Homework 4 Solutions

Problem 1

$\vec{B} = B\hat{z}$

$H = -\vec{\mu}\cdot\vec{B}$ $\qquad \vec{\mu} = -\frac{e}{m}\vec{S}$

$H = \frac{e}{m} B S_z \rightarrow \quad \mathcal{E}_\uparrow = \frac{eB}{m}\frac{\hbar}{2} \; ; \; \mathcal{E}_\downarrow = -\frac{eB}{m}\frac{\hbar}{2}$

$|\psi(t=0)\rangle = |\uparrow_{\hat{n}}\rangle = c_{\frac{\theta}{2}}\, e^{-i\varphi/2}\,|\uparrow\rangle + s_{\frac{\theta}{2}}\, e^{i\varphi/2}\,|\downarrow\rangle$

a) $|\psi(t)\rangle = c_{\frac{\theta}{2}}\, e^{-i\varphi/2}\, e^{-i\frac{eB}{2m}t}\,|\uparrow\rangle + s_{\frac{\theta}{2}}\, e^{i\varphi/2}\, e^{i\frac{eB}{2m}t}\,|\downarrow\rangle$

Clearly $\theta \rightarrow$ constant $\quad \boxed{\varphi(t) = \varphi + \frac{eB}{m}t}$ precession.

b) $|\uparrow_{\hat{x}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) \; ; \; |\downarrow_{\hat{x}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$

$|\uparrow_{\hat{y}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + i|\downarrow\rangle) \; ; \; |\downarrow_{\hat{y}}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - i|\downarrow\rangle)$

$$P(\uparrow_{\hat{x}}) = |\langle\uparrow_{\hat{x}}|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\left(c_{\frac{\theta}{2}}\, e^{-i\frac{\varphi(t)}{2}} + s_{\frac{\theta}{2}}\, e^{i\frac{\varphi(t)}{2}}\right)\right|^2$$

$$= \frac{1}{2}\left(c_{\frac{\theta}{2}} + s_{\frac{\theta}{2}}\, e^{i\varphi}\right)\left(c_{\frac{\theta}{2}} + s_{\frac{\theta}{2}}\, e^{-i\varphi}\right)$$

$$= \frac{1}{2}\Big(c^2_{\frac{\theta}{2}} + s_{\frac{\theta}{2}}c_{\frac{\theta}{2}}\underbrace{(e^{i\varphi} + e^{-i\varphi})}_{2c\varphi} + s^2_{\frac{\theta}{2}}\Big) = \frac{1}{2}(1 + s\theta\, c\varphi)$$

②

$$P(\downarrow \hat{x}) = |\langle \downarrow_{\hat{x}} | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \left(c\tfrac{\theta}{2}\, e^{-i\varphi/2} - s\tfrac{\theta}{2}\, e^{i\varphi/2} \right) \right|^2$$

$$= \frac{1}{2} (1 - s\theta\, c\varphi)$$

$$\boxed{\begin{array}{lll} \text{probability of measuring } S_x = \frac{\hbar}{2} & \rightarrow & \frac{1}{2}(1 + s\theta\, c\varphi(t)) \\ \text{" " " } S_x = -\frac{\hbar}{2} & \rightarrow & \frac{1}{2}(1 - s\theta\, c\varphi(t)) \end{array}}$$

$$\boxed{\varphi(t) = \varphi + \frac{eB}{m} t}$$

$$P_{\uparrow \hat{y}} = |\langle \uparrow_{\hat{y}} | \psi \rangle|^2 = \left| \frac{1}{\sqrt{2}} \left(c\tfrac{\theta}{2}\, e^{-i\varphi/2} - i\, s\tfrac{\theta}{2}\, e^{i\varphi/2} \right) \right|^2$$

$$= \frac{1}{2} \left(c\tfrac{\theta}{2} - i s\tfrac{\theta}{2}\, e^{i\varphi} \right) \left(c\tfrac{\theta}{2} + i s\tfrac{\theta}{2}\, e^{-i\varphi} \right)$$

$$= \frac{1}{2} \left(c^2\tfrac{\theta}{2} + s^2\tfrac{\theta}{2} + i s\tfrac{\theta}{2}\, c\tfrac{\theta}{2}\, \underbrace{(e^{-i\varphi} - e^{i\varphi})}_{-2i s\varphi} \right)$$

$$= \frac{1}{2} (1 + s\varphi\, s\theta)$$

$$P_{\downarrow \hat{y}} = \frac{1}{2} (1 - s\varphi\, s\theta)$$

$$\boxed{\begin{array}{lll} \text{Prob. of measuring } S_y = \hbar/2 & \rightarrow & \frac{1}{2}(1 + s\theta\, s\varphi(t)) \\ \text{" " " } S_y = -\frac{\hbar}{2} & \rightarrow & \frac{1}{2}(1 - s\theta\, s\varphi(t)) \end{array}}$$

