# Homework V

## Problem 1

a) $$R(\alpha,\hat{y})\,|l, l_z = l\rangle = e^{-\frac{i}{\hbar}\alpha L_y}\,|l, l_z = l\rangle$$

$$\underset{\text{small }\alpha}{\simeq} \left(1 - \frac{i}{\hbar}\alpha L_y + \frac{1}{2}\left(-\frac{i\alpha}{\hbar}\right)^2 L_y^2\,|l, l_z = l\rangle + \mathcal{O}(\alpha^3)\right)$$

$$R(\alpha,\hat{y})\,|l,l\rangle \cong |l,l\rangle - \frac{i\alpha}{\hbar} L_y\,|ll\rangle - \frac{\alpha^2}{2}\left(\frac{L_y}{\hbar}\right)^2 |l,l\rangle + \mathcal{O}(\alpha^3)$$

$$\left.\begin{aligned} L_+ &= L_x + iL_y \\ L_- &= L_x - iL_y \end{aligned}\right\} \Rightarrow L_y = \frac{L_+ - L_-}{2i}$$

$l_+|ll\rangle = 0$

$$R(\alpha,\hat{y})\,|l,l\rangle \simeq |l,l\rangle - \frac{\alpha}{2}(l_+ - l_-)\,|l,l\rangle + \frac{\alpha^2}{8}(l_+ - l_-)^2\,|l,l\rangle + \mathcal{O}(\alpha^3)$$

$$\simeq |l,l\rangle \mp \frac{\alpha}{2} l_-\,|l,l\rangle + \frac{\alpha^2}{8}\left(l_-^2 - l_+l_-\right)|l,l\rangle + \ldots \qquad \underbrace{l_+l_+}_{\to 0} - l_+l_- - \underbrace{l_-l_+}_{\to 0} + l_-l_-$$

$$l_+l_- = \underbrace{[l_+, l_-]}_{2l_z} + l_-l_+$$

$$\simeq |l,l\rangle + \frac{1}{2}\alpha l_-\,|l,l\rangle + \frac{\alpha^2}{8}\left(l_-^2 - 2l_z\right)|l,l\rangle + \mathcal{O}(\alpha^3)$$

$$l_-\,|l,l\rangle = \sqrt{l(l+1) - l(l-1)}\;|l, l-1\rangle = \sqrt{2l}\;|l, l-1\rangle$$

$$\cancel{l^2} + l - \cancel{l^2} + l$$

$$l_-^2\,|l,l\rangle = \sqrt{2l}\;l_-\,|l,l-1\rangle = \sqrt{2l}\sqrt{l(l+1) - (l-1)(l-2)}\;|l,l-2\rangle = \sqrt{2l}\sqrt{4l-2}\;|l,l-2\rangle$$

$$\cancel{l^2} + l - \cancel{l^2} + 3l - 2 \qquad = 2\sqrt{l(2l-1)}\;|l,l-2\rangle$$

thus

$$\boxed{\begin{aligned} R(\alpha,\hat{y}) &\simeq |l,l\rangle + \alpha\sqrt{\frac{l}{2}}\,|l,l-1\rangle + \frac{\alpha^2}{8}\,2\sqrt{l(2l-1)}\,|l,l-2\rangle - \frac{\alpha^2}{4}l\,|l,l\rangle + \mathcal{O}(\alpha^3) \\ &\simeq |l,l\rangle + \alpha\sqrt{\frac{l}{2}}\,|l,l-1\rangle + \frac{1}{4}\alpha^2\sqrt{l(2l-1)}\,|l,l-2\rangle - \frac{\alpha^2}{4}l\,|l,l\rangle + \mathcal{O}(\alpha^3)\end{aligned}}$$

or

$$\langle l,l|\,d^{(l)}(\alpha)\,|l\,l\rangle \simeq 1 - \frac{\alpha^2}{4}l \quad\longrightarrow\quad \boxed{\begin{aligned}P(l_z=l) &\simeq \left(1-\frac{\alpha^2}{4}l\right)^2 \\ &\simeq 1-\frac{\alpha^2}{2}l\end{aligned}}$$

