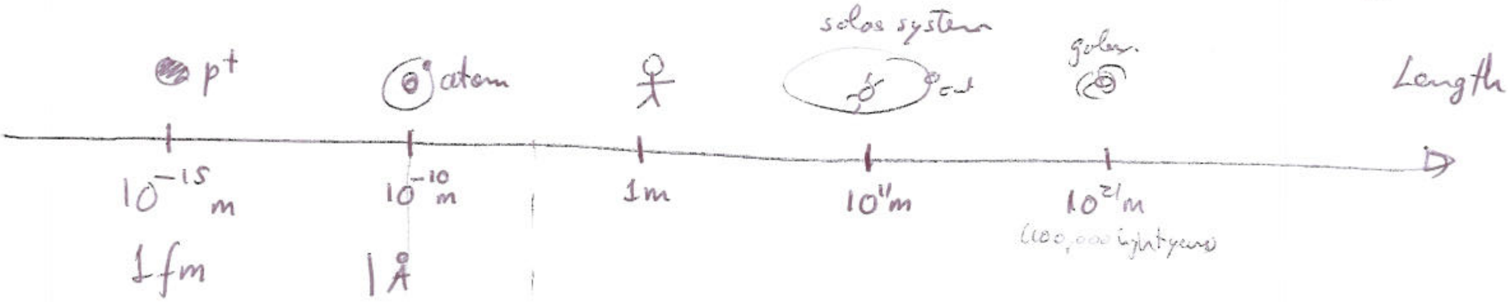
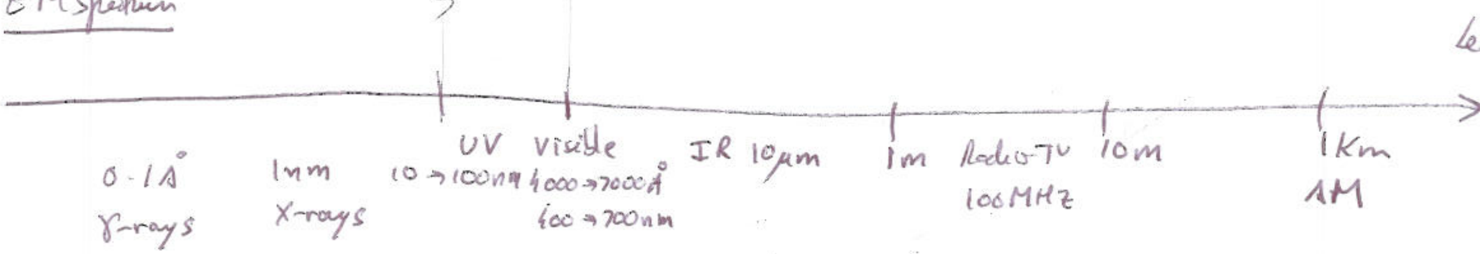
Lecture 1

Scales



EM spectrum



Constants

$\hbar c = 197$ MeV fm $\qquad$ $1 eV = 1.6\times10^{-19}\underbrace{C.V}_{J}$

$e^2 = 1.44$ MeV fm $\left(\frac{e^2}{4\pi\varepsilon_0}\right)$

$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$

$K_B T$ = energy $\qquad$ $300°K \rightarrow 0.025\,eV$

$\mathcal{E}_\gamma = h\nu = \frac{2\pi\hbar c}{\lambda} \approx \frac{1200\,\text{MeV fm}}{\lambda} \rightarrow$ $\quad 500\,nm \rightarrow 500\times10^{-9}m$

$\mathcal{E}_\gamma = \frac{1200\times10^{-15}\,\cancel{m}\,\text{MeV}}{500\,10^{-9}\,\cancel{m}} \approx 2\times10^{-6}\,\text{MeV}$ $\boxed{\approx 2eV}$ visible

$\overset{+}{\circ}\ \ \overset{-}{\circ}$ $\underbrace{\quad}_{1Å}$ $\quad \mathcal{E} = \frac{e^2}{r} = \frac{1.44\,\text{MeV}\,\overset{10^{-15}m}{\text{fm}}}{10^{-10}m} = 1.44\times10^{6}\,eV\,10^{-5} = 14\,eV \longleftrightarrow \sim 100nm \sim \underline{\underline{UV}}$