$P_{a\hat{z}} = \cancel{\phi}\, c^2\theta/_2$ $\qquad$ $P_{b\hat{z}} = \cancel{\phi}\, s^2\theta/_2$ $\qquad$ conserved.

c)

$$\langle S_x \rangle = \frac{\hbar}{4}(1 + s\theta\, c\varphi) - \frac{\hbar}{4}(1 - s\theta\, c\varphi)$$

$$= \frac{\hbar}{2} s\theta\, c\varphi(t)$$

$$\langle S_y \rangle = \frac{\hbar}{4}(1 + s\theta\, s\varphi) - \frac{\hbar}{4}(1 - s\theta\, s\varphi)$$

$$= \frac{\hbar}{2} s\theta\, s\varphi(t)$$

$$\langle S_y \rangle = \frac{\hbar}{2}\left(c^2\theta/_2 - s^2\theta/_2\right) = \frac{\hbar}{2} c\theta$$

recall $\vec{n}(t) = (s\theta\, c\varphi(t),\ s\theta\, s\varphi(t),\ c\theta)$ agrees.

Problem 2

$$\langle m|x|n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \quad \langle m|\, a + a^\dagger \,|n\rangle =$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \quad \langle m| \left(\sqrt{n}\, |n-1\rangle + \sqrt{n+1}\, |n+1\rangle \right) =$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\, \delta_{m,n-1} + \sqrt{n+1}\, \delta_{m,n+1} \right)$$

$$\langle m|p|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}} \quad \langle m|\, (a^\dagger - a)\,|n\rangle$$

$$= i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{n+1}\, \delta_{m,n+1} - \sqrt{n}\, \delta_{m,n-1} \right)$$

$$\langle m|xp+px|n\rangle = \frac{i}{2}\hbar \quad \langle m|\, (a+a^\dagger)(a^\dagger - a) + (a^\dagger - a)(a + a^\dagger)\,|n\rangle$$

$$\cancel{aa^\dagger} - aa + a^\dagger a^\dagger - \cancel{a^\dagger a} + \cancel{a^\dagger a} + a^\dagger a^\dagger - aa - \cancel{aa^\dagger}$$

$$2(a^\dagger a^\dagger - aa)$$

$$= i\hbar \quad \langle m|\, (a^\dagger a^\dagger - aa)\,|n\rangle$$

$$= i\hbar \left(\sqrt{n+2}\sqrt{n+1}\, \delta_{m,n+2} - \sqrt{n}\sqrt{n-1}\, \delta_{m,n-2} \right)$$

$$\langle m|x^2|n\rangle = \frac{\hbar}{2m\omega} \quad \langle m|\, (a+a^\dagger)(a+a^\dagger)\,|n\rangle =$$

$$= \frac{\hbar}{2m\omega} \quad \langle m|\, aa + a^\dagger a^\dagger + a^\dagger a + aa^\dagger \,|n\rangle =$$

$$aa^\dagger = a^\dagger a + [a, a^\dagger] = a^\dagger a + 1$$

$$= \frac{\hbar}{2m\omega}\Big(\sqrt{n}\sqrt{n-1}\;\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\;\delta_{m,n+2} +$$

$$+ (n + n+1)\;\delta_{m,n}\Big)$$

$$= \frac{\hbar}{2m\omega}\left(\sqrt{n}\sqrt{n-1}\;\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\;\delta_{m,n+2} + (2n+1)\,\delta_{m,n}\right)$$

$$\langle m|p^2|n\rangle = -\frac{m\hbar\omega}{2}\;\langle m|\,(a^\dagger-a)(a^\dagger-a)\,|n\rangle$$

$$a^\dagger a^\dagger + aa - aa^\dagger - a^\dagger a$$

$$= -\frac{m\hbar\omega}{2}\left(\sqrt{n}\sqrt{n-1}\;\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\;\delta_{m,n+2} - (2n+1)\,\delta_{m,n}\right)$$