$$\langle l,l-1|\,d^{(l)}(\alpha)\,|l\,l\rangle \simeq \alpha\sqrt{\frac{l}{2}}$$

$$\langle l,l-2|\,d^{(l)}(\alpha)\,|l\,l\rangle \simeq \frac{1}{4}\alpha^2\sqrt{l(2l-1)}$$

checks

$$c\theta \simeq 1-\frac{1}{2}\theta^2 \qquad s\theta \simeq \theta$$

take $l=2$

$$d^{(2)}_{22} \simeq 1-\frac{\alpha^2}{4}\cdot 2 = 1-\frac{1}{2}\alpha^2 \quad ; \quad d^2_{22} = \left(\frac{1+c\theta}{2}\right)^2 \simeq \left(\frac{1}{2}+\frac{1}{2}-\frac{1}{4}\theta^2\right)^2 = (1-\frac{1}{4}\theta^2)^2 \simeq 1-\frac{1}{2}\theta^2 \checkmark$$

$$d^{(2)}_{12} \simeq \alpha \quad ; \quad d^{(2)}_{12} = -d^{(2)}_{21} = \frac{1+c\theta}{2}s\theta \simeq \theta \checkmark$$

$$d^{(2)}_{02} \simeq \frac{\alpha^2}{4}\sqrt{2\times 3} = \frac{\sqrt{6}}{4}\alpha^2 = \sqrt{\frac{3}{8}}\alpha^2 \; ; \quad d^{(2)}_{02} = \frac{\sqrt{6}}{4}\sin^2\theta \simeq \frac{\sqrt{6}}{4}\theta^2 \checkmark$$

From Sakurai

$$d^j_{jj} = \frac{\sqrt{\cancel{(2j)!\,(2j)!}}}{\cancel{(2j)!}}\left(\cos\frac{\beta}{2}\right)^{2j}\left(s\frac{\beta}{2}\right)^{\cancel{0}\,1} = \left(1-\frac{1}{8}\beta^2\right)^{2j} \simeq 1-\frac{j}{4}\beta^2 \checkmark$$

$$d^j_{j-1\,j} = \frac{\sqrt{(2j)!\,(2j-1)!}}{(2j-1)!}\left(c\frac{\beta}{2}\right)^{2j-2+1}\left(s\frac{\beta}{2}\right)^{2-1} \simeq \sqrt{2j}\,\frac{\beta}{2} = \sqrt{\frac{j}{2}}\beta \checkmark$$

$$d^j_{j-2\,j} = \frac{\sqrt{(2j)!\,(2j-2)!\,2!}}{(2j-2)!\,2!}\left(c\frac{\beta}{2}\right)^{2j-4+2}\left(s\frac{\beta}{2}\right)^{4-2} = \frac{\sqrt{2j(2j-1)}}{\sqrt{2}}\left(s\frac{\beta}{2}\right)^2\cdot\left(c\frac{\beta}{2}\right)^{2j-2} = \frac{\sqrt{j(2j-1)}}{4}\beta^2$$

Problem 2

$$\psi(\vec{r}) = (x+y+3z)\,f(r)$$

$$= (s\theta c\phi + s\theta s\phi + 3c\theta)\, f(r)\cdot r$$

$$= s\theta\left(\frac{e^{i\phi}+e^{-i\phi}}{2} + \frac{e^{i\phi}-e^{-i\phi}}{2i}\right) r f(r) + 3c\theta\, r f(r)$$

$$= \frac{1-i}{2}\, s\theta\, e^{i\phi}\, r f(r) + \frac{1+i}{2}\, s\theta\, e^{-i\phi}\, r f(r) + 3c\theta\, r f(r)$$

$$= \frac{1-i}{2}\left(-\sqrt{\frac{8\pi}{3}}\right) r f(r)\, Y_{11}(\theta,\varphi) + \frac{1+i}{2}\sqrt{\frac{8\pi}{3}}\, r f(r)\, Y_{1-1}(\theta,\varphi)$$

$$+ 3\sqrt{\frac{4\pi}{3}}\, Y_{10}\, r f(r)$$

a) Since it only contains $Y_{\ell m}$ with $\ell = 1$ $[m=-1,0,1]$ then it is an eigenstate of $L^2$; eigenvalue $\hbar^2 1(1+1) = 2\hbar^2$

