Phys 660 HW 2 Student Solutions

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Problem 1

a) Given $\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

We know: $|\uparrow\rangle_n = \cos\frac{\theta}{2} e^{-i\frac{\varphi}{2}}|\uparrow\rangle + \sin\frac{\theta}{2} e^{i\frac{\varphi}{2}}|\downarrow\rangle$

Result of measurment of S_x gets its eigenvalues which are $\pm\frac{\hbar}{2}$.

$$P_{\hbar/2} = |\langle S_x^+ | \uparrow_n\rangle|^2 = \left|\left(\frac{1}{\sqrt{2}}\langle\uparrow| + \frac{1}{\sqrt{2}}\langle\downarrow|\right)\left|\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}|\downarrow\rangle\right.\right|^2$$

$$= \frac{1}{2}\left|\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} + \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\right|^2$$

$$= \frac{1}{2}\left(\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} + \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\right)\left(\cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} + \sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\right)$$

$$= \frac{1}{2}\left[\cos^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) + \sin^2\frac{\theta}{2}\right]$$

$$= \frac{1}{2}\left[1 + 2\cos\varphi\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right] = \boxed{\frac{1}{2}\left[1 + \cos\varphi\sin\theta\right]}$$

$$P_{-\hbar/2} = |\langle S_x^- | \uparrow_n\rangle|^2 = \left|\left(\frac{1}{\sqrt{2}}\langle\uparrow| - \frac{1}{\sqrt{2}}\langle\uparrow|\right)\left(\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}|\uparrow\rangle + \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}|\downarrow\rangle\right)\right|^2$$

$$= \frac{1}{2}\left|\left(\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} - \sin\theta e^{i\frac{\varphi}{2}}\right)\right|^2$$

$$= \frac{1}{2}\left|\left(\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} - \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\right)\right|^2$$

$$= \frac{1}{2}\left(\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} - \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\right)\left(\cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} - \sin e^{-i\frac{\varphi}{2}}\right)$$

$$= \frac{1}{2}\left(\underbrace{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}_{1} - \underbrace{\cos\frac{\theta}{2}\sin\frac{\theta}{2}}_{\frac{1}{2}\sin\theta}\underbrace{\left(e^{i\varphi} + e^{-i\varphi}\right)}_{2\cos\varphi}\right)$$

$$= \boxed{\frac{1}{2}\left(1 - \cos\varphi\sin\theta\right)}$$

(b) $\sigma_x^2 = {}_{\hat{n}}\langle\uparrow|(S_x - \bar{S}_x)^2|\uparrow\rangle_{\hat{n}}$, where $\bar{S}_x = {}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}$

$$\langle(S_x - \bar{S}_x)^2\rangle = \langle(S_x^2 + \bar{S}_x^2 - 2S_x\bar{S}_x)\rangle$$

$$= \langle S_x^2\rangle + \langle S_x\rangle^2 - 2\langle S_x\rangle\langle S_x\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2$$

$$\Rightarrow \sigma_x^2 = {}_{\hat{n}}\langle\uparrow|S_x^2|\uparrow\rangle_{\hat{n}} - \left({}_{\hat{n}}\langle\uparrow|S_x|\uparrow\rangle_{\hat{n}}\right)^2$$

$$= \frac{\hbar^2}{4}\left(\cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \quad \sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\end{pmatrix}$$

$$- \frac{\hbar^2}{4}\left[\left(\cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \quad \sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\end{pmatrix}\right]^2$$

$$= \frac{\hbar^2}{4}\left[\underbrace{\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)}_{1} - \left[\left(\cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \quad \sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\right)\begin{pmatrix}\sin\frac{\theta}{2}e^{i\frac{\varphi}{2}}\\ \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}}\end{pmatrix}\right]^2\right.$$

$$= \frac{\hbar^2}{4}\left[1 - \left(2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\varphi\right)^2\right]$$

$$= \boxed{\frac{\hbar^2}{4}\left[1 - \sin^2\theta\cos^2\varphi\right]}$$

(C) For $\theta = 0$, $\theta = \pi$ & $\theta = \frac{\pi}{2}$, $\varphi = 0$

- for $\theta = 0$ & $\theta = \pi$ we have $\begin{cases} P_{+\frac{\hbar}{2}} = \frac{1}{2}(1 + \cos\varphi\sin\theta) \\ P_{-\frac{\hbar}{2}} = \frac{1}{2}(1 - \cos\varphi\sin\theta). \end{cases}$