Check

$$\langle m|\,\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2\,|n\rangle = -\frac{\hbar\omega}{4}\left(\sqrt{n}\sqrt{n-1}\;\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\;\delta_{m,n-2} - (2n+1)\,\delta_{m,n}\right)$$

$$+ \frac{1}{4}\hbar\omega\left(\sqrt{n}\sqrt{n-1}\;\delta_{m,n-2} + \sqrt{n+1}\sqrt{n+2}\;\delta_{m,n+2} + (2n+1)\,\delta_{m,n}\right)$$

$$= \frac{\hbar\omega}{2}(2n+1)\;\delta_{m,n} = \hbar\omega\left(n+\tfrac{1}{2}\right)\delta_{m,n} \quad ✓.$$

b) $\langle\alpha|x|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\;\langle\alpha|a+a^\dagger|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\;\underbrace{(\alpha+\alpha^*)}_{2\mathrm{Re}\,\alpha}$

$a|\alpha\rangle = \alpha|\alpha\rangle \Rightarrow \langle\alpha|a|\alpha\rangle = \alpha$

$\langle\alpha|a^\dagger = \langle\alpha|\alpha^* \Rightarrow \langle\alpha|a^\dagger|\alpha\rangle = \alpha^*$

$\langle\alpha|p|\alpha\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\;\langle\alpha|a^\dagger-a|\alpha\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\;\underbrace{(\alpha^*-\alpha)}_{-2i\,\mathrm{Im}\,\alpha}$

$\langle\alpha|x|\alpha\rangle = \sqrt{\frac{2\hbar}{m\omega}}\;\mathrm{Re}(\alpha)$

$\langle\alpha|p|\alpha\rangle = \sqrt{2m\hbar\omega}\;\mathrm{Im}\,\alpha$

$\langle\alpha|x^2|\alpha\rangle = \frac{\hbar}{2m\omega}\;\langle\alpha|\,(a+a^\dagger)(a+a^\dagger)\,|\alpha\rangle$

$aa+a^\dagger a^\dagger + \underbrace{aa^\dagger}_{a^\dagger a+[a,a^\dagger]=a^\dagger a+1} + a^\dagger a = aa+a^\dagger a^\dagger + 2a^\dagger a + 1$

$= \frac{\hbar}{2m\omega}\left(\alpha^2 + (\alpha^*)^2 + 2\alpha^*\alpha + 1\right)$

$\langle\alpha|a^\dagger a|\alpha\rangle = \alpha^*\alpha.$

$\langle\alpha|p^2|\alpha\rangle = -\frac{m\hbar\omega}{2}\;\langle\alpha|\,(a^\dagger-a)(a^\dagger-a)\,|\alpha\rangle$

$a^\dagger a^\dagger + aa - aa^\dagger - a^\dagger a = a^\dagger a^\dagger + aa - 2a^\dagger a - 1$

$= -\frac{m\hbar\omega}{2}\left(\alpha^{*2} + \alpha^2 - 2\alpha^*\alpha - 1\right)$

c) $\langle\alpha|(\Delta x)^2|\alpha\rangle = \langle\alpha|x^2|\alpha\rangle - \langle\alpha|x|\alpha\rangle^2$

$$= \frac{\hbar}{2m\omega}\left(\cancel{\alpha^2} + \cancel{(\alpha^*)^2} + \cancel{2\alpha^*\alpha} + 1\right) - \frac{\hbar}{2m\omega}\left(\cancel{\alpha^2} + \cancel{\alpha^{*2}} + \cancel{2\alpha\alpha^*}\right)$$

$$= \frac{\hbar}{2m\omega}$$

$\langle\alpha|(\Delta p)^2|\alpha\rangle = \langle\alpha|p^2|\alpha\rangle - \langle\alpha|p|\alpha\rangle^2$

$$= -\frac{m\hbar\omega}{2}\left(\cancel{\alpha^{*2}} + \cancel{\alpha^2} - \cancel{2\alpha^*\alpha} - 1\right) + \frac{m\hbar\omega}{2}\left(\cancel{\alpha^{*2}} + \cancel{\alpha^2} - \cancel{2\alpha^*\alpha}\right)$$

$$= \frac{m\hbar\omega}{2}$$

$$\Delta x\,\Delta p = \sqrt{\frac{\hbar}{2\cancel{m\omega}}\,\frac{\cancel{m}\hbar\cancel{\omega}}{2}} = \frac{\hbar}{2}$$ minimum uncertainty.

