Phys 660 HW 1 Student Solutions

Problem 1

Farhan

LHS =

~~LHS~~ $[AB, CD] = ABCD - CDAB$

$RHS = -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB$

$= -\cancel{ACDB} \mp \cancel{ACBD} + \cancel{ACBD} + ABCD - CDAB \mp \cancel{CADB}$
$\quad + \cancel{CADB} + \cancel{ACDB}$

$= ABCD - CDAB$

$= R.H.S.$ [Proved]

Problem 2

Ralph

Proof. (a) Since the product is over a complete set, the operator $\prod_{i=1}^{N}(A - a_i) = 0$ will always encounter an element $|a_j\rangle$ such that $a_i = a_j$ in which case the result is zero. Thus, for any state $|\psi\rangle$, we have

$$\begin{aligned}
\prod_{i=1}^{N}(A - a_i)\,|\psi\rangle &= \prod_{i=1}^{N}(A - a_i)\sum_{j=1}^{N}|a_j\rangle\langle a_j|\psi\rangle \\
&= \sum_{j=1}^{N}\prod_{i=1}^{N}(a_j - a_i)\,|a_j\rangle\langle a_j|\psi\rangle \\
&= \sum_{j=1}^{N} 0 \\
&= 0
\end{aligned}$$

(b) If the product instead is over all $a_i \neq a_j$, then the only surviving term in the sum is

$$\prod_{i=1}^{N}(a_j - a_i)\,|a_j\rangle\,\langle a_j|\psi\rangle$$

and dividing by the factors $(a_j - a_i)$ just gives the projection of $|\psi\rangle$ on the direction $|a_i\rangle$. Therefore, it is like a projection operator which projects the $|a_i\rangle$ component of $|\psi\rangle$.

(c) For the operator $A = S_z$ and $|a_i\rangle = \{|+\rangle, |-\rangle\}$, we have

$$\prod_{i=1}^{N}(A - a_i) = (S_z - |+\rangle)(S_z - |-\rangle) = \left(S_z - \frac{\hbar}{2}\right)\left(S_z + \frac{\hbar}{2}\right)$$

$$\prod_{j=1, j\neq i}^{N}\left(\frac{A - a_j}{a_i - a_j}\right) = \begin{cases} \left(\frac{S_z - |+\rangle}{|-\rangle - |+\rangle}\right) = \left(\frac{S_z - \hbar/2}{-\hbar}\right), & \text{for } a_j = |+\rangle \\ \left(\frac{S_z - |-\rangle}{|+\rangle - |-\rangle}\right) = \left(\frac{S_z + \hbar/2}{\hbar}\right), & \text{for } a_j = |-\rangle\,. \end{cases}$$

For the first equation, it is easy to see we get $S_z^2 - \frac{\hbar^2}{4} = 0$. For the second equation, we can work them out explicitly

$$\left(\frac{S_z - \hbar/2}{-\hbar}\right) = -\frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - \mathbb{I}\right] = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \text{ projection on } |-\rangle$$

$$\left(\frac{S_z + \hbar/2}{\hbar}\right) = \frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \mathbb{I}\right] = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \text{ projection on } |+\rangle$$

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Problem 3

Tiahra

$$H = E\left(\begin{pmatrix}1\\0\end{pmatrix}(1\ 0) - \begin{pmatrix}0\\1\end{pmatrix}(0\ 1)\right) + \Delta\left(\begin{pmatrix}1\\0\end{pmatrix}(0\ 1) + \begin{pmatrix}0\\1\end{pmatrix}(1\ 0)\right)$$

$$= E\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right) + \Delta\left(\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right)$$

$$= E\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \Delta\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$H = \begin{pmatrix}E & \Delta\\ \Delta & -E\end{pmatrix}$$

To find the eigenvalues

$$\det \begin{pmatrix} E-\lambda & \Delta \\ \Delta & -E-\lambda \end{pmatrix} = 0$$

$$(E-\lambda)(-E-\lambda) - (\Delta)^2 = 0$$
$$-E^2 - E\lambda + E\lambda + \lambda^2 - \Delta^2 = 0$$
$$\lambda^2 = \Delta^2 + E^2$$

$$\lambda = \pm\sqrt{\Delta^2 + E^2}$$ energy eigenvalues

To find the energy eigenstate

$$\begin{pmatrix} E-\lambda & \Delta \\ \Delta & -E-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$(E-\lambda)v_1 + \Delta v_2 = 0 \qquad \Delta v_1 - (E+\lambda)v_2 = 0$$