$\boxed{\ell = 1}$

b) $\hat{l}_z \to -1, 0, 1$

$$|\psi\rangle = -\sqrt{\frac{8\pi}{3}}\,\frac{1-i}{2}\, r f\, |11\rangle + \frac{1+i}{2}\sqrt{\frac{8\pi}{3}}\, r f\, |1-1\rangle + 3\sqrt{\frac{4\pi}{3}}\, r f\, |10\rangle$$

We can integrate $\int r^2 dr\, r f(r) = A$

$$|\psi\rangle = \alpha|11\rangle + \beta|1-1\rangle + \gamma|10\rangle$$

$$\alpha = -\sqrt{\frac{8\pi}{3}}\,\frac{1-i}{2}\,A$$

$$\beta = \sqrt{\frac{8\pi}{3}}\,\frac{1+i}{2}\,A$$

$$\gamma = 3\sqrt{\frac{4\pi}{3}}\,A$$

$$P(l_z=+1) = |\alpha|^2 = \frac{8\pi}{3}\,\frac{1}{2}\,|A|^2 = \frac{4\pi}{3}|A|^2$$ (4)

$$P(l_z=-1) = |\beta|^2 = \frac{4\pi}{3}|A|^2$$

$$P(l_z=0) = 9\,\frac{4\pi}{3}\,|A|^2 = 12\pi\,|A|^2$$

$$\left(\frac{4\pi}{3}+\frac{4\pi}{3}+12\pi\right)|A|^2 = 1 \quad\Rightarrow\quad \frac{8\pi}{3}+12\pi = \frac{8+36}{3}\pi = \frac{44}{3}\pi$$

$$|A| = \sqrt{\frac{3}{44\pi}}$$

$$P_{+1} = \frac{4\cancel{\pi}}{\cancel{3}}\,\frac{\cancel{3}}{44\cancel{\pi}} = \frac{1}{11} \qquad P_{-1} = \frac{1}{11} \qquad P_0 = \underset{3}{\cancel{12}}\cancel{\pi}\,\frac{3}{\underset{11}{\cancel{44}}\cancel{\pi}} = \frac{9}{11}$$

$$\boxed{P(l_z=+1) = \frac{1}{11}\;;\quad P(l_z=-1) = \frac{1}{11}\;;\quad P(l_z=0) = \frac{9}{11}}$$

c) $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \quad\Rightarrow\quad V\psi = E\psi + \frac{\hbar^2}{2m}\nabla^2\psi$

$$\boxed{V = E + \frac{\hbar^2}{2m}\,\frac{\nabla^2\psi}{\psi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\,\partial_r(r^2\partial_r\psi) + \frac{\hat{L}^2}{\hbar^2 r^2}$$

Also $\quad -\frac{\hbar^2}{2m}\partial_r^2\chi + \left(V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}\right)\chi = E\chi$

$\underline{l=1}$ $\qquad \psi = \frac{1}{r}\chi$

$\chi = r^2 f$

$$-\frac{\hbar^2}{2m}\partial_r^2 \chi + V\chi + \frac{\hbar^2}{mr^2}\chi = E\chi$$

$$V = E - \frac{\hbar^2}{mr^2} + \frac{\hbar^2}{2m}\frac{\partial_r^2 (r^2 f)}{r^2 f}$$

$$\partial_r^2 (r^2 f) = \partial_r (2rf + r^2 f') = 2f + 2rf' + 2rf' + r^2 f''$$
$$= 2f + 4rf' + r^2 f''$$

$$V = E - \frac{\hbar^2}{mr^2} + \frac{\hbar^2}{2m}\left(\frac{2f + 4rf' + r^2 f''}{r^2 f}\right)$$
$$= E - \cancel{\frac{\hbar^2}{mr^2}} + \cancel{\frac{\hbar^2}{mr^2}} + \frac{2\hbar^2 f'}{mrf} + \frac{\hbar^2}{2m}\frac{f''}{f}$$

$$V = E + \frac{\hbar^2}{2m}\left(\frac{f''}{f} + \frac{4}{r}\frac{f'}{f}\right)$$

