660, Fall 2017, Homework I, (5 problems)

Based on problems 1.1, 1.7, 1.10/1.11, 1.14,1.23 of Sakurai's book

Problem 1

Prove that

$$[AB, CD] = -AC\{D, B\} + A\{C, B\}D - C\{D, A\}B + \{C, A\}DB \tag{0.1}$$

Problem 2

Consider a Hermitian operator A, (*i.e.* $A = A^\dagger$). Let $\{|a_i\rangle, i = 1 \ldots N\}$ be a basis of eigenstates $|a_i\rangle$ of A, with eigenvalues a_i. Assume for simplicity that there is no degeneracy, namely all the a_i are different.

a) Prove that

$$\prod_{i=1}^{N} (A - a_i) = 0 \tag{0.2}$$

b) For a given value of i consider the operator

$$P_i = \prod_{\substack{j=1 \\ j \neq i}}^{N} \left(\frac{A - a_j}{a_i - a_j} \right) \tag{0.3}$$

What does P_i do when applied to an arbitrary state?

c) Illustrate points **a)** and **b)** by using the operator S_z of a spin 1/2 system.

d) Discuss how to modify the formulas if there is a degenracy in the spectrum of A.

Problem 3

Consider the following Hamiltonian of a two-state system

$$H = E(|1\rangle\langle 1| - |2\rangle\langle 2|) + \Delta(|1\rangle\langle 2| + |2\rangle\langle 1|) \tag{0.4}$$

where E, Δ have dimension of energy. Find the energy eigenvalues and the corresponding eigenstates as linear combinations of $|1\rangle, |2\rangle$.

Problem 4

Consider the following Hamiltonian of a three-state system

$$H = \frac{\epsilon}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{0.5}$$

where ϵ has dimension of energy. Find the energy eigenvalues and the corresponding eigenstates.

Problem 5

Consider the following observables in a three-state system:

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}, \quad B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}, \tag{0.6}$$

where a, b are real numbers.

a) The spectrum of A is degenerate. How about the spectrum of B?.

b) Show that A, and B commute.

c) Find a new orthonormal basis where both A and B are diagonal. Do A and B form a complete set of observables for this system?