660, Fall 2017, Homework III, (3 problems)

Problem 1

Consider an inverted square-well potential

$$V(x) = \begin{cases} V_0 & \text{if} \quad |x| < \frac{a}{2} \\ 0 & \text{if} \quad |x| > \frac{a}{2} \end{cases} \tag{0.1}$$

with $V_0 > 0$. Compute the scattering matrix S and the reflection and transmission coefficients. Consider both cases, $0 < E < V_0$ and $V_0 < E$. Check the unitarity and symmetries of S. Plot the transmission coefficient as a function of energy. Choose the parameters such that there are well-defined resonance peaks.

Note: Problems 2 and 3 are not going to be tested in the final exam.

Problem 2

Consider the spin $\frac{1}{2}$ Heisenberg chain and the operator $\hat{n} = \frac{1}{2}(1+\sigma_z)$ that has expectation value 0 for a spin down and 1 for a spin up. It can be thought as the particle number in the interpretation where the vacuum are all spins down and a few spins are up.

- Compute the mean value of $\langle k|n^{(i)}|k\rangle$ for a given site i in the one particle states assuming the spin chain has finite length $N \gg 1$ for normalization purposes (but ignoring boundary effects).

- Compute the correlation

$$C_{ij} = \langle\psi|n^{(i)}n^{(j)}|\psi\rangle, \qquad i < j \tag{0.2}$$

  as a function of $|i-j|$ for the two particle states, *i.e.* $|\psi\rangle = |k_1, k_2\rangle$ in the infinite chain. Pay particular attention to the bound states and argue that the two particles are bounded.

Problem 3

Consider a particle in the one dimensional potential $V(x) = \lambda x^4$ such that the Hamiltonian is

$$H = \frac{p^2}{2m} + \lambda x^4 \tag{0.3}$$

- Write the corresponding Schrödinger equation for the (possible) wave function of energy E. Rescale the variable x by a constant a, namely define $z = ax$ and find a to eliminate λ from the equation (after appropriately rescaling E).

- Solve the resulting equation numerically for different values of E. [Use the boundary condition $\psi(0) = 1$, $\psi'(0) = 0$]. By considering the behavior of ψ at infinity determine the lowest eigenvalue of the energy. After that, compute two other eigenvalues.

- Consider the wave function

  $$\psi = Ae^{-\alpha x^2} \tag{0.4}$$

  Choose A such the $\psi(x)$ is normalized and then compute $E(\alpha) = \langle\psi|H|\psi\rangle$. Minimize $E(\alpha)$ with respect to α and compare the minimum value of $E(\alpha)$ with the result of the previous point to see how good the approximation is.

Note: This problem requires the use of a compute algebra program such as Mathematica, Maple, Matlab etc. (or the coding the integration in some computer language such as C, fortran etc.). It is more difficult and requires some extra research but playing with the numerics is always useful to understand what one is doing analytically.