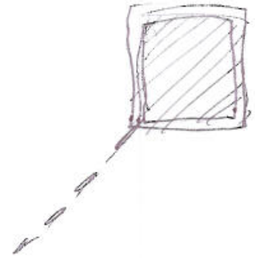
Change of orthonormal basis



$$\{|e_i\rangle\} \longrightarrow \{|\tilde{e}_j\rangle\}$$

Define U / $\quad U|e_1\rangle = |\tilde{e}_1\rangle$

$\vdots$

$$U|e_n\rangle = |\tilde{e}_n\rangle$$

$$\cancel{U|v\rangle = U\sum_n v_n |e_n\rangle = \sum_n v_n |\tilde{e}_n\rangle}$$

$$|v\rangle = \sum_n v_n |e_n\rangle = \sum_n \tilde{v}_n |\tilde{e}_n\rangle$$

$$\tilde{v}_n = \langle \tilde{e}_n | v\rangle = \langle e_n | U^\dagger |v\rangle = \cancel{\langle \tilde{e}_n|\tilde{v}\rangle} = \langle e_n|U^\dagger|e_m\rangle\langle e_m|v\rangle$$

$$\cancel{|\tilde{v}\rangle = U^\dagger}|v\rangle = U^{-1}_{nm} v_m$$

$$\langle e_j|U|e_i\rangle = U_{ji} = \langle e_j|\tilde{e}_i\rangle$$

$$UU^\dagger = \sum_{ij} U|e_i\rangle\langle e_i|U^\dagger|e_j\rangle\langle e_j|$$

$$= \sum_{ij} |\tilde{e}_i\rangle\langle e_j|U|e_i\rangle^* \langle e_j|$$

$$= \sum_{ij} |\tilde{e}_i\rangle\langle e_j|\tilde{e}_i\rangle^*\langle e_j| = \sum_{ij} |\tilde{e}_i\rangle\langle \tilde{e}_i|e_j\rangle\langle e_j| = \mathbb{1}.$$

$\boxed{U \text{ is unitary.}}$

Example

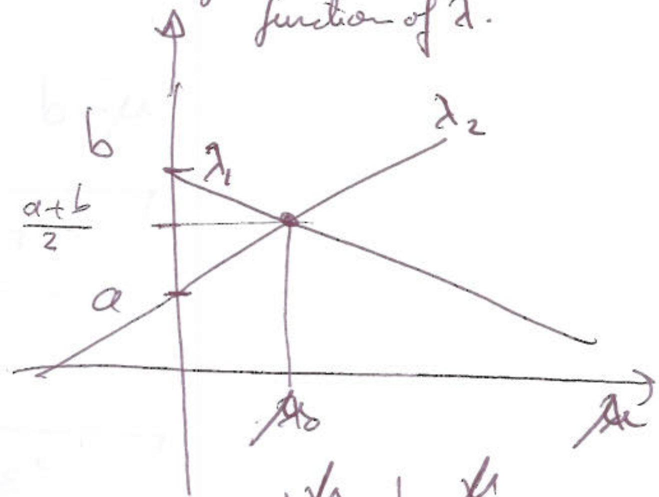
(no) Eigenvalue Crossing

Consider

$$H = \begin{pmatrix} a+\mu & 0 \\ 0 & b-\mu \end{pmatrix} \qquad b>a \text{ say}$$

$$\begin{cases} \lambda_1 = a+\mu \\ \lambda_2 = b-\mu \end{cases}$$

eigenvalues as function of λ.



$$a+\mu_0 = b-\mu_0$$

$$\boxed{\mu_0 = \frac{b-a}{2}}$$

Suppose

$$H = \begin{pmatrix} a+\mu & \varepsilon \\ \varepsilon & b-\mu \end{pmatrix}$$

ε very small.

$$\det(H-\lambda I) = (a+\mu-\lambda)(b-\mu-\lambda) - \varepsilon^2 = 0$$

$$\lambda^2 - (b-\cancel{\mu}+a+\cancel{\mu})\lambda + (a+\mu)(b-\mu) - \varepsilon^2 = 0$$

$$\lambda^2 - (a+b)\lambda + \cancel{(a+\mu}\ (ab+\mu(b-a) - \mu^2 - \varepsilon^2) = 0$$

$$\lambda = \frac{a+b \pm \sqrt{(a+b)^2 - 4ab - 4\mu(b-a) + 4\mu^2 + 4\varepsilon^2}}{2}$$

$$\lambda = \frac{a+b \pm \sqrt{(a-b)^2 - 4\mu(b-a) + 4\mu^2 + 4\varepsilon^2}}{2} = \frac{a+b \pm \sqrt{(a-b+2\mu)^2 + 4\varepsilon^2}}{2}$$

Spin operators

two states.

basis $\{|\uparrow\rangle, |\downarrow\rangle\}$

eigenvector of S_z $\quad S_z = \pm\frac{\hbar}{2}$

$$\boxed{S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Pauli Matrix.