$0.025eV \rightarrow 300K$
$14eV \rightarrow 170{,}000\,K$

Uncertainty principle

$\boxed{\Delta x . \Delta p \geq \hbar}$

$\mathcal{E}_e = \frac{p^2}{2m} - \frac{e^2}{r}$

on round 

$p \sim \hbar/r$

$\mathcal{E} \sim \frac{\hbar^2}{2mr^2} - \frac{e^2}{r}$

$-\frac{\hbar^2}{mr^3} + \frac{e^2}{\cancel{r^2}} \sim 0$

$\frac{\hbar^2}{mr} \sim e^2$

$r \sim \frac{\hbar^2}{me^2}$

$\mathcal{E} \sim \frac{\hbar^2}{2m\frac{\hbar^4}{m^2e^4}} - \frac{e^2\,me^2}{\hbar^2}$

$\mathcal{E} \sim \frac{me^4}{2\hbar^2} - \frac{me^4}{\hbar^2} = -\frac{me^4}{2\hbar^2}$

$$\mathcal{E}_{g.s} \sim -\frac{me^4}{2\hbar^2} \; ; \; \frac{0.5\ \text{MeV}\ (1.44)^2\ \text{MeV}^2\text{fm}^2}{2\,(197)^2\ \text{MeV}^2\text{fm}^2}$$

$$-\frac{mc^2}{2}\frac{e^4}{(\hbar c)^2} = -\alpha^2\frac{mc^2}{2} = -\frac{0.5\ \text{MeV}}{2}\frac{1}{(137)^2} = -13.6\times 10^{-5}\ \text{MeV} = -13.6\ \text{eV}$$

$$r_0 = \frac{\hbar^2}{me^2} = \frac{(\hbar c)^2}{mc^2e^2} = \frac{(197)^2\ \text{MeV}^2\text{fm}^2}{0.5\ \text{MeV}\ 1.44\ \text{MeV fm}} = 5.4\times 10^4\times 10^{-15}\ \text{m}$$

$$= 5.4\times 10^{-11}\ \text{m}$$

$$= 0.5\times 10^{-10}\ \text{m}$$

$$\simeq \underline{0.5\ \text{Å}}$$

$$\frac{\hbar^2 c}{mc e^2} = \frac{\hbar}{mc\alpha}$$

a) $\frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$

$$\frac{\hbar^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2$$

$$-\frac{\hbar^2}{mr^3} + \frac{1}{2}m\omega^2 r = 0 \qquad \frac{1}{2}m\omega^2 r = \frac{\hbar^2}{mr^3}$$

$$r^4 = \frac{\hbar^2}{m^2\omega^2}$$

$$\mathcal{E} = \frac{\hbar^2}{2m}\frac{m\omega}{\hbar} + \frac{1}{2}\frac{m\omega^2}{m\omega}\hbar = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{2} = \hbar\omega$$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

$$\frac{\hbar^2}{2mr^2} + \frac{1}{2}m\omega^2 r^2 \rightarrow -\frac{\hbar^2}{mr^3} + m\omega^2 r = 0$$

$$m^2\omega^2 r^4 = \hbar^2 \qquad r^4 = \frac{\hbar^2}{m^2\omega^2}$$

$$\mathcal{E} = \frac{\hbar^2 m\omega}{2m\hbar} + \frac{1}{2}\frac{m\omega^2\hbar}{m\omega} = \hbar\omega \quad \left(\text{instead of } \frac{3}{2}\hbar\omega\right)$$

Stern Gerlach

Atoms of Ag (silver)

$[Kr]\ 4d^{10}\ 5s^1$; $\underbrace{[1s^2]}_{He}\underbrace{[2s^2 2p^6]}_{Ne}\underbrace{[3s^2 3p^6]}_{Ar}\underbrace{[3d^{10} 4s^2 4p^6]}_{Kr}$

spin 1/2 $\quad \vec{\mu} = -\frac{e}{mc}\vec{S} \qquad \vec{S} = \hbar/2$

$\mathcal{E} = -\mu \cdot B$

$B \sim 0.1\,T$

$\nabla B \quad 10T/cm \qquad = 0.2mm$ splitting.

magnet $\longrightarrow$ 2.5cm long.

oven heated 1000°C

$= 1273°K$

$\boxed{1T = 1\frac{V.S}{m^2}}$

$\mathcal{E}_k = \frac{1}{2}mv^2 \simeq \frac{1}{2}kT \qquad v \simeq \sqrt{\frac{kT}{m}}$