For the eigenvalue $\lambda_1 = -\sqrt{E^2+\Delta^2}$ the eigenvector can be expressed as

$$\left(v_2 \cdot \left(\frac{E}{\Delta} - v_2\right) \cdot \left(\frac{\sqrt{E^2+\Delta^2}}{\Delta}\right), v_2\right)$$

choosing $v_2 = 1$

our eigenvector looks like $$\begin{pmatrix} \frac{E}{\Delta} - \frac{\sqrt{E^2+\Delta^2}}{\Delta} \\ 1 \end{pmatrix}$$

For $\lambda_2 = \sqrt{E^2+\Delta^2}$ $$\begin{pmatrix} \frac{E}{\Delta} + \frac{\sqrt{E^2+\Delta^2}}{\Delta} \\ 1 \end{pmatrix}$$

Problem 4:

$$H = \frac{\epsilon}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon/\sqrt{2} & 0 \\ \epsilon/\sqrt{2} & 0 & \epsilon/\sqrt{2} \\ 0 & \epsilon/\sqrt{2} & 0 \end{pmatrix}$$

$$\det(H - \lambda I) = 0$$

$$\det \begin{pmatrix} -\lambda & \epsilon/\sqrt{2} & 0 \\ \epsilon/\sqrt{2} & -\lambda & \epsilon/\sqrt{2} \\ 0 & \epsilon/\sqrt{2} & -\lambda \end{pmatrix} = 0$$

$$-\lambda\left(\lambda^2 - \frac{\epsilon^2}{2}\right) - \frac{\epsilon}{\sqrt{2}}\left(-\frac{\lambda\epsilon}{\sqrt{2}}\right) = 0$$

$$-\lambda^3 + \frac{\epsilon^2\lambda}{2} + \frac{\epsilon^2}{2}\lambda = 0$$

Eigen values

$$\lambda(\lambda^2 - \epsilon^2) = 0 \implies \boxed{\lambda_1 = 0, \quad \lambda_2 = +\epsilon \text{ and } \lambda_3 = -\epsilon}$$

or Energy, $\boxed{\lambda = 0, \pm 1}$

$\boxed{\lambda_1 = 0 :}$

$$\begin{pmatrix} 0 & \epsilon/\sqrt{2} & 0 \\ \epsilon/\sqrt{2} & 0 & \epsilon/\sqrt{2} \\ 0 & \epsilon/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\frac{\epsilon}{\sqrt{2}} x_2 = 0 \implies x_2 = 0$$

$$\frac{\epsilon x_1}{\sqrt{2}} + \frac{\epsilon x_3}{\sqrt{2}} = 0 \implies x_3 = -x_1$$

$$\frac{\epsilon}{\sqrt{2}} x_2 = 0 \implies x_2 = 0$$

Normalized Eigenstate

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$x_1^2 + 0 + x_1^2 = 1$$

$$x_1 = \frac{1}{\sqrt{2}}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}$$

$\boxed{\lambda_2 = +\epsilon :}$

$$\begin{pmatrix} -\epsilon & \epsilon/\sqrt{2} & 0 \\ \epsilon/\sqrt{2} & -\epsilon & \epsilon/\sqrt{2} \\ 0 & \epsilon/\sqrt{2} & -\epsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-\epsilon x_1 + \frac{\epsilon x_2}{\sqrt{2}} = 0 \implies x_2 = \sqrt{2}\, x_1$$

$$\frac{\epsilon x_1}{\sqrt{2}} - \epsilon x_2 + \frac{\epsilon x_3}{\sqrt{2}} = 0 \implies -x_1 + x_3 = 0$$

or $x_3 = x_1$

$$\frac{\epsilon}{\sqrt{2}} x_2 - \epsilon x_3 = 0 \implies x_2 = \sqrt{2}\, x_3$$

Normalized eigenstate

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$x_1^2 + 2x_1^2 + x_1^2 = 1$$

$$\implies x_1^2 = \frac{1}{4} \quad \text{or} \quad x_1 = \frac{1}{2}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}}$$

$\boxed{\lambda_2 = -\epsilon:}$

$$\begin{pmatrix} \epsilon & \epsilon/\sqrt{2} & 0 \\ \epsilon/\sqrt{2} & \epsilon & \epsilon/\sqrt{2} \\ 0 & \epsilon/\sqrt{2} & \epsilon \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\epsilon x_1 + \frac{\epsilon}{\sqrt{2}} x_2 = 0 \Rightarrow x_2 = -\sqrt{2}\, x_1$$

$$\frac{\epsilon}{\sqrt{2}} x_1 + \epsilon x_2 + \frac{\epsilon}{\sqrt{2}} x_3 = 0 \Rightarrow -x_1 + x_3 = 0$$
or $x_3 = x_1$

$$\epsilon/\sqrt{2}\, x_2 + \epsilon x_3 = 0 \Rightarrow -\epsilon x_1 + \epsilon x_3 = 0$$
or $x_3 = x_1$