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{n} = (s\theta c\varphi,\ s\theta s\varphi,\ c\theta)$$

$$\vec{S}.\hat{n} = S_n = \frac{\hbar}{2}\begin{pmatrix} c\theta & s\theta e^{-i\varphi} \\ s\theta e^{i\varphi} & -c\theta \end{pmatrix}$$ hermitian.

eigenvalues: $(c\theta - \lambda)(-c\theta - \lambda) - s^2\theta = 0$.

$\cancel{c^2\theta}\ -(c^2\theta - \lambda^2) - s^2\theta = 0 \qquad \lambda^2 - 1 = 0 \qquad \lambda = \pm 1$

$\boxed{\lambda_{1,2} = \pm\frac{\hbar}{2}}$ ok all directions are equivalent.

Position operator

$\hat{x}$, eigenvalues x.

$$\hat{x}|x\rangle = x|x\rangle$$

$\langle x|\psi\rangle = \psi(x)$ wave function.

$$\||\psi\rangle\|^2 \overset{?}{=} \int_{-\infty}^{\infty} \psi^*(x)\psi(x)\,dx = \int_{-\infty}^{\infty} \langle\psi|x\rangle\langle x|\psi\rangle\,dx$$

So we have $\int_{-\infty}^{\infty} |x\rangle\langle x|\,dx = \mathbb{1}$ we replaced sum by integrals

but $\int_{-\infty}^{\infty} |x\rangle\langle x|x'\rangle\,dx = |x'\rangle$

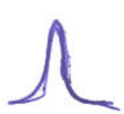
we need $\langle x|x'\rangle = \delta(x-x')$ Dirac delta function.

$\int_{-\infty}^{\infty} \delta(x-x')f(x)\,dx = f(x')$ we can think $\boxed{\begin{array}{ll}\delta(x-x')=0 & x\neq x' \\ \delta(x-x')=\infty & x=x'\end{array}}$ not really

δ is called a distribution, a linear operator acting on functions.

Can be approximated by ordinary functions:

$$\delta_\varepsilon(x) = \frac{1}{\sqrt{\pi\varepsilon}} e^{-x^2/\varepsilon}$$



$$\int_{-\infty}^{\infty} e^{-x^2/\varepsilon} = \sqrt{\pi\varepsilon}$$

$\lim \varepsilon \to 0$

$$\delta_\varepsilon(x) = \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$$

$$\int_{-\infty}^{\infty} \frac{\varepsilon\,dx}{x^2+\varepsilon^2} \overset{x=\varepsilon u}{=} \int_{-\infty}^{\infty} \frac{\varepsilon^2\,du}{\varepsilon^2(1+u^2)} = \pi$$

$$\delta_N(x) = \frac{1}{2\pi}\int_{-N}^{N} e^{ikx}\,dk \quad N\to\infty$$

2

We can define a translation operator:

$$U(a)|x\rangle = |x+a\rangle$$ preserves the norm $\Rightarrow$ unitary.

$$U(a)|\psi\rangle = \int_{-\infty}^{\infty} dx\ U(a)|x\rangle\langle x|\psi\rangle = \int_{-\infty}^{\infty} dx\ \psi(x)|x+a\rangle =$$

$$= \int_{-\infty}^{\infty} dx\ \psi(x-a)|x\rangle$$

$$\boxed{\psi(x) \to \psi(x-a)}$$

$$\hat{x}U_a - U_a\hat{x} = aU_a$$

$$\boxed{[\hat{x}, U_a] = aU_a}$$

$a \to 0$

$$U(a)|\psi\rangle \simeq \int_{-\infty}^{\infty} dx\ (\psi(x) - a\psi'(x))|x\rangle$$

$$\langle x|U(a)|\psi\rangle = \psi(x) - a\psi'(x)$$

$$\psi_0(x) - \psi(x) = -a\psi'(x)$$

In fact: $\psi(x-a) = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}\psi^{(n)}(x) = \left(\sum_{n=0} \frac{(-a)^n}{n!}\partial_x^n\right)\psi(x)$

$$= e^{-a\partial_x}\psi(x) = e^{i(ia\partial_x)}\psi(x)$$

$i\partial_x$: hermitian operator

$U = e^{-\frac{ipa}{\hbar}}$ p hermitian $\rightarrow$ $p^+ = p$ momentum. fundamental commutator relation. no finite dim rep

$$\boxed{[\hat{x}, \hat{p}] = i\hbar}$$

in fact if $[\hat{x}, \hat{p}] = \#$ (c-number) $\Rightarrow$ $[\hat{x}, U_a] = aU_a$ $(p \cdot x \to \hbar)$