Problem 3

a) $$\hat{x}_H(t) = e^{\frac{iHt}{\hbar}}\, \hat{x}_s\, e^{-\frac{iHt}{\hbar}}$$

•) one way.

$$C(t) = \langle 0|\, e^{\frac{iHt}{\hbar}}\, x_s\, e^{-\frac{iHt}{\hbar}}\, \underbrace{x_s}_{x_H(0) = x_s}\, |0\rangle$$

$$= e^{i\frac{\omega t}{2}} \langle 0|\, x_s\, e^{-\frac{iHt}{\hbar}}\, x_s\, |0\rangle$$

$$= e^{i\frac{\omega t}{2}}\, \frac{\hbar}{2m\omega} \langle 0|(a+a^\dagger)\, e^{-\frac{iHt}{\hbar}}\, \underbrace{(\overset{0}{a}+a^\dagger)|0\rangle}_{\underbrace{e^{-\frac{iHt}{\hbar}}|1\rangle}_{e^{-i\frac{3}{2}\omega t}|1\rangle}}$$

$$= e^{-i\omega t}\, \frac{\hbar}{2m\omega} \langle 0|\, \underbrace{a+a^\dagger|1\rangle}_{|0\rangle + \sqrt{2}|2\rangle} = \frac{\hbar}{2m\omega}\, e^{-i\omega t}$$

$$\boxed{C(t) = \frac{\hbar}{2m\omega}\, e^{-i\omega t}}$$

ii) Another way.

$$x_H = \cos\omega t \; x_S + \sin\omega t \; \frac{p_S}{m\omega}$$

$$\langle 0|\, X_H(t)\, X_H(0)\, |0\rangle = \langle 0| \left(\cos\omega t \; x_S + \sin\omega t \; \frac{p_S}{m\omega} \right) x\, |0\rangle$$

$$= \cos\omega t \; \langle 0| x^2 |0\rangle + \frac{\sin\omega t}{m\omega} \; \langle 0|\, p\, x\, |0\rangle$$

$$= \cos\omega t \; \frac{\hbar}{2m\omega} + \frac{\sin\omega t}{m\omega} \; i\frac{\hbar}{2} \; \underbrace{\langle 0|\, (\cancel{a^\dagger} - a)(\cancel{a} + a^\dagger)\, |0\rangle}_{\langle 0| -aa^\dagger |0\rangle = -1}$$

$$= \frac{\hbar}{2m\omega} \left(\cos\omega t - i \sin\omega t \right)$$

$$= \frac{\hbar}{2m\omega} \; e^{-i\omega t} \qquad \text{(same)}$$

Problem 4

$V(x) = -\lambda\delta(x) \qquad \lambda > 0$

$$-\frac{\hbar^2}{2m}\partial_x^2\psi - \lambda\delta(x)\psi = E\psi$$

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\partial_x^2\psi\, dx - \lambda\int_{-\varepsilon}^{\varepsilon}\delta(x)\psi\, dx = E\int_{-\varepsilon}^{\varepsilon}\psi\, dx$$

$$-\frac{\hbar^2}{2m}\left(\psi'(0^+) - \psi'(0^-)\right) - \lambda\psi(0) = 0 \qquad (\varepsilon \to 0)$$

$$\psi'(0^+) - \psi'(0^-) = -\frac{2m\lambda}{\hbar^2}\psi(0)$$

$\underline{x \neq 0}$ $$\partial_x^2\psi = -\frac{2mE}{\hbar^2}\psi = K^2\psi \qquad K = \sqrt{-\frac{2mE}{\hbar^2}}$$

