B^{(l)}_{j\, s+1-j} =$$

$$= \sum_{m,n} a_{m,n} \tilde{F}^{(m,n)}_l \Big|_{\text{coff } \Delta^s}$$