(5)

a small $U|\psi\rangle \simeq (1 - \frac{ipa}{\hbar})|\psi\rangle$

$$\langle x|U|\psi\rangle = \psi(x) - i\frac{a}{\hbar}\langle x|p|\psi\rangle$$

we need ~~$p = -i\hbar\partial$~~

$$\boxed{\langle x|p|\psi\rangle = -i\hbar\partial_x\langle x|\psi\rangle}$$

$$\langle x|U|\psi\rangle = \langle x|e^{-i\frac{a}{\hbar}p}|\psi\rangle$$

$$= e^{-\cancel{i}\frac{a}{\hbar}(+\cancel{i}\cancel{\hbar}\partial_x)}\psi(x) = \boxed{e^{-a\partial_x}\psi(x)}$$

What is $\langle x|p\rangle$?

we need to diagonalize p.

Use a basis

$$|p\rangle = \int_{-\infty}^{\infty}\psi_p(x)\,|x\rangle$$

$$\langle x|\hat{p}|p\rangle = -i\hbar\,\partial_x\psi_p(x) = p\,\psi_p(x)$$

$$\psi_p = Ae^{i\frac{px}{\hbar}}$$

$$\langle x|p\rangle = Ae^{\frac{ipx}{\hbar}} \Rightarrow \boxed{|p\rangle = A\int_{-\infty}^{\infty}e^{\frac{ipx}{\hbar}}|x\rangle}$$

$$\langle p'|p\rangle = \int_{-\infty}^{\infty}dx\,\langle p'|x\rangle\langle x|p\rangle = \int_{-\infty}^{\infty}dx\,e^{\frac{ix(p-p')}{\hbar}} = 2\pi\hbar\,\delta(p-p')$$

$$\boxed{|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{ipx/\hbar}\,|x\rangle}$$

Proof of δ function approximation.

$$\delta_N(x) = \frac{1}{2\pi}\int_{-N}^{N} e^{ikx}\,dk \qquad N\to\infty.$$

$$\int_{-\infty}^{\infty} f(x)\,\frac{1}{2\pi}\int_{-N}^{N} e^{ikx}\,dk = \cancel{\frac{1}{2\pi}\int_{-N}^{N}}$$

$$\int_{-N}^{N} e^{ikx}\,dk = 2\,\frac{e^{i\cancel{k}N}-e^{-i\cancel{k}N}}{2i\cancel{k}} = 2\,\frac{\sin(\cancel{k}N)}{\cancel{k}}$$

$$\frac{N}{\pi}\int_{-\infty}^{\infty} f(x)\,\frac{\sin(xN)}{xN}\,dx = \frac{\cancel{N}}{\pi}\int_{-\infty}^{\infty} f\left(\frac{u}{N}\right)\frac{\sin(u)}{u}\,du$$

$N\to\infty \qquad f\left(\frac{u}{N}\right)\simeq f(0) \qquad = f(0)\,\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin u}{u}\,du = f(0)$

$$\cancel{\int_{M}^{\infty} f\left(\frac{u}{N}\right)\frac{\sin u}{u}\,du} \qquad f = f(0) + \left(f - f(0)\right)$$

$$= f(0) + g(x) \qquad g(0) = 0.$$

$$\int_{-\infty}^{\infty} g\left(\frac{u}{N}\right)\frac{\sin u}{N\frac{u}{N}}\,du = \frac{1}{N}\int_{-\infty}^{\infty} h\left(\frac{u}{N}\right)\sin u\,du \quad \cancel{\leq \frac{1}{N}\int_{-\infty}^{\infty} h\left(\frac{u}{N}\right)du}$$

$$= -\frac{1}{N}\int_{-\infty}^{\infty} h\left(\frac{u}{N}\right)d\cos u = -\frac{1}{N}\int_{-\infty}^{\infty}\cos u\,\frac{1}{N}\,h'\left(\frac{u}{N}\right)du = -\frac{1}{N^2}\int_{-\infty}^{\infty}\cos u\,h'\left(\frac{u}{N}\right)du$$

$$\leq -\frac{N}{N^2}\int_{-\infty}^{\infty} h'(x)\,dx = -\frac{1}{N}\left(h(\infty)-h(-\infty)\right) = 0.$$