$\Rightarrow$ for both of angles : $P_{+\frac{\hbar}{2}} = \frac{1}{2}$ because $\sin\theta = 0$
$P_{-\frac{\hbar}{2}} = \frac{1}{2}$

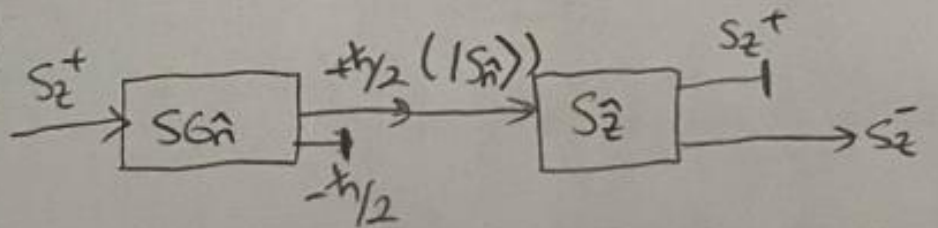
- for $\theta = \frac{\pi}{2}$ & $\varphi = 0$ $\begin{cases} P_{+\frac{\hbar}{2}} = \frac{1}{2}(1 + \underbrace{\cos 0\sin\frac{\pi}{2}}_{1}) = 1 \\ P_{-\frac{\hbar}{2}} = \frac{1}{2}(1 - \underbrace{\cos 0\sin\frac{\pi}{2}}_{1}) = 0 \end{cases}$

- This is what expected.

- for $\theta = \frac{\pi}{2}$ & $\varphi = 0$ $\Rightarrow \sigma_x^2 = \frac{\hbar^2}{4}\left[1 - \sin^2\underbrace{\theta}_{\frac{\pi}{2}}\cos^2\underbrace{\varphi}_{0}\right] = 0$

Zero Variance means there is no other values obtained in the measurment.

Q-2:



a)

Intensity (I) that gets through successive apparatuses is the same as the probabilities of a particle getting through in successive experiments. Since the experiments are successive we multiply the probability of $|S_n^+\rangle$ given the initial $|S_z^+\rangle$ state (i.e. $|\langle S_z^+|S_n^+\rangle|^2$), and the probability of a $|S_z^-\rangle$ outcome given that $|S_n^+\rangle$ intermediate outcome ($|\langle S_n^+|S_z^-\rangle|^2$):

$$I = |\langle S_n^+|S_z^-|^2 \, |\langle S_z^+|S_n^+|^2$$

We know that $|S_n^+\rangle = \cos\left(\frac{\beta}{2}\right)|+\rangle + \sin\left(\frac{\beta}{2}\right) e^{i\alpha}$

$$|S_n^+\rangle\langle S_n^+| = \left[\cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle\right]\left[\cos\frac{\beta}{2}\langle+| + \sin\frac{\beta}{2}\langle-|\right]$$

$$I = \langle S_n^+|S_z^-\rangle\langle S_z^-|S_n^+\rangle\langle S_z^+|S_n^+\rangle\langle S_n^+|S_z^+\rangle$$

$$= \left[\cos\frac{\beta}{2}\langle+|-\rangle + \sin\frac{\beta}{2}e^{-i\alpha}\langle-|-\rangle\right]$$

$$\times\left[\cos\frac{\beta}{2}\langle-|+\rangle + \sin\frac{\beta}{2}e^{i\alpha}\langle-|-\rangle\right]$$

$$\times\left[\cos\frac{\beta}{2}\langle+|+\rangle + \sin\frac{\beta}{2}e^{i\alpha}\langle+|-\rangle\right]$$

$$\times\left[\cos\frac{\beta}{2}\langle+|+\rangle + \sin\frac{\beta}{2}e^{-i\alpha}\langle-|+\rangle\right]$$

$$= \left(\sin\frac{\beta}{2}e^{-i\alpha}\right)\left(\sin\frac{\beta}{2}e^{i\alpha}\right)\left(\cos\frac{\beta}{2}\right)\left(\cos\frac{\beta}{2}\right)$$

$$= \sin^2\left(\frac{\beta}{2}\right)\cos^2\left(\frac{\beta}{2}\right) = \frac{1-\cos\beta}{2}\cdot\frac{1+\cos\beta}{2}$$

$$= \frac{(1-\cos\beta)^2}{4} \Rightarrow \boxed{I = \frac{\sin^2\beta}{4}}$$

b)

We should orient $SG\hat{n}$, at $\beta = \frac{\pi}{2}$ which is $90°$ relative to the z-axis. This maximizes the intensity to 0.25 of the beam that entered the $SG\hat{n}$ apparatus.

