660, Fall 2017, Homework II, (5 problems)

Based on problems 1.12, 1.13, 1.18c, 1.30,1.32, 1.33 of Sakurai's book

Problem 1

A spin $\frac{1}{2}$ system is in the state $|\uparrow\rangle_{\hat{n}}$, namely in a spin up eigenstate in an arbitrary direction defined by the unit vector $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

a) If the component S_x is measured, find the possible results of the measurement and their probabilities.

b) Evaluate the dispersion σ_x given by

$$\sigma_x^2 = {}_{\hat{n}}\langle\uparrow|(S_x - \bar{S}_x)^2|\uparrow\rangle_{\hat{n}} \tag{0.1}$$

where $\bar{S}_x = {}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}$.

c) Check your answers for the cases $\theta = 0, \pi$ and $\theta = \pi/2, \phi = 0$.

Problem 2

Continuing from problem 1, a series of Stern-Gerlach experiments are done to measure different components of the spin in succession. The beams are directed along direction $\hat{x}$ and the experiments are done as follows:

- The first device accepts only $s_z = \hbar/2$ states, (*i.e.* those with $s_z = -\hbar/2$ are blocked) thus creating a polarized beam for the next devices.
- The second device accepts only states with $s_{\hat{n}} = \hbar/2$, where $\hat{n}$ is a unit vector perpendicular to $\hat{x}$.
- The third device accepts only $s_z = -\hbar/2$.

a) What is the ratio of the intensities of the final $s_z = -\hbar/2$ beam and the initial $s_z = \hbar/2$ polarized beam?.

b) How should the orientation of the second device, namely the vector $\hat{n}$, should be chosen to maximize the ratio computed in **a)**.

Problem 3

Consider a particle in a state with a Gaussian wave-function

$$\langle x|\psi\rangle = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ikx - \frac{1}{4\sigma^2}(x-x_0)^2} \tag{0.2}$$

a) Compute $\langle\psi|\hat{x}|\psi\rangle$, $\langle\psi|\hat{p}|\psi\rangle$, $\langle\psi|(\Delta x)^2|\psi\rangle$, $\langle\psi|(\Delta p)^2|\psi\rangle$

b) Check that such state has minimal uncertainty, namely

$$\sqrt{\langle\psi|(\Delta x)^2|\psi\rangle}\ \sqrt{\langle\psi|(\Delta p)^2|\psi\rangle} = \frac{\hbar}{2} \tag{0.3}$$

c) Show that for this state

$$\langle x|\Delta x|\psi\rangle = i\lambda\langle x|\Delta p|\psi\rangle \tag{0.4}$$

where $\lambda \in \mathbb{R}$. How does this relate to the minimal uncertainty property? **Hint:** Recall the proof of the uncertainty principle based on defining an operator $\mathcal{O} = \Delta x + i\mu\Delta p$ and computing $\langle\psi|\mathcal{O}^\dagger\mathcal{O}|\psi\rangle \geq 0$.

Problem 4

Continuing from problem 3.

a) Compute the momentum wave function $\tilde{\psi}(p) = \langle p|\psi\rangle$ for state $|\psi\rangle$.

b) Using $\tilde{\psi}(p) = \langle p|\psi\rangle$, compute $\langle\psi|\hat{p}|\psi\rangle$, $\langle\psi|(\Delta p)^2|\psi\rangle$ and check that you obtain the same results as in problem 3.

Problem 5

Given the translation operator

$$U(a) = e^{-i\frac{\hat{p}a}{\hbar}} \tag{0.5}$$

a) Use the fundamental commutation relation

$$[\hat{p}, \hat{x}] = -i\hbar \tag{0.6}$$

to compute the commutator

$$[\hat{x}, U(a)] =? \tag{0.7}$$

b) Given a state $|\psi\rangle$ such that $\langle\psi|\hat{x}|\psi\rangle = \bar{x}$, what is the mean value of $\hat{x}$ in the state $|\phi\rangle = U(a)|\psi\rangle$?