## Problem 3

$$|l=2, l_z=0\rangle \rightarrow |\psi\rangle = e^{-\frac{i}{\hbar}\beta L_y} |l=2, l_z=0\rangle$$

a) Possible values $l_z = -2, -1, 0, 1, 2$ (in principle)

$$P_{l_z} = \left|\langle 2, l_z| e^{-\frac{i}{\hbar}\beta L_y} |l=2, 0\rangle\right|^2 = \left| d^{(2)}_{l_z 0}(\beta) \right|^2$$

$$P_{-2} = |d^{(2)}_{-20}(\beta)|^2 = \frac{6}{16} \sin^4\beta = P_2$$

$$P_{-1} = |d^{(2)}_{-10}(\beta)|^2 = \cancel{\left(\frac{1+c\beta}{2}\right)^2 \sin^2\beta = P_1} = \frac{3}{2} \sin^2\beta\, c^2\beta = P_1$$

$$P_0 = |d_{00}(\beta)|^2 = \left(\frac{3}{2} c^2\beta - \frac{1}{2}\right)^2$$

**Check**

$$P_{-2} + P_2 + P_{-1} + P_1 + P_0 = \frac{12}{16} s^4\beta + 3 s^2\beta c^2\beta + \left(\frac{3}{2} c^2\beta - \frac{1}{2}\right)^2$$

$$= \frac{3}{4} s^4\beta + 3 s^2\beta c^2\beta + \frac{9}{4} c^4\beta - \frac{3}{2} c^2\beta + \frac{1}{4}$$

$$= \frac{1}{4}\left(3 s^4\beta + 12 s^2\beta c^2\beta + 9 c^4\beta - 6 c^2\beta + 1\right)$$

$$= \frac{1}{4}\left(3(1-c^2\beta)^2 + 12(1-c^2\beta)c^2\beta + 9c^4\beta - 6c^2\beta + 1\right)$$

$$= \frac{1}{4}\left[3 - \cancel{6c^2\beta} + \cancel{3c^4\beta} + \cancel{12c^2\beta} - \cancel{12c^4\beta} + \cancel{9c^4\beta} - \cancel{6c^2\beta} + 1\right]$$

$$= \frac{1}{4}[4] = 1 \checkmark$$

Problem 4

$l_{1z}\ l_{2z}$

$|+1\ +1\rangle = |2\ +2\rangle$

$l_- \rightarrow$

7

$l_-|1+1\rangle = \sqrt{1\times2-0}\ |1\,0\rangle = \sqrt{2}\,|1\,0\rangle$

$l_-|1\,0\rangle = \sqrt{2-0}\ |1-1\rangle = \sqrt{2}\,|1-1\rangle$

$l_-|2+2\rangle = \sqrt{6-2}\ |2+1\rangle = 2\,|2+1\rangle$

$l_-|2+1\rangle = \sqrt{6-0}\ |2\,0\rangle = \sqrt{6}\,|2\,0\rangle$

$l_-|2\,0\rangle = \sqrt{6-0}\ |2-1\rangle = \sqrt{6}\,|2-1\rangle$

$l_-|2-1\rangle = \sqrt{6-2}\ |2-2\rangle = 2\,|2-2\rangle$

$l_-|+1+1\rangle = \sqrt{2}\,|1\,0\rangle + \sqrt{2}\,|0\,1\rangle$ ; $l_-|2+2\rangle = 2\,|2+1\rangle$

$|2+1\rangle = \frac{1}{\sqrt{2}}|1\,0\rangle + \frac{1}{\sqrt{2}}|0\,1\rangle$

$l_-$

$\sqrt{6}\,|2\,0\rangle = \frac{1}{\sqrt{2}}\left(\sqrt{2}\,|0\,0\rangle + \sqrt{2}\,|1-1\rangle + \sqrt{2}\,|-1\,1\rangle + \sqrt{2}\,|0\,0\rangle\right) = |1-1\rangle + |-1\,1\rangle + 2|0\,0\rangle$