P3 Ralph

Proof. (a) In the position basis, $\hat{x} = x$, and we have

$$\begin{aligned}
\langle\psi|\hat{x}|\psi\rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-ikx} e^{ikx} x e^{-\frac{(x-x_0)^2}{2\sigma^2}} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x e^{-\frac{(x-x_0)^2}{2\sigma^2}} \, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left(\sqrt{2\sigma^2} t + x_0\right) e^{-t^2} \sqrt{2\sigma^2} \, \mathrm{d}t \\
&= \frac{1}{\sqrt{\pi}} \left[\sqrt{2\sigma^2} \int_{-\infty}^{\infty} t e^{-t^2} \, \mathrm{d}t + x_0 \int_{-\infty}^{\infty} e^{-t^2} \, \mathrm{d}t\right] \\
&= \frac{1}{\sqrt{\pi}} \left[0 + x_0 \sqrt{\pi} \, \mathrm{d}t\right] \\
&= x_0,
\end{aligned}$$

where the final jump was due to two things: the first integral is odd over an even domain and so evaluates to zero, and the second integral is equal to $\sqrt{\pi}$. Our solution makes sense as the Gaussian wave-function is centered at the mean $\mu = x_0$.

$$
\begin{aligned}
\langle\psi|\hat{p}|\psi\rangle &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \left(-i\hbar\frac{\partial}{\partial x}\right) \left(e^{ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}}\right) \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \frac{\partial}{\partial x} \left(e^{ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}}\right) \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \left(ik - \frac{2(x-x_0)}{4\sigma^2}\right) e^{ikx} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left(ik - \frac{2(x-x_0)}{4\sigma^2}\right) e^{-\frac{(x-x_0)^2}{2\sigma^2}} \mathrm{d}x \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \left(ik - \frac{2t}{4\sigma^2}\right) e^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \left[ik \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t - \frac{1}{2\sigma^2} \int_{-\infty}^{\infty} t e^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t\right] \\
&= -\frac{i\hbar}{\sqrt{2\pi\sigma^2}} \left[ik\sqrt{2\pi\sigma^2} - 0\right] \\
&= \hbar k.
\end{aligned}
$$

$$
\begin{aligned}
\big\langle\psi\big|(\Delta x)^2\big|\psi\big\rangle &= \big\langle\psi\big|(\hat{x} - \langle x\rangle)^2\big|\psi\big\rangle \\
&= \big\langle\psi\big|x^2\big|\psi\big\rangle - \langle x\rangle^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}} \mathrm{d}x - x_0^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (t+x_0)^2 e^{-\frac{t^2}{2\sigma^2}} \mathrm{d}t - x_0^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \left[-\frac{t}{\sigma^2}(t+x_0)^2 e^{-\frac{t^2}{2\sigma^2}}\bigg|_{-\infty}^{\infty} - \frac{2t^2}{\sigma^4}(t+x_0) e^{-\frac{t^2}{2\sigma^2}}\bigg|_{-\infty}^{\infty} + \frac{2t^3}{\sigma^6} e^{-\frac{t^2}{2\sigma^2}}\bigg|_{-\infty}^{\infty}\right] - x_0^2 \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \sqrt{2\pi\sigma^2}\left(x_0^2 + \sigma^2\right) - x_0^2 \\
&= \left(x_0^2 + \sigma^2\right) - x_0^2 \\
&= \sigma^2.
\end{aligned}
$$