$(E<0)$

$$\psi = Ae^{-Kx} \qquad (x>0)$$

$$\psi = Be^{Kx} \qquad (x<0)$$

$$\psi(0^+) = \psi(0^-) \Rightarrow A = B$$

$$\psi'(0^+) = -KA \qquad \psi'(0^-) = KB = KA$$

$$\psi'(0^+) - \psi'(0^-) = -2K\cancel{A} = -\frac{2m\lambda}{\hbar^2}\cancel{A}$$

$$K = \frac{m\lambda}{\hbar^2}$$

$$\psi = A\, e^{-\frac{m\lambda}{\hbar^2}|x|}$$

$$\int_{-\infty}^{\infty} |\psi|^2 dx = A^2\, 2\int_0^{\infty} e^{-\frac{2m\lambda}{\hbar^2}x} dx = \cancel{2}A^2 \frac{\hbar^2}{\cancel{2}m\lambda} = 1$$

$$\frac{2m\lambda x}{\hbar^2}$$

$$A = \frac{\sqrt{m\lambda}}{\hbar}$$

$$\boxed{\psi = \frac{\sqrt{m\lambda}}{\hbar}\, e^{-\frac{m\lambda}{\hbar^2}|x|}}$$

check of units $\quad \delta(x) \sim 1/L \quad \Rightarrow [\lambda] = E \cdot L$

$$\frac{m\lambda}{\hbar^2} = \frac{mc^2\lambda}{\hbar^2 c^2} = \frac{MeV\ MeV fm}{(MeV fm)^2} = \frac{1}{fm} = \frac{1}{L} \quad ✓$$

$$\left[\frac{\sqrt{m\lambda}}{\hbar}\right] = \sqrt{L} \quad ✓ \qquad \int |\psi|^2 dx = 1$$

$[\psi] = \sqrt{L}$ in 1-dim.

Now $\quad H = \dfrac{p^2}{2m}$

$$\psi(t=0) = \frac{\sqrt{m\lambda}}{\hbar} e^{-\frac{m\lambda}{\hbar^2}|x|} = \frac{1}{\sqrt{x_0}} e^{-\frac{|x|}{x_0}}$$

$$\boxed{x_0 = \frac{\hbar^2}{m\lambda}} \qquad [x_0] = L.$$

$\psi(t) = ?$

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle$$

$$\langle x|\psi(t)\rangle = \langle x| e^{-\frac{iHt}{\hbar}} |\psi(0)\rangle$$

$$= \langle x| e^{-\frac{iHt}{\hbar}} \int_{-\infty}^{\infty} dp \, |p\rangle \langle p|\psi(0)\rangle$$

$$e^{-\frac{iHt}{\hbar}} |p\rangle = e^{-\frac{ip^2 t}{2m\hbar}} |p\rangle$$

$$\langle x|\psi(t)\rangle = \int_{-\infty}^{\infty} dp \, e^{-\frac{ip^2 t}{2m\hbar}} \langle x|p\rangle \int_{-\infty}^{\infty} dx' \, \langle p|x'\rangle \langle x'|\psi(0)\rangle$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}}$$

$$\langle x|\psi(t)\rangle = \int_{-\infty}^{\infty} \frac{dp\, dx'}{2\pi\hbar}\, e^{\frac{ip(x-x')}{\hbar}}\, e^{-\frac{ip^2 t}{2m\hbar}}\, \psi(x', t=0)$$

$$= \int_{-\infty}^{\infty} \frac{dp dx'}{2\pi\hbar}\, e^{\frac{ip(x-x')}{\hbar}}\, e^{-\frac{ip^2 t}{2m\hbar}}\, \frac{1}{\sqrt{x_0}}\, e^{-\frac{|x'|}{x_0}}$$

2 options: we integrate first x' then p; " " " p " x'

$$\int_{-\infty}^{\infty} dx'\, e^{-\frac{ipx'}{\hbar} - \frac{|x'|}{x_0}} = \int_0^{\infty} dx'\, e^{-\frac{ipx'}{\hbar} - \frac{x'}{x_0}} + \int_{-\infty}^{0} dx'\, e^{-\frac{ipx'}{\hbar} + \frac{x'}{x_0}}$$

$$= \left. \frac{e^{-\frac{ipx'}{\hbar} - \frac{x'}{x_0}}}{-\frac{ip}{\hbar} - \frac{1}{x_0}} \right|_0^{\infty} + \left. \frac{e^{-\frac{ipx'}{\hbar} + \frac{x'}{x_0}}}{-\frac{ip}{\hbar} + \frac{1}{x_0}} \right|_{-\infty}^{0}$$

$$= \frac{1}{\frac{ip}{\hbar} + \frac{1}{x_0}} + \frac{1}{-\frac{ip}{\hbar} + \frac{1}{x_0}} = \frac{2/x_0}{\frac{1}{x_0^2} + p^2/\hbar^2} = \frac{2x_0}{1 + \frac{x_0^2}{\hbar^2} p^2}$$