$F = -\mu\frac{\partial B}{\partial z}$

$\Delta t = \frac{\Delta x}{v} \qquad \frac{1}{2}at^2$

$\Delta z = \frac{1}{2}\frac{\mu\,\partial B/\partial z}{M}\frac{(\Delta x)^2}{v^2} = \frac{1}{2}\frac{\mu\,\partial B/\partial z}{kT}(\Delta x)^2$

$\Delta z = \frac{1}{2}\frac{e\hbar}{2m\cancel{c}}\frac{(\partial B/\partial z)}{(kT)}(\Delta x)^2$

$\frac{1}{4}\frac{\hbar c}{mc^2}\frac{1}{\partial z}\frac{e.\delta B}{(kT)}(\Delta x)^2$

$\frac{1}{4}\ e.\overset{1T}{\delta B} = 1.6\times10^{-19}C.\frac{V.S}{m^2}$

$\simeq 1eV\frac{S}{m^2}$

$\frac{1}{4}\frac{197\cancel{MeV}fm\,c}{0.5\cancel{MeV}}\frac{10}{cm}\frac{eVS}{m^2}\frac{(3.5cm)^2}{0.083\cancel{eV}}$

$\frac{10\times197\times(3.5)^2}{4\times0.5\times0.083}3\times10^8\frac{fm\,\cancel{m}\,\cancel{s}\,cm^2}{\cancel{s}\,cm\,m^2}$

$4.4\times10^5\times10^8\times10^{-15}$ cm

4.4×10^{-2} cm $\rightarrow$ 0.4mm

Stern-Gerlach

Atoms of Ag (silver)

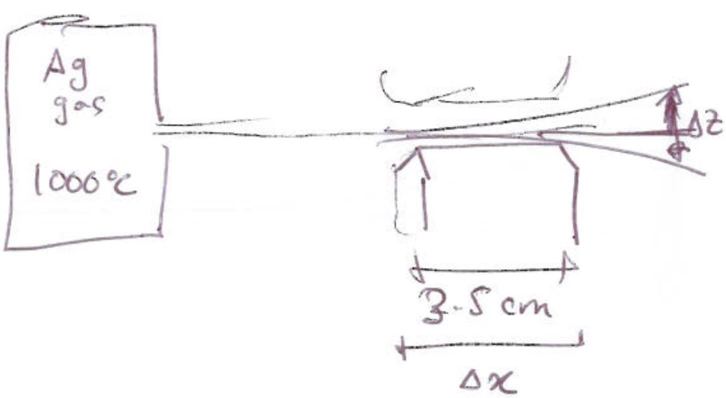
$[Kr]\,4d^{10}\,5s^1$; $\underset{He}{[1s^2]}\,\underset{Ne}{[2s^2 2p^6]}\,\underset{Ar}{[3s^2 3p^6]}\,\underset{Kr}{[3d^{10} 4s^2 4p^6]}$

Spin 1/2

$$\mu = -\frac{e}{m_e}\vec{S} \quad \text{(SI units)} \qquad \text{or} \quad \frac{e}{m_e c}\vec{S}.$$

$|\vec{S}| = \hbar/2$.

$$E = -\mu \cdot B$$



$B \sim 0.1\,T$

$\frac{\partial B}{\partial z} \sim 10\,T/cm$

$\Delta t = \frac{\Delta x}{V}$

$\frac{1}{2} M V^2 = \frac{3}{2} k_B T$

$1\,T = 1 V \cdot \frac{s}{m^2}$

$$\Delta z = \frac{1}{2} a \Delta t^2 = \frac{1}{2}\frac{F}{M}\Delta t^2 = \frac{1}{2}\mu\frac{\partial B}{\partial z}\frac{1}{M}\frac{\Delta x^2}{v^2}$$

$$= \frac{1}{2}\frac{1}{(k_B T)}\mu\frac{\partial B}{\partial z}\Delta x^2$$

$$= \frac{1}{2}\frac{1}{(k_B T)}\frac{e\hbar}{2m_e}\cdot 10\frac{T}{cm}\Delta x^2$$

$$= \frac{1}{2\cdot 3(k_B T)}\frac{e\hbar c\,c}{2mc^2}\,10\,\frac{1eV\,s}{cm\;m^2}(\Delta x)^2$$

Quantum Mechanics

Postulates or principles.