$$
\begin{aligned}
\langle\psi|(\Delta p)^2|\psi\rangle &= \langle\psi|(\hat{p}-\langle p\rangle)^2|\psi\rangle \\
&= \langle\psi|p^2|\psi\rangle - \langle p\rangle^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(e^{-ikx-\frac{(x-x_0)^2}{4\sigma^2}}\right)\frac{\partial^2}{\partial x^2}\left(e^{ikx-\frac{(x-x_0)^2}{4\sigma^2}}\right)\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(e^{-ikx-\frac{(x-x_0)^2}{4\sigma^2}}\right)\left(ik-\frac{2(x-x_0)}{4\sigma^2}\right)^2 e^{ikx-\frac{(x-x_0)^2}{4\sigma^2}}\,\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(ik-\frac{2(x-x_0)}{4\sigma^2}\right)^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}}\,\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(-k^2-\frac{ik(x-x_0)}{\sigma^2}-\frac{(x-x_0)^2}{4\sigma^4}\right)e^{-\frac{(x-x_0)^2}{2\sigma^2}}\,\mathrm{d}x - (\hbar k)^2 \\
&= -\frac{\hbar^2}{\sqrt{2\pi\sigma^2}}\left(-\frac{\sqrt{\pi}\left(4k^2\sigma^2+1\right)}{2\sqrt{2\sigma^2}}\right) - (\hbar k)^2 \\
&= \frac{\hbar^2\left(4k^2\sigma^2+1\right)}{4\sigma^2} - (\hbar k)^2 \\
&= \hbar^2k^2 + \frac{\hbar^2}{4\sigma^2} - (\hbar k)^2 \\
&= \frac{\hbar^2}{4\sigma^2}.
\end{aligned}
$$

(b) The uncertainty is

$$
\sqrt{\langle\psi|(\Delta x)^2|\psi\rangle}\sqrt{\langle\psi|(\Delta p)^2|\psi\rangle} = \sqrt{\sigma^2}\sqrt{\frac{\hbar^2}{4\sigma^2}} = \frac{\hbar}{2},
$$

and it fulfils the minimal uncertainty, as needed. This was expected as the condition of a Gaussian wave-functions for position and momentum creates the minimum uncertainty state.

(c) We have

$$
\begin{aligned}
\langle x|\Delta p|\psi\rangle &= \langle x|\hat{p} - \langle p\rangle|\psi\rangle \\
&= \langle x|\hat{p}|\psi\rangle - \langle x|\langle p\rangle|\psi\rangle \\
&= \hat{p}\,\langle x|\psi\rangle - \langle p\rangle\,\langle x|\psi\rangle \\
&= [\hat{p} - \langle p\rangle]\,\langle x|\psi\rangle \\
&= \left[-i\hbar\frac{\partial}{\partial x} - \langle p\rangle\right]\langle x|\psi\rangle \\
&= \left[-i\hbar\left(ik - \frac{2(x-x_0)}{4\sigma^2}\right) - \hbar k\right]\langle x|\psi\rangle \\
&= \left[\hbar k + \frac{i\hbar(x-x_0)}{2\sigma^2} - \hbar k\right]\langle x|\psi\rangle \\
&= \left[\frac{i\hbar(x-x_0)}{2\sigma^2}\right]\langle x|\psi\rangle \\
&= \left[\frac{i\hbar}{2\sigma^2}\right](\hat{x} - \langle x\rangle)\,\langle x|\psi\rangle \\
&= \left[\frac{i\hbar}{2\sigma^2}\right]\langle x|\hat{x} - \langle x\rangle|\psi\rangle \\
&= \left[\frac{i\hbar}{2\sigma^2}\right]\langle x|\Delta x|\psi\rangle\,.
\end{aligned}
$$

Thus,

$$
\langle x|\Delta x|\psi\rangle = i\lambda\,\langle x|\Delta p|\psi\rangle = -\frac{2i\sigma^2}{\hbar}\,\langle x|\Delta p|\psi\rangle \implies \lambda = -\frac{2\sigma^2}{\hbar}.
$$

If we rearrange the terms, we get

$$
\sqrt{\langle\psi|(\Delta x)^2|\psi\rangle}\sqrt{\langle\psi|(\Delta p)^2|\psi\rangle} = \frac{\hbar}{2} = -\frac{\sigma^2}{\lambda}.
$$

P4

Proof. (a) We have that

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}} = \langle p|x\rangle^* .$$

Checking the momentum wave function, we get

$$\begin{aligned}
\tilde{\psi}(p) &= \langle p|\psi\rangle \\
&= \int_{-\infty}^{\infty} \langle p|x\rangle \langle x|\psi\rangle \,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ipx}{\hbar}}\right)\left(\frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ikx-\frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} \left(e^{-\frac{ipx}{\hbar}}\right)\left(e^{ikx-\frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} \left(e^{ix\left(k-\frac{p}{\hbar}\right)-\frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} \int_{-\infty}^{\infty} \left(e^{i(x-x_0)\left(k-\frac{p}{\hbar}\right)-\frac{1}{4\sigma^2}(x-x_0)^2}\right) \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} 2\sqrt{\pi\sigma^2} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2} \\
&= \sqrt{\frac{2\sigma}{\hbar}} \frac{1}{(2\pi)^{\frac{1}{4}}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2} \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}.
\end{aligned}$$