$$\langle x|\psi(t)\rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{\frac{ipx}{\hbar}}\, e^{-\frac{ip^2 t}{2m\hbar}}\, \frac{1}{\sqrt{x_0}}\, \frac{2x_0}{1 + \frac{x_0^2}{\hbar^2} p^2}$$

define $k = \frac{p x_0}{\hbar}$

$$\langle x|\psi(t)\rangle = \frac{\cancel{\hbar}}{\cancel{x_0}}\, \frac{\cancel{2} \cancel{x_0}}{\sqrt{x_0}}\, \frac{1}{\cancel{2}\pi \cancel{\hbar}} \int_{-\infty}^{\infty} dk\, \frac{e^{ik\frac{x}{x_0} - i\frac{k^2 \hbar t}{2m x_0^2}}}{1 + k^2}$$

$$\int_0^\infty d\lambda\, e^{-\alpha\lambda} = \frac{-e^{-\alpha\lambda}}{\alpha}\Big|_0^\infty = \frac{1}{\alpha} \Rightarrow \frac{1}{1+k^2} = \int_0^\infty d\lambda\, e^{-\lambda-\lambda k^2}$$

$$\langle x|\psi(t)\rangle = \frac{\cancel{\hbar}}{\pi\sqrt{x_0}} \int_{-\infty}^{\infty} dk \int_0^\infty d\lambda\; e^{-\lambda-\lambda k^2 + ik\frac{x}{x_0} - \frac{i\hbar t}{2m x_0^2}k^2}$$

$$= \frac{\cancel{\hbar}}{\pi\sqrt{x_0}} \int_0^\infty d\lambda\, e^{-\lambda}\sqrt{\frac{\pi}{\lambda+\frac{i\hbar t}{2mx_0^2}}}\; e^{-\frac{x^2}{x_0^2}\frac{1}{4\left(\lambda+\frac{i\hbar t}{2mx_0^2}\right)}}$$

$$\lambda + \frac{i\hbar t}{2mx_0^2} = \mu^2 \qquad d\lambda = 2\mu\, d\mu$$

$$\langle x|\psi(t)\rangle = \frac{\cancel{\hbar}}{\pi\sqrt{x_0}}\int_{\bullet} 2\cancel{\mu}\,d\mu\, \frac{\sqrt{\pi}}{\cancel{\mu}}\; e^{-\mu^2+\frac{i\hbar t}{2mx_0^2}}\; e^{-\frac{x^2}{4x_0^2\mu^2}}$$

Consider the error function $\Phi(y) = \frac{2}{\sqrt{\pi}}\int_0^y e^{-(y')^2}dy'$; $\boxed{\partial_y\Phi(y) = \frac{2e^{-y^2}}{\sqrt{\pi}}}$

$\boxed{\Phi(\infty) = 1}$

$$\partial_\mu \Phi\left(\mu+\frac{x}{2x_0\mu}\right) = \frac{2}{\sqrt{\pi}} e^{-\left(\mu+\frac{x}{2x_0\mu}\right)^2}\left(1-\frac{x}{2x_0\mu^2}\right) = \frac{2}{\sqrt{\pi}}e^{-\mu^2-\frac{x^2}{4x_0^2\mu^2}} e^{-\frac{x}{x_0}}\left(1-\frac{x}{2x_0^2\mu^2}\right)$$

$$\partial_\mu\Phi\left(\mu-\frac{x}{2x_0\mu}\right) = \frac{2}{\sqrt{\pi}}\, e^{-\left(\mu-\frac{x}{2x_0\mu}\right)^2}\left(1+\frac{x}{2x_0\mu^2}\right) = \frac{2}{\sqrt{\pi}}\, e^{-\mu^2-\frac{x^2}{4x_0^2\mu^2}}\, e^{\frac{x}{x_0}}\left(1+\frac{x}{2x_0^2\mu^2}\right)$$

$$\frac{\sqrt{\pi}}{2}e^{x/x_0}\,\partial_\mu\Phi\left(\mu+\frac{x}{2x_0\mu}\right) + \frac{\sqrt{\pi}}{2}\, e^{-\frac{x}{x_0}}\,\partial_\mu\Phi\left(\mu-\frac{x}{2x_0\mu}\right) = 2\,e^{-\mu^2-\frac{x^2}{4x_0^2\mu^2}}$$