-) States of a physical system are vectors in a complex vector space.
-) The states evolves in time according to the Schrödinger eqn.
-) Observables are hermitian linear operators in such space.
-) The Only possible result of measuring an observable are the eigenvalues of the corresponding linear operator.
-) The process of measuring projects a state onto the eigenstate (or eigenspace) corresponding to the measured eigenvalue.
-) The probability of measuring a given eigenvalue is the modulus square of the projection over the corresponding eigenspace

Linear Algebra and Dirac Notation

-) vector space (complex)
-) linear operator (hermitian, unitary)
-) eigenvalues and eigenvector

-) Vector space V, set with $+$, 0 and multiplication by a scalar in $\mathbb{C}$.

$|\psi\rangle \in V \qquad |\psi\rangle$ is called a <u>ket</u>

$|\psi\rangle, \ |\phi\rangle \in V \ \rightarrow \ |\psi\rangle + |\phi\rangle \in V$

$\alpha \in \mathbb{C}, \ |\psi\rangle \in V \ \rightarrow \ \alpha|\psi\rangle \in V$

$|\psi\rangle + 0 = |\psi\rangle$

<u>example</u>

$\mathbb{C}^n$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\alpha\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} + \beta\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \beta w_1 \\ \vdots \\ \alpha v_n + \beta w_n \end{pmatrix}$$

Basically given $\alpha, \beta \in \mathbb{C}$, $|\psi\rangle, |\phi\rangle \in V \rightarrow \alpha|\psi\rangle + \beta|\phi\rangle$ is defined

<u>Linear operator</u> $\quad A : V \rightarrow V$

$\qquad |\psi\rangle \rightarrow A|\psi\rangle \in V$

and $A(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha A|\psi\rangle + \beta A|\phi\rangle$

$A(|\psi\rangle) = A|\psi\rangle$

[anti-linear $\quad A(\alpha|\psi\rangle + \beta|\phi\rangle) = \alpha^* A|\psi\rangle + \beta^* A|\phi\rangle$]

<u>example</u> matrix

$$(A.v)_i = \sum_{j=1}^{n} A_{ij} v_j \quad ; \quad \begin{pmatrix} & A & \end{pmatrix}\begin{pmatrix} v \end{pmatrix} = \begin{pmatrix} w \end{pmatrix}$$

Basis

Set of <u>linearly independent</u> vectors $|e_i\rangle$ $i=1 \ldots n$

$\hookrightarrow$ i.e. if $\exists\, \alpha_i \in \mathbb{C} \,/\, \sum \alpha_i |e_i\rangle = 0 \Rightarrow \boxed{\alpha_i = 0}$

and such that for <u>any</u> $|v\rangle \in V \;\exists\, v_i \in \mathbb{C} \,/\, |v\rangle = \sum_{i=1}^{n} v_i |e_i\rangle$

(complete)

$v_{i=1-n}$ are the components of $|v\rangle$ in the basis $|e_i\rangle$

v_i are <u>unique</u> $\quad |v\rangle = \sum \alpha_i |e_i\rangle = \sum \beta_i |e_i\rangle$

$\Rightarrow \sum (\alpha_i - \beta_i) |e_i\rangle = 0 \Rightarrow \alpha_i = \beta_i$

<u>three cases of interest:</u>

.) $i=1 \ldots n$ finite dimensional space <u>easier</u>

.) $i \in \mathbb{Z}_{>0}$ ∞ dimension but denumerable or countable basis

e.g. $\psi : \mathbb{R} \to \mathbb{C} \,/\, \int_{-\infty}^{\infty} |\psi(x)|^2 dx < \infty$

or $\psi : S^1 \to \mathbb{C}$ (periodic function on a circle).

$\sum_{i=1}^{n} \alpha_i |e_i\rangle \xrightarrow{} \;\; \sum_{i=1}^{\infty} \alpha_i |e_i\rangle$ — problems with convergence
$n \to \infty$

.) $i \to x \in \mathbb{R}$ $\quad \sum \to \int_{-\infty}^{\infty} \underline{\underline{dx}}\, \psi(x) |x\rangle$

$\hookrightarrow$ measure of integration.

We consider finite dimension and then extend by analogy.

Linear operator in a basis

we only need to know how A acts on a basis.

$$A|v\rangle = \sum_i v_i A|e_i\rangle = \sum_j \left(\sum_i a_{ji} v_i\right)|e_j\rangle$$

$$A|e_i\rangle = \sum_j a_{ji} |e_j\rangle$$

$a_{ji} \rightarrow$ $n \times n$ matrix

$v_i \rightarrow$ column vector

} so example was quite general.

However we do not need to specify a basis, or we choose different bases.

