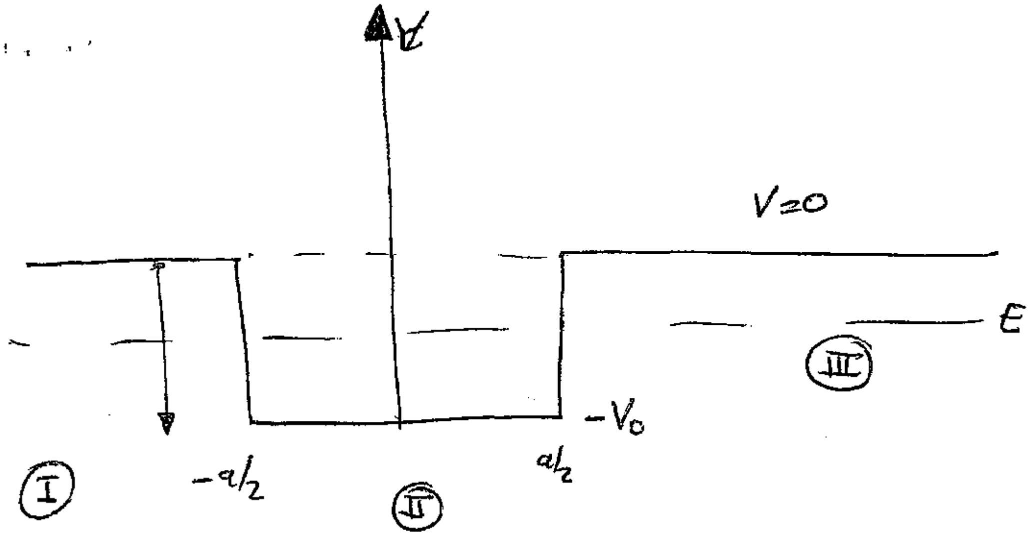


$$\psi_{I} = Ae^{kx} + Be^{-kx}$$
$$\psi_{II} = Ce^{iKx} + De^{-iKx}$$
$$\psi_{III} = Fe^{kx} + Ge^{-kx}$$

$$k = \frac{\sqrt{-2mE}}{\hbar} > 0$$

$$K = \frac{\sqrt{2m(V_0+E)}}{\hbar} > 0$$

$$Ae^{-ka/2} + Be^{ka/2} = Ce^{-iK\frac{a}{2}} + De^{iK\frac{a}{2}}$$

$$kAe^{-ka/2} - kBe^{ka/2} = iKCe^{-iK\frac{a}{2}} - iKDe^{iK\frac{a}{2}}$$

$$\begin{pmatrix} e^{-ka/2} & e^{ka/2} \\ ke^{-ka/2} & -ke^{ka/2} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} e^{-iK\frac{a}{2}} & e^{iK\frac{a}{2}} \\ iKe^{-iK\frac{a}{2}} & -iKe^{iK\frac{a}{2}} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix}$$

$$Ce^{iK\frac{a}{2}} + De^{-iK\frac{a}{2}} = \cancel{ik}\, Fe^{\frac{ka}{2}} + Ge^{-\frac{ka}{2}}$$

$$iKCe^{iK\frac{a}{2}} - iKDe^{-iK\frac{a}{2}} = kFe^{\frac{ka}{2}} - kGe^{-\frac{ka}{2}}$$

$$\begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iKe^{iK\frac{a}{2}} & -iKe^{-iK\frac{a}{2}} \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} e^{\frac{ka}{2}} & e^{-\frac{ka}{2}} \\ ke^{\frac{ka}{2}} & -ke^{-\frac{ka}{2}} \end{pmatrix}\begin{pmatrix} F \\ G \end{pmatrix}$$

1a

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{4kK} \begin{pmatrix} (-k-iK)\, e^{(iK-k)\frac{a}{2}} & (-k+iK)\, e^{-(k+iK)\frac{a}{2}} \\ -(k-iK)\, e^{(iK+k)\frac{a}{2}} & -(k+iK)\, e^{(k-iK)\frac{a}{2}} \end{pmatrix}$$