(b) We have

$$
\begin{aligned}
\langle\psi|\hat{p}|\psi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\psi|p\rangle \langle p|\hat{p}|p'\rangle \langle p'|\psi\rangle \,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\, p\delta(p-p')\, \tilde{\psi}(p') \,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty} \tilde{\psi}^*(p) p \tilde{\psi}(p)\,\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p \left(e^{ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) \mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) \mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} p e^{-2\sigma^2\left(\frac{p}{\hbar}-k\right)^2} \mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}\hbar^2 k}{\sqrt{2\sigma^2}} \\
&= \hbar k.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\langle\psi|\hat{p}^2|\psi\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle\psi|p\rangle \langle p|\hat{p}^2|p'\rangle \langle p'|\psi\rangle \,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\, p^2\delta(p-p')\, \tilde{\psi}(p') \,\mathrm{d}p\,\mathrm{d}p' \\
&= \int_{-\infty}^{\infty} \tilde{\psi}^*(p) p^2 \tilde{\psi}(p)\,\mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p^2 \left(e^{ix_0\left(k-\frac{p}{\hbar}\right)} e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) \mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) p^2 \left(e^{-\sigma^2\left(\frac{p}{\hbar}-k\right)^2}\right) \mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} p^2 e^{-2\sigma^2\left(\frac{p}{\hbar}-k\right)^2} \mathrm{d}p \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}\hbar^3\left(4k^2\sigma^2+1\right)}{2^{\frac{5}{2}}\sigma^3} \\
&= \frac{\hbar^2\left(4k^2\sigma^2+1\right)}{4\sigma^2} \\
&= \hbar^2 k^2 + \frac{\hbar^2}{4\sigma^2}.
\end{aligned}
$$

Checking, we have

$$
\langle\psi|\Delta p|\psi\rangle = \langle\psi|\hat{p}^2|\psi\rangle - \langle\psi|\hat{p}|\psi\rangle^2 = \hbar^2k^2 + \frac{\hbar^2}{4\sigma^2} - \hbar^2k^2 = \frac{\hbar^2}{4\sigma^2}.
$$

P5 Adriel

a) Use the fundamental relation $[\hat{p},\hat{x}] = -i\hbar$ to compute the commutator

$$[\hat{x}, U(a)] = \ ?$$

Because we have an operator of the form $U(a) = e^{-i\hat{p}\frac{a}{\hbar}}$ if we do a Taylor expansion of it we'll get that $e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Since $x^n = \left(-i\hat{p}\frac{a}{\hbar}\right)^n$ we need to see how this power "n" affects our inner terms.

Therefore, recall that $[\hat{p},\hat{x}] = -i\hbar$, $[\hat{x},\hat{p}] = +i\hbar$.

For $n=0$,

$[\hat{x}, \hat{p}^{n=0}] = [\hat{x}, 1] = \hat{x} - \hat{x} = \underline{0}$

Note: We see from $n=0,1,2,3$ that $\underline{[\hat{x},\hat{p}^n] = n i\hbar \hat{p}^{n-1}}$, it's general form

For $n=1$,

$[\hat{x}, \hat{p}^1] = \hat{x}\hat{p} - \hat{p}\hat{x} = \underline{i\hbar}$

For $n=2$,

$[\hat{x}, \hat{p}^2] = [\hat{x}, \hat{p}\cdot\hat{p}] = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x} = \hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x} + \hat{p}\hat{x}\hat{p} - \hat{p}\hat{x}\hat{p}$

$[\hat{x}, \hat{p}^2] = \hat{p}\hat{x}\hat{p} - \hat{p}\hat{p}\hat{x} + \hat{x}\hat{p}\hat{p} - \hat{p}\hat{x}\hat{p} = \hat{p}[\hat{x},\hat{p}] + [\hat{x},\hat{p}]\hat{p} = \hat{p}(i\hbar) + (i\hbar)\hat{p}$