Eigenvectors of A

$|v\rangle \in V \;/\; A|v\rangle = \lambda|v\rangle \;,\; \lambda \in \mathbb{C}.$

in a basis $\quad a_{ij} v_j - \lambda \delta_{ij} v_j = 0.$

$(A - \lambda \mathbb{I}) \cdot v = 0 \qquad \det(A - \lambda \mathbb{I}) = 0 \rightarrow$ polynomial in $\mathbb{C}$ has a solution.

every matrix has at least one eigenvalue.

Inner Product (scalar product)

We construct it in two steps

First we define a new vector space. V^*, co-vectors or bras

$$|v\rangle \xrightarrow{+} (|v\rangle)^{+} = \langle v| \in V^*$$

dagger is an antilinear map $V \to V^*$

$$\alpha|v\rangle + \beta|\omega\rangle \xrightarrow{+} \alpha^*\langle v| + \beta^*\langle\omega|$$

example

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}; \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}$$

$$(\omega_1^* \cdots \omega_n^*)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \omega_1^* v_1 + \cdots + \omega_n^* v_n.$$

$$\omega^+ \cdot v$$

$$\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} \xrightarrow{+} \omega^+ = (\omega_1^* \cdots \omega_n^*)$$

column row

Now we define a bilinear map. $V^* \times V \to \mathbb{C}$

denoted as $\langle v|\omega\rangle$

$$\langle v|\left(\alpha|\omega_1\rangle + \beta|\omega_2\rangle\right) = \alpha\langle v|\omega_1\rangle + \beta\langle v|\omega_2\rangle$$

$$\left(\alpha\langle v_1| + \beta\langle v_2|\right)|\omega\rangle = \alpha\langle v_1|\omega\rangle + \beta\langle v_2|\omega\rangle$$

with the properties

$$\langle v|\omega\rangle = \langle\omega|v\rangle^* \qquad (\Rightarrow \langle v|v\rangle \text{ is real})$$

and such that

$$\langle v|v\rangle \geq 0 \quad \text{and} \quad \langle v|v\rangle = 0 \text{ only if } |v\rangle = 0.$$

Defines a positive norm $\| |v\rangle \|^2 = \langle v|v\rangle$

Example

$$\| |v\rangle \| = \sum |v_i|^2.$$

if $\langle v|v\rangle = 1$ we say $|v\rangle$ is normalized.

Orthonormal bases

basis such that $\langle e_j | e_i \rangle = \delta_{ji}$ $\nearrow \langle e_i|e_i\rangle = 1$, $\searrow \langle e_i|e_j\rangle = 0 \quad i \neq j$

given $\langle v|v\rangle \neq 1$ we can normalize $\dfrac{|v\rangle}{\| |v\rangle \|} = \dfrac{|v\rangle}{\sqrt{\langle v|v\rangle}}$

$$|v\rangle = \sum_i v_i |e_i\rangle$$

$$\boxed{\langle e_j | v\rangle = v_j}$$

exterior product $\quad |v\rangle\langle w|$

$$\left(\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} (w_1 \dots w_n) = \underbrace{\begin{pmatrix} - & - \\ - & - \\ - & - \end{pmatrix}}_{\text{matrix}} \right)$$

is an operator

$$|v\rangle \langle w | \phi \rangle = \langle w|\phi\rangle \, |v\rangle$$

what operator is

$$\sum_i |e_i\rangle\langle e_i|$$

$$\sum_i |e_i\rangle\langle e_i | v\rangle = \sum_i v_i |e_i\rangle = |v\rangle$$

So $\sum_i |e_i\rangle\langle e_i|$ is the identity! $\boxed{\mathbb{1} = \sum_i |e_i\rangle\langle e_i|}$

extremely important and useful.
in this notation..

consider A.

$$A = \sum_{ji} |e_j\rangle\langle e_j | A | e_i\rangle\langle e_i| = \sum_{ji} a_{ji} \, |e_j\rangle\langle e_i|$$

$$\boxed{a_{ji} = \langle e_j | A | e_i\rangle}$$

given a matrix we can construct: $A, A^*, A^t, A^\dagger$.

the same with an operator.

given $A : V \to V$ we can define and $A : V^\vee \to V^\wedge$ by.

$\langle \omega | A = ?$ $\quad (\langle \omega | A) | v \rangle = \langle \omega | (A | v \rangle)$.

$A(\langle \omega |) = \langle \omega | A$.