$k_\pm = k \pm iK$

$$\begin{pmatrix} -(k+iK) e^{(iK-k)\frac{a}{2}} & (-iK+k)\, e^{(iK+k)\frac{a}{2}} \\ (-iK+k) e^{-(iK+k)\frac{a}{2}} & (-iK-k)\, e^{(k-iK)\frac{a}{2}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$= -\frac{i}{4kK} \begin{pmatrix} -k_+ e^{-k_-\frac{a}{2}} & -k_- e^{-k_+\frac{a}{2}} \\ -k_- e^{k_+\frac{a}{2}} & -k_+ e^{k_-\frac{a}{2}} \end{pmatrix} \begin{pmatrix} -k_+ e^{-k_-\frac{a}{2}} & +k_- e^{k_+\frac{a}{2}} \\ +k_- e^{-k_+\frac{a}{2}} & -k_+ e^{k_-\frac{a}{2}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$= -\frac{i}{4kK} \begin{pmatrix} k_+^2 e^{-k_- a} - k_-^2 e^{-k_+ a} & -k_+k_-\left(e^{(k_+-k_-)\frac{a}{2}} + e^{-(k_+-k_-)\frac{a}{2}}\right) \\ k_+k_-\left(e^{(k_+-k_-)\frac{a}{2}} - e^{(k_--k_+)\frac{a}{2}}\right) & -k_-^2 e^{k_+ a} + k_+^2 e^{k_- a} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$k_+ - k_- = (k+iK) - (k-iK) = 2iK$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{4kK} \begin{pmatrix} k_+^2 e^{-k_- a} - k_-^2 e^{-k_+ a} & -k_+k_-\left(e^{iKa} + e^{-iKa}\right) \\ k_+k_-\left(e^{iKa} - e^{-iKa}\right) & -k_-^2 e^{k_+ a} + k_+^2 e^{k_- a} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{4kK} \begin{pmatrix} k_+^2 e^{-k_- a} - k_-^2 e^{-k_+ a} & -2ik_+k_- \sin Ka \\ 2iK_+k_- \sin Ka & -k_-^2 e^{k_+ a} + k_+^2 e^{k_- a} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

Bound state $B=0$ $F=0$

$B=0 \Rightarrow F = -\frac{i}{4kK} \underbrace{\left(k_+^2 e^{-k_- a} - k_-^2 e^{-k_+ a}\right)}_{0} A$

$$k_+^2 e^{-k_- a} = k_-^2 e^{-k_+ a}$$

$$\left(\frac{k_-}{k_+}\right)^2 = e^{(k_+ - k_-)a} = e^{2iKa}$$

$$\left(\frac{k - iK}{k + iK}\right)^2 = e^{2iKa}$$

$$G = \cancel{\pm}\frac{\cancel{i}}{4kK}\, \cancel{2i}\, k_+ k_- \sin Ka \; A = \frac{k^2 + K^2}{2kK} \sin(Ka)\; A$$

$$\frac{k - iK}{k + iK} = \pm e^{iKa} \qquad \pm e^{-iKa} = \frac{k + iK}{k - iK}$$

$$\sin Ka = \pm \frac{(e^{iKa} - e^{-iKa})}{2i} = \pm \frac{1}{2i}\left(\frac{k - iK}{k + iK} - \frac{k + iK}{k - iK}\right)$$

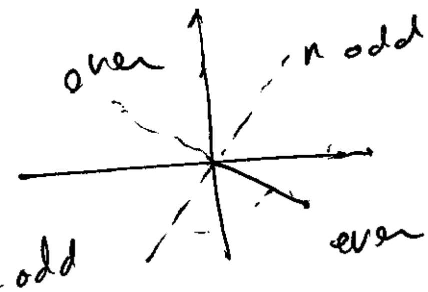
$$= \pm \frac{1}{2i} \frac{(k - iK)^2 - (k + iK)^2}{k^2 + K^2} = \pm \frac{1}{2i} \frac{-4ikK}{k^2 + K^2} = \mp \frac{2kK}{k^2 + K^2}$$

$$\boxed{G = \pm A}$$ symmetric & anti-symmetric

1c

$k - iK = k_0 e^{i\varphi}$ $\qquad -\frac{\pi}{2} < \varphi < 0$

$$e^{4i\varphi} = e^{2iKa}$$



$$4\varphi = 2Ka + 2n\pi$$

$$2\varphi = Ka + n\pi$$

$$\varphi = \frac{Ka}{2} + \frac{n\pi}{2}$$

$$\sin\left(\frac{Ka}{2}\right) = \sin\left(\varphi - \frac{n\pi}{2}\right) = \begin{cases} n \text{ odd} & c\varphi,\ -c\varphi \\ n \text{ even} & s\varphi;\ -s\varphi \end{cases}$$

$$s\varphi = -\frac{K}{\sqrt{k^2+K^2}} = -\frac{K}{k_0}$$

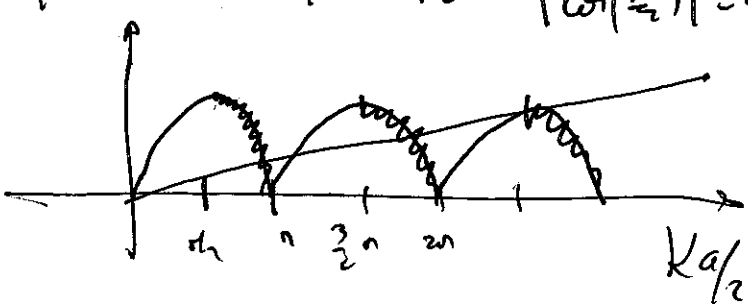
$$k^2 + K^2 = \frac{1}{\hbar^2}\left(\cancel{-2mE} + 2mV_0 + \cancel{2mE}\right) = \frac{2mV_0}{\hbar^2} = k_0^2$$