Commutation relations:

$$\hat{x}\hat{p} - \hat{p}\hat{x} = ?$$

$$(xp - px)|a\rangle = \cancel{\hat{x}p|a\rangle - a\,p|a\rangle}$$

$$= \cancel{x\int dy\, |y\rangle\langle y|}$$

$$= \cancel{x\int_{-\infty}^{\infty} dx'\,|x'\rangle\langle x'|p|a\rangle}$$

$$\langle x|(xp - px)|\psi\rangle = \int_{-\infty}^{\infty} \cancel{\langle x|x|x'\rangle\langle x'|p|\psi\rangle - p|x'\rangle\langle x'|x|\psi\rangle}$$

$$\cancel{= \int_{-\infty}^{\infty} x'\,(-i\hbar\,\partial_{x'}\psi(x'))}$$

$$\cancel{= \int_{-\infty}^{\infty} x'\,(-i\hbar\,\partial_{x'}\langle x'|\psi\rangle) - x'\,p|x'\rangle}\,\langle x'|\psi\rangle$$

$$\langle x|(xp - px)|\psi\rangle = x\,\langle x|p|\psi\rangle - \langle x|p\int_{-\infty}^{\infty}|x'\rangle\langle x'|\psi\rangle$$

$$\hat{x}\,|x+a\rangle = (x+a)\,|x+a\rangle$$

$$\hat{x}|x\rangle = x|x\rangle$$

$$\hat{x}\,U_a|x\rangle = (x+a)\,|x+a\rangle = (x+a)\,U_a|x\rangle$$

$$U_a\hat{x}|x\rangle = x\,|x+a\rangle$$

$$(\hat{x}U_a - U_a\hat{x})|x\rangle = a\,U_a|x\rangle$$

$$\boxed{\hat{x}U_a - U_a\hat{x} = aU_a}$$

U_a as a translation

$$U_a \simeq 1 - iP/_{\hbar}\,a$$

$$[\hat{x}, U_a] = [\hat{x}, 1 - iP/_{\hbar}a] = -\frac{i}{\hbar}a\,[\hat{x},\hat{p}] = a + O(a^2)$$

fundamental commutation relation

$$\boxed{[\hat{x},\hat{p}] = i\hbar}$$

Uncertainty relation

$\langle A \rangle = \langle \psi | A | \psi \rangle$ mean value, expectation value.

$\langle A^2 \rangle = \langle \psi | A^2 | \psi \rangle$

$\Delta A = A - \langle A \rangle$

$\langle (\Delta A)^2 \rangle$ = variance = $\langle A^2 \rangle - \langle A \rangle^2$

•) $\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \left| \langle [\Delta A, \Delta B] \rangle \right|^2 = \frac{1}{4} \left| \langle [A, B] \rangle \right|^2$.

given $\mathcal{O}$ not necessarily hermitian.

$\langle \psi | \mathcal{O}^+ \mathcal{O} | \psi \rangle \geq 0$

Proof. $\sum_i \langle \psi | \mathcal{O}^+ | e_i \rangle \langle e_i | \mathcal{O} | \psi \rangle = \sum_i | \langle e_i | \mathcal{O} | \psi \rangle |^2 \geq 0$.

take $\mathcal{O} = \Delta A + i\mu \Delta B$ $\mu \in \mathbb{R}$.

$\mathcal{O}^+ = \Delta A - i\mu \Delta B$

$\langle \psi | \mathcal{O}^+ \mathcal{O} | \psi \rangle = \langle (\Delta A)^2 \rangle - i\mu \langle \Delta A \Delta B \rangle + i\mu \langle \Delta B \Delta A \rangle + \mu^2 \langle \Delta B^2 \rangle \geq 0$

$ax^2 + bx + c \geq 0$ iff $b^2 - 4ac \leq 0$. $-i\mu \langle \underbrace{[\Delta A, \Delta B]}_{} \rangle$

imaginary. here $\langle AB \rangle = \langle BA \rangle^*$

$- \langle [\Delta A, \Delta B] \rangle^2 - 4 \langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \leq 0$

$\boxed{\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} \left| \langle [\Delta A, \Delta B] \rangle \right|^2}$ $\boxed{\Delta x \Delta p \geq \hbar/2}$