Suppose $\langle \omega | = \sum_i \omega_i \langle e_i |$

$$\langle \omega | A = \sum_i \omega_i \langle e_i | A \sum_j | e_j \rangle \langle e_j | = \sum_j \left(\sum_i \omega_i a_{ij} \right) \langle e_j |$$

$$= \sum_j (A^t \omega)_j \langle e_j |$$

Hermitian conjugate.

$(A | v \rangle)^\dagger = \langle v | A^\dagger$

Suppose $| v \rangle = \sum v_i | e_i \rangle$

$A | v \rangle = \sum a_{ij} v_j | e_i \rangle$

$$(A | v \rangle)^\dagger = \sum a_{ij}^* v_j^* \langle e_i | = \sum v_j^* \langle e_j | A^\dagger | e_i \rangle \langle e_i |$$

$$\Rightarrow \quad a_{ij}^* = \langle e_j | A^\dagger | e_i \rangle = (a^\dagger)_{ji}$$

is hermitian conjugate

$A^\dagger$ can act also on $| v \rangle \in V$

$A^\dagger | v \rangle$ is such that $\langle \omega | (A^\dagger | v \rangle) = (\langle \omega | A^\dagger) | v \rangle = (A | \omega \rangle)^\dagger | v \rangle$

$= \langle v | A | \omega \rangle^*$

A has an eigenvector.

$$A|v\rangle = \lambda|v\rangle$$

Consider $V_\perp$, namely all $|w\rangle$ / $\langle v|w\rangle = 0$.

$$A \cdot V_\perp = V_\perp$$

$$\langle v|A|w\rangle = \langle w|\underset{\widetilde{A}}{A^\dagger}|v\rangle^* = \lambda^* \langle w|v\rangle^* = \lambda^* \langle v|w\rangle = 0.$$

Then we find a new eigenvector on $V_\perp$ and so on until we covers all space.

So the eigenvector of a hermitian operator form a bases.

~~For a given eigenvalue~~ (Several eigenvectors can have the same eigenvalue)

$$\sum_i |\lambda_i\rangle\langle\lambda_i| = \mathbb{1}$$

sum with multiplicities.

If two observables commute then they can be diagonalized simultaneously.

$$A \cdot B = B \cdot A$$

Consider diagonalizing A.

$$A = \begin{pmatrix} \lambda_1 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_1 & & & & \\ & & & \lambda_2 & & & \\ & & & & \ddots & & \\ & & & & & \lambda_2 & \\ & & & & & & \ddots \\ & & & & & & & \lambda_m \\ & & & & & & & & \ddots \\ & & & & & & & & & \lambda_m \end{pmatrix}$$

We claim B is

$$\begin{pmatrix} \boxed{B_1} & 0 & 0 \\ 0 & \boxed{B_2} & 0 \\ & & \ddots \\ 0 & 0 & \boxed{B_n} \end{pmatrix}.$$

namely $V = V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$

are invariant subspaces under B.

$|v\rangle \in V_{\lambda_1} \quad A|v\rangle = \lambda_1 |v\rangle$

$B|v\rangle \in V_{\lambda_1}$ indeed $AB|v\rangle = BA|v\rangle = \lambda_1 B|v\rangle$.

now we can diagonalize B on each subspace.

since $|\omega\rangle = \sum \alpha_i |v_i\rangle$ with $|v_i\rangle \in V_{\lambda_1}$ $A|\omega\rangle = \lambda_1 \sum \alpha_i |v_i\rangle$. etc.

Viceverse if they can be diagonalized simultaneously then they commute.

the commutator tests if two operators commute.

$$AB - BA = [A, B]$$

Properties

$$[A, B] = -[B, A]$$

$$[[A, B]C] + [[C, A]B] + [[B, C]A] = 0$$ Jacobi identity

$$[AB, C] = [A, C]B + [A, C]B$$

(Also anticommutators $\{A, B\} = AB + BA$).

Complete set of commuting observables.
the set of eigenvalues completely determine the state.

Unitary operators

$$U^\dagger = U^{-1} \qquad U^\dagger U = \mathbb{1}$$

Preserve the norm $\quad \| U|v\rangle \|^2 = \langle v|U^\dagger U|v\rangle = \langle v|v\rangle = \||v\rangle\|^2$

if H is hermitian e^{iH} is unitary $U = e^{iH}$ $\qquad \underline{e^{iH} e^{-iH} = \mathbb{1}}$ ✓

$$U^\dagger = e^{-iH^\dagger} = e^{-iH}$$