$\left|\sin\left(\frac{Ka}{2}\right)\right| = \frac{K}{k_0}$ and n even

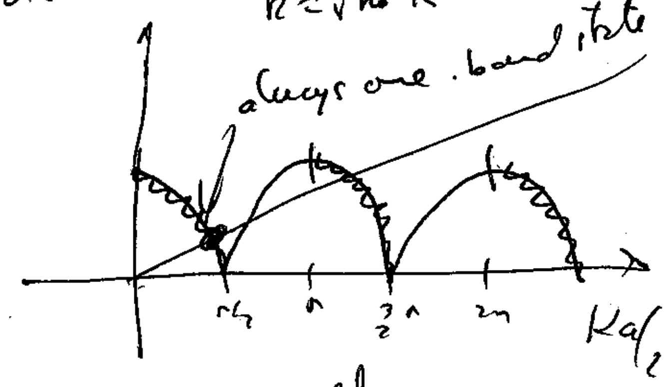
$k^2 + K^2 = k_0^2$

$\left|\sin\left(\frac{Ka}{2}\right)\right| = \frac{k}{k_0}$ " n odd. $\Rightarrow \left|\cos\left(\frac{K_a}{2}\right)\right| = K/k_0$

$k = \sqrt{k_0^2 - K^2}$



n even



n odd.

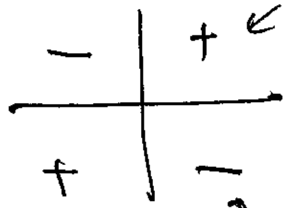
1d

$$\overset{+1}{\text{even}}, \ \overset{-1}{\text{odd}}$$

$S_g(\sin Ka) \rightarrow$ even, odd

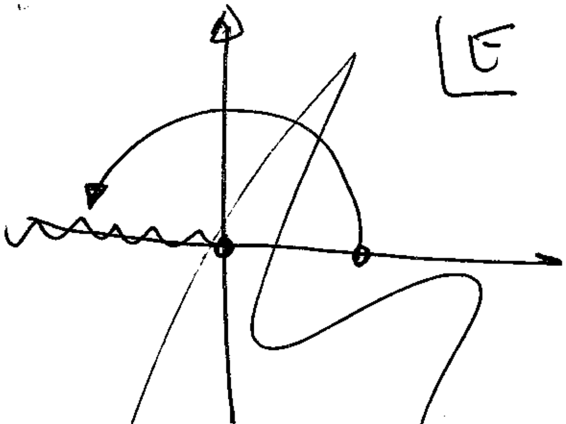
$2 \sin \frac{Ka}{2} \cos \frac{Ka}{2}$

n odd → even $\psi(x)$

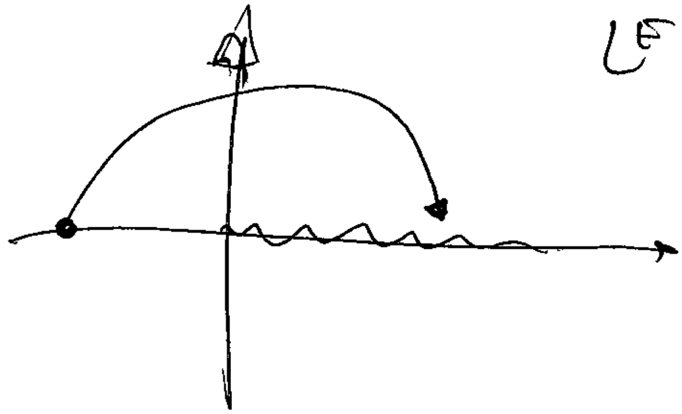


n even → odd $\psi(x)$.