$$e^{-\mu^2-\frac{x^2}{4x_0^2\mu^2}} = \partial_\mu\left[\frac{\sqrt{\pi}}{4}\left(e^{\frac{x}{x_0}}\,\Phi\left(\mu+\frac{x}{2x_0\mu}\right) + e^{-\frac{x}{x_0}}\,\Phi\left(\mu-\frac{x}{2x_0\mu}\right)\right)\right]$$

$$\langle x|\psi(t)\rangle = \frac{2}{\sqrt{\cancel{\pi}x_0}}\, e^{\frac{i\hbar t}{2mx_0^2}}\, \frac{\cancel{\sqrt{\pi}}}{4}\left[e^{\frac{x}{x_0}}\,\Phi\left(\mu+\frac{x}{2x_0\mu}\right)+e^{-\frac{x}{x_0}}\,\Phi\left(\mu-\frac{x}{2x_0\mu}\right)\right]$$

$$= \frac{e^{\frac{i\hbar t}{2mx_0^2}}}{2\sqrt{x_0}}\left[e^{\frac{x}{x_0}}\,\Phi\left(\frac{2x_0\left(\lambda+\frac{i\hbar t}{2mx_0^2}\right)+x}{2x_0\sqrt{\lambda+\frac{i\hbar t}{2mx_0^2}}}\right)+(x\to -x)\right]_{\lambda=0}^{\lambda=\infty}$$

$$\lambda\to\infty \quad \Phi(\infty)=1.$$

$$= \frac{e^{\frac{i\hbar t}{2mx_0^2}}}{2\sqrt{x_0}}\left[e^{\frac{x}{x_0}}\left(1-\Phi\left(\frac{\frac{i\hbar t}{mx_0}+x}{2x_0\sqrt{\frac{i\hbar t}{2mx_0^2}}}\right)\right)+(x\to -x)\right]$$

$$\longrightarrow \sqrt{\frac{i\hbar t\,4\cancel{x_0^2}}{2m\cancel{x_0^2}}}=\sqrt{\frac{2i\hbar t}{m}}$$

$$\boxed{\langle x|\psi(t)\rangle = \frac{e^{\frac{i\hbar t}{2mx_0^2}}}{2\sqrt{x_0}}\left[e^{\frac{x}{x_0}}\left(1-\Phi\left(\frac{x+\frac{i\hbar t}{mx_0}}{\sqrt{\frac{2i\hbar t}{m}}}\right)\right)+e^{-\frac{x}{x_0}}\left(1-\Phi\left(\frac{-x+\frac{i\hbar t}{mx_0}}{\sqrt{\frac{2i\hbar t}{m}}}\right)\right)\right]}$$

Alternatively we integrate p first.

$$\int_{-\infty}^{\infty} dp\, e^{\frac{ip(x-x')}{\hbar}} e^{-\left(\frac{it}{2m\hbar}\right)p^2} = \sqrt{\frac{\pi\, 2m\hbar}{it}}\; e^{-\frac{(x-x')^2}{\hbar^{\cancel{2}}}\frac{2m\cancel{\hbar}}{4it}}$$

$$= \sqrt{\frac{2\pi m\hbar}{it}}\; e^{-\frac{m}{2i\hbar t}(x-x')^2} = \sqrt{\frac{2\pi m\hbar}{it}}\; e^{\frac{im}{2\hbar t}(x-x')^2}$$

$$\langle x|\psi(t)\rangle = \int_{-\infty}^{\infty}\frac{dx'}{2\pi\hbar}\frac{1}{\sqrt{x_0}}e^{-\frac{|x'|}{x_0}}\sqrt{\frac{2\pi m\hbar}{it}}\; e^{\frac{im}{2\hbar t}(x-x')^2}$$

$$= \sqrt{\frac{m}{2\pi\hbar i t x_0}}\left[\int_0^{\infty} dx'\, e^{-\frac{x'}{x_0}} e^{\frac{im}{2\hbar t}(x-x')^2} + \int_{-\infty}^{0} dx'\, e^{\frac{x'}{x_0}+\frac{im}{2\hbar t}(x-x')^2}\right.$$

$$x' \to -x' \quad \int_0^{\infty} e^{-\frac{x'}{x_0}+\frac{im}{2\hbar t}(x+x')^2}$$