$|2\,0\rangle = \frac{1}{\sqrt{6}}|1-1\rangle + \frac{2}{\sqrt{6}}|0\,0\rangle + \frac{1}{\sqrt{6}}|-1\,1\rangle$

$\sqrt{6}\,|2-1\rangle = \frac{1}{\sqrt{6}}\left(\sqrt{2}\,|0-1\rangle + \underbrace{2\sqrt{2}}\,|-1\,0\rangle + 2\sqrt{2}\,|0-1\rangle + \underbrace{\sqrt{2}\,|-1\,0\rangle}\right)$

$= \frac{3\sqrt{2}}{\sqrt{6}}|-1\,0\rangle + \frac{3\sqrt{2}}{\sqrt{6}}|0-1\rangle = \sqrt{3}\left(|-1\,0\rangle + |0-1\rangle\right)$ $\qquad \frac{3\sqrt{2}}{\sqrt{6}} = \sqrt{\frac{9\times2}{6\ 3}} = \sqrt{3}$

$|2-1\rangle = \frac{1}{\sqrt{2}}|-1\,0\rangle + \frac{1}{\sqrt{2}}|0-1\rangle$

$2|2-2\rangle = |-1-1\rangle + |-1-1\rangle \Rightarrow |2-2\rangle = |-1-1\rangle$

$|1\,+1\rangle = \frac{1}{\sqrt{2}}|10\rangle - \frac{1}{\sqrt{2}}|01\rangle$ orthogonal to $|2\,+1\rangle$

$L_-$ again

$$\sqrt{2}\,|10\rangle = \cancel{|00\rangle} + |1\,-1\rangle - |-1\,1\rangle - \cancel{|00\rangle}$$

$$|10\rangle = \frac{1}{\sqrt{2}}|1\,-1\rangle - \frac{1}{\sqrt{2}}|-1\,1\rangle$$ (is orthogonal to $|20\rangle$ ✓)

$$\sqrt{2}\,|1\,-1\rangle = |0\,-1\rangle - |-1\,0\rangle$$

$$|1\,-1\rangle = \frac{1}{\sqrt{2}}\left(|0\,-1\rangle - |-1\,0\rangle\right)$$

Finally

$$|00\rangle = \alpha|1\,-1\rangle + \beta|00\rangle + \gamma|-1\,1\rangle$$

$\langle 20|00\rangle = 0 \Rightarrow \alpha + 2\beta + \gamma = 0$, $2\alpha + 2\beta = 0$, $\beta = -\alpha$

$\langle 10|00\rangle = 0 \Rightarrow \alpha - \gamma = 0 \Rightarrow \gamma = \alpha$

$$|00\rangle = \frac{1}{\sqrt{3}}\left(|1\,-1\rangle - |00\rangle + |-1\,1\rangle\right)$$

normalization

$|2\,+2\rangle = |+1\,+1\rangle$

$|2\,+1\rangle = \frac{1}{\sqrt{2}}|+1\,0\rangle + \frac{1}{\sqrt{2}}|0\,+1\rangle$

$|20\rangle = \frac{1}{\sqrt{6}}|+1\,-1\rangle + \frac{2}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{6}}|-1\,+1\rangle$

$|2\,-1\rangle = \frac{1}{\sqrt{2}}|-1\,0\rangle + \frac{1}{\sqrt{2}}|0\,-1\rangle$

$|2\,-2\rangle = |-1\,-1\rangle$

$|1\,+1\rangle = \frac{1}{\sqrt{2}}|+1\,0\rangle - \frac{1}{\sqrt{2}}|0\,+1\rangle$

$|10\rangle = \frac{1}{\sqrt{2}}|1\,-1\rangle - \frac{1}{\sqrt{2}}|-1\,1\rangle$

$|1\,-1\rangle = \frac{1}{\sqrt{2}}|0\,-1\rangle - |-1\,0\rangle$

$|00\rangle = \frac{1}{\sqrt{3}}\left(|1\,-1\rangle - |00\rangle + |-1\,1\rangle\right)$