Normalized eigenstate

$$x_1^2 + x_2^2 + x_3^2 = 1$$

$$x_1^2 + 2x_1^2 + x_1^2 = 1 \Rightarrow x_1 = \frac{1}{2}$$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}}$$

Problem 5: 3 state system $|1\rangle, |2\rangle, |3\rangle$

(a) B degenerate ?

$$B = \begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}$$

Finding if eigenvalues are distinct.

$$\det(B - \lambda I) = 0$$

$$\det \begin{pmatrix} b-\lambda & 0 & 0 \\ 0 & -\lambda & -ib \\ 0 & ib & -\lambda \end{pmatrix} = 0$$

$$(b-\lambda)(\lambda^2 - b^2) = 0$$

$$\lambda = b \quad \text{and} \quad \lambda = \pm b$$

Thus, B has $\lambda = +b$ as eigen values twice!

$\therefore$ Yes, B also has degenerate spectrum!

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix} :$$

$$\det(A - \lambda I) = 0$$

$$(a-\lambda)(a+\lambda)(a+\lambda) = 0$$

$$\lambda = a \; ; \; \lambda = -a \; ; \; \lambda = -a$$

Eigenvalues repeat and are not distinct.

$\therefore$ A clearly has degenerate spectrum!

(b) Show that A and B commute:

i.e., $[A,B]=0$

or $AB-BA=0$ or $\boxed{AB=BA}$ → show that

$$AB=\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}=\begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

equal

$$BA=\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{pmatrix}=\begin{pmatrix} ab & 0 & 0 \\ 0 & 0 & iab \\ 0 & -iab & 0 \end{pmatrix}$$

$\therefore AB=BA \Rightarrow [A,B]=0$ or A & B commute!

(c) Since A & B commute, they have simultaneous eigenvectors

Matrix B:

$$x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$\lambda_1=b$: $Bx=\lambda_1 x$

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=b\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$bx_1=bx_1 \Rightarrow x_1=x_1$ choose $x_1=1$

$-ibx_3=bx_2 \Rightarrow ix_2=x_3$ then if $x_2=0 \Rightarrow x_3=0$

$ibx_2=bx_3 \Rightarrow ix_2=x_3$

$\therefore \boxed{x=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ → normalized

$\lambda_2=b$ (degenerate): $By=\lambda_2 y$

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}=b\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$by_1=by_1$ orthonormal to x $\therefore$ chose $y_1=0$

$-iby_3=by_2$ if $y_2=1$, then $y_3=i$ $\therefore \boxed{y=\begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}}$

$iby_2=by_3$

Normalize y: $\quad y_2^2(0^2+1^2+|i|^2)=1$

$$y_2 = \frac{1}{\sqrt{2}} \qquad \therefore \boxed{y = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix}}$$

$\lambda_3 = -b$: $\quad Bz = \lambda_3 z$

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 0 & -ib \\ 0 & ib & 0 \end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix} = -b\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix}$$

$bz_1 = -bz_1$

$-ibz_3 = -bz_2 \Rightarrow z_2 = iz_3$

$ibz_2 = -bz_3 \Rightarrow z_2 = iz_3$

$z \rightarrow$ orthogonal to x & y $\therefore$ if $z_1 = 0$ & $z_3 = -i$, then $z_2 = 1$

$$\boxed{z = \begin{pmatrix}0\\1\\-i\end{pmatrix}}$$

Normalize : $z_2^2(0^2+1^2+|-i|^2)=1$

$$z_2 = \frac{1}{\sqrt{2}}$$

$$\therefore \boxed{z = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\end{pmatrix}}$$

$\therefore$ The normalized orthogonal basis are: (orthonormal)

$$\boxed{\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\end{pmatrix}\right\}}$$

We can write $|1\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\1\\0\end{pmatrix}$ & $|3\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Then y can be written as:

$$y = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|2\rangle + i|3\rangle\right)$$

and z can be written as:

$$z = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|2\rangle - i|3\rangle\right)$$

Also holds for A: $Ax = \underset{\substack{\uparrow\\ \lambda_1}}{a}x$, $Ay = -\underset{\substack{\uparrow\\ \lambda_2}}{a}y$, $Az = -\underset{\substack{\uparrow\\ \lambda_3}}{a}z$

$\boxed{\text{Yes, A \& B form a complete set of observables.}}$

Since A & B commute, they have a common eigenbasis, and any state can be expressed using these common eigenbasis. Each eigenvector is identified by the eigenvalues of A & B.