$$= \sqrt{\frac{m}{2\pi\hbar i t x_0}}\left[\int_0^{\infty} dx'\, e^{-\frac{x'}{x_0}+\frac{im}{2\hbar t}(x-x')^2} + (x \to -x)\right].$$

$$\int_0^{\infty} dx'\, e^{-\frac{x'}{x_0}+\frac{im}{2\hbar t}(x-x')^2} = ?$$

$$\frac{im}{2\hbar t}(x-x')^2 - \frac{x'}{x_0} = \frac{im}{2\hbar t}\left[x^2 - 2xx' + x'^2 + \frac{2i\hbar t}{m x_0}x'\right] =$$

$$= \frac{im}{2\hbar t}\left[\left(x'-x+\frac{i\hbar t}{mX_0}\right)^2 - \left(\frac{i\hbar t}{mX_0}\right)^2 + \frac{2x i\hbar t}{mX_0}\right]$$

$$= \frac{im}{2\hbar t}\left(x'-x+\frac{i\hbar t}{mX_0}\right)^2 + \frac{i\cancel{m}}{2\cancel{\hbar t}}\frac{\hbar^{\cancel{2}} t^{\cancel{2}}}{m^2 X_0^2} - \frac{x}{X_0}$$

$$\int_0^\infty dx'\, e^{-\frac{x'}{X_0}+\frac{im}{2\hbar t}(x-x')^2} = \int_0^\infty dx'\, e^{\frac{im}{2\hbar t}\left(x'-x+\frac{i\hbar t}{mX_0}\right)^2} e^{-\frac{x}{X_0}+\frac{i\hbar t}{2mX_0^2}} = ⓐ$$

$$\partial_{x'}\Phi\left(\sqrt{\frac{m}{2i\hbar t}}\left(x'-x+\frac{i\hbar t}{mX_0}\right)\right) = \frac{2}{\sqrt{\pi}} e^{-\frac{m}{2i\hbar t}\left(x'-x+\frac{i\hbar t}{mX_0}\right)^2}\sqrt{\frac{m}{2i\hbar t}}$$

$$ⓐ = \sqrt{\frac{2i\hbar t}{m}}\, e^{-\frac{x}{X_0}+\frac{i\hbar t}{2mX_0^2}}\frac{\sqrt{\pi}}{2}\,\Phi\left(\sqrt{\frac{m}{2i\hbar t}}\left(x'-x+\frac{i\hbar t}{mX_0}\right)\right)\Bigg|_0^\infty$$

$$= \sqrt{\frac{2i\hbar t}{m}}\,\frac{\sqrt{\pi}}{2}\, e^{-\frac{x}{X_0}+\frac{i\hbar t}{2mX_0^2}}\left(1-\Phi\left(\sqrt{\frac{m}{2i\hbar t}}\left(-x+\frac{i\hbar t}{mX_0}\right)\right)\right)$$

$$\langle x|\psi(t)\rangle = \sqrt{\frac{\cancel{m}}{\cancel{2\pi\hbar i t}\, X_0}}\sqrt{\frac{\cancel{2i\hbar t}}{\cancel{m}}}\,\frac{\sqrt{\cancel{\pi}}}{2}\, e^{-\frac{x}{X_0}+\frac{i\hbar t}{2mX_0^2}}\left(1-\Phi\right) + (x\to -x)$$

$$= \frac{e^{\frac{i\hbar t}{2mX_0^2}}}{2\sqrt{X_0}}\left[e^{-\frac{x}{X_0}}\left(1-\Phi\left(\frac{-x+\frac{i\hbar t}{mX_0}}{\sqrt{\frac{2i\hbar t}{m}}}\right)\right) + (x\to -x)\right]$$

$$\langle x|\phi(t)\rangle = \frac{e^{\frac{i\hbar t}{2mx_0^2}}}{2\sqrt{x_0}}\left[e^{-\frac{x}{x_0}}\left(1-\Phi\left(\frac{-x+\frac{i\hbar t}{mx_0}}{\sqrt{\frac{2i\hbar t}{m}}}\right)\right) + \right.$$

$$\left. + e^{\frac{x}{x_0}}\left(1-\Phi\left(\frac{x+\frac{i\hbar t}{mx_0}}{\sqrt{\frac{2i\hbar t}{m}}}\right)\right)\right]$$

Same as before.