$[\hat{x}, \hat{p}^2] = \underline{2i\hbar\hat{p}}$

For $n=3$,

$[\hat{x}, \hat{p}^3] = \hat{x}\hat{p}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{p}\hat{x} = \hat{x}\hat{p}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{p}\hat{x} + \hat{p}\hat{x}\hat{p}\hat{p} - \hat{p}\hat{x}\hat{p}\hat{p} + \hat{p}\hat{p}\hat{x}\hat{p} - \hat{p}\hat{p}\hat{x}\hat{p}$

$[\hat{x}, \hat{p}^3] = \hat{p}\hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{p}\hat{x} + \hat{p}\hat{p}\hat{x}\hat{p} - \hat{p}\hat{p}\hat{x}\hat{p} + \hat{x}\hat{p}\hat{p}\hat{p} - \hat{p}\hat{x}\hat{p}\hat{p}$

$[\hat{x}, \hat{p}^3] = \hat{p}(\hat{x}\hat{p}\hat{p} - \hat{p}\hat{p}\hat{x} + \hat{p}\hat{x}\hat{p} - \hat{p}\hat{x}\hat{p}) + (\hat{x}\hat{p} - \hat{p}\hat{x})\hat{p}^2$

$[\hat{x}, \hat{p}^3] = \hat{p}\left(\hat{p}(\hat{x}\hat{p} - \hat{p}\hat{x}) + (\hat{x}\hat{p} - \hat{p}\hat{x})\hat{p}\right) + [\hat{x},\hat{p}]\hat{p}^2$

$[\hat{x}, \hat{p}^3] = \hat{p}\left(\hat{p}[\hat{x},\hat{p}] + [\hat{x},\hat{p}]\hat{p}\right) + [\hat{x},\hat{p}]\hat{p}^2 = \hat{p}\underbrace{[\hat{x},\hat{p}^2]}_{2i\hbar\hat{p}} + \underbrace{[\hat{x},\hat{p}]}_{i\hbar}\hat{p}^2 = 2i\hbar\hat{p}^2 + i\hbar\hat{p}^2 = \underline{3i\hbar\hat{p}^2}$

Let's see what $[\hat{x}, U(a)]$ is.

$$[\hat{x}, U(a)] = \left[\hat{x}, \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i\hat{p}a}{\hbar}\right)^n\right] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{ia}{\hbar}\right)^n [\hat{x}, \hat{p}^n] = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{ia}{\hbar}\right)^n (n i\hbar \hat{p}^{n-1})$$

$$[\hat{x}, U(a)] = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \cdot n i\hbar \left(\frac{-ia}{\hbar}\right)^{n-1} \cdot \left(\frac{-ia}{\hbar}\right) \hat{p}^{n-1} = a \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-ia}{\hbar}\right)^{n-1} \hat{p}^{n-1} = a\, e^{-i\hat{p}a/\hbar}$$

$$\boxed{[\hat{x}, U(a)] = a U(a)}$$

Note: $n-1 = 0 = n$

$n \to n-1$

$n = (1)-(1) = 0$

b) Given the state $|\psi\rangle$ such that $\langle\psi|\hat{x}|\psi\rangle = \bar{x}$, what is the mean value of $\hat{x}$ in the state $|\phi\rangle = U(a)|\psi\rangle$?

Know that $[\hat{x}, U(a)] = \hat{x}U(a) - U(a)\hat{x} = aU(a)$ So, $\underline{\hat{x}U(a)} = U(a)\hat{x} + aU(a)$

The mean value of $\hat{x}$ in the state $|\phi\rangle = U(a)|\psi\rangle$ would be,

$$\langle\phi|\hat{x}|\phi\rangle = \langle\psi|U^\dagger(a)\,\hat{x}\,U(a)|\psi\rangle = \langle\psi|U^\dagger(a)(U(a)\hat{x} + aU(a))|\psi\rangle$$

$$\langle\phi|\hat{x}|\phi\rangle = \langle\psi|U^\dagger(a)U(a)\hat{x}|\psi\rangle + \langle\psi|U^\dagger(a)\,a\,U(a)|\psi\rangle$$

$$\langle\phi|\hat{x}|\phi\rangle = \langle\psi|\hat{x}|\psi\rangle + a\langle\psi|U^\dagger(a)U(a)|\psi\rangle = \bar{x} + a$$

$$\boxed{\langle\phi|\hat{x}|\phi\rangle = \bar{x} + a}$$