LE



LE (1e)



$$k = \frac{\sqrt{-2mE}}{\hbar} =$$

$$k = \frac{\sqrt{-2mE}}{\hbar} \qquad \tilde{k} = \frac{\sqrt{2mE}}{\hbar}$$

$$\tilde{k}^2 - K^2 = \frac{2mE}{\hbar^2} - \frac{2mV_0}{\hbar^2}$$

$$E + i\varepsilon \qquad k = \frac{\sqrt{2m(-E - i\varepsilon)}}{\hbar} = -i\tilde{k} \qquad -\frac{2mV_0}{\hbar^2} = -k_0^2$$

$$k_\pm = k \pm iK \rightarrow -i\tilde{k} \pm iK = -i(\tilde{k} \mp K) = -i\tilde{k}_\mp$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = + \frac{1}{+4i\tilde{k}K} \begin{pmatrix} -\tilde{k}_-^2 e^{i\tilde{k}_+ a} + \tilde{k}_+^2 e^{i\tilde{k}_- a} & 2i\tilde{k}_+\tilde{k}_- \sin Ka \\ -2i\tilde{k}_+\tilde{k}_- \sin Ka & \tilde{k}_+^2 e^{-i\tilde{k}_- a} - \tilde{k}_-^2 e^{-i\tilde{k}_+ a} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\tilde{k} \rightarrow k$$

$$= \frac{1}{4kK} \begin{pmatrix} e^{ika}\left(-(k-K)^2 e^{iKa} + (k+K)^2 e^{-iKa}\right) & 2i(k^2-K^2) \sin Ka \\ -2i(k^2-K^2) \sin Ka & e^{-ika}\left((k+K)^2 e^{+iKa} - (k-K)^2 e^{-iKa}\right) \end{pmatrix}$$

$$= \frac{1}{4kK} \begin{pmatrix} e^{ika}\left(\cos Ka\, 4kK + i \sin Ka\, 2(k^2+K^2)\right) & 2i(k^2-K^2) \sin Ka \\ -2i(k^2-K^2) \sin Ka & e^{-ika}\left(\cos Ka\, 4kK + i \sin Ka\, 2(k^2+K^2)\right) \end{pmatrix}$$

$$= \frac{-2i}{4kK} \begin{pmatrix} e^{ika}\left(s_{Ka}(k^2+K^2) + 2ikK\cos Ka\right) & k_0^2 \sin Ka \\ -k_0^2 \sin Ka & e^{-ika}\left(-s_{Ka}(k^2+K^2) + 2ikK\cos Ka\right) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

(4f)

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} G \\ A \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} B \\ F \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \frac{1}{ad-bc}$$ ②

$$ad-bc = -iK - iK = -2iK$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{i}{2K} \begin{pmatrix} -iK e^{iK\frac{a}{2}} & -e^{iK\frac{a}{2}} \\ -iK e^{-iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} e^{-\frac{ka}{2}} & e^{\frac{ka}{2}} \\ k e^{-\frac{ka}{2}} & -k e^{\frac{ka}{2}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{1}{2k} \begin{pmatrix} -k e^{-\frac{ka}{2}} & -e^{-\frac{ka}{2}} \\ -k e^{\frac{ka}{2}} & e^{\frac{ka}{2}} \end{pmatrix} \begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iK e^{iK\frac{a}{2}} & -iK e^{-iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{4kK} \begin{pmatrix} -k e^{-ka/2} & -e^{-ka/2} \\ -k e^{ka/2} & e^{ka/2} \end{pmatrix} \underbrace{\begin{pmatrix} e^{iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \\ iK e^{iK\frac{a}{2}} & -iK e^{-iK\frac{a}{2}} \end{pmatrix} \begin{pmatrix} -iK e^{iK\frac{a}{2}} & -e^{iK\frac{a}{2}} \\ -iK e^{-iK\frac{a}{2}} & e^{-iK\frac{a}{2}} \end{pmatrix}}_{\mathbb{A}} \begin{pmatrix} e^{-\frac{ka}{2}} & e^{\frac{ka}{2}} \\ k e^{-\frac{ka}{2}} & -k e^{\frac{ka}{2}} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

$B = 0$, $F = 0$ $\Rightarrow$ eigenstate.

$$\begin{pmatrix} 0 \\ G \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} A \\ 0 \end{pmatrix} = \begin{pmatrix} A t_{11} \\ A t_{21} \end{pmatrix} \Rightarrow \boxed{t_{11} = 0}$$

$$\mathbb{A} = \begin{pmatrix} -iK(e^{iKa} + e^{-iKa}) & -e^{iKa} + e^{-iKa} \\ K^2 e^{iKa} - K^2 e^{-iKa} & -iK e^{iKa} - iK e^{-iKa} \end{pmatrix} =$$

$$\mathbb{A} = \begin{pmatrix} -2iK \cos Ka & -2i \sin Ka \\ 2iK^2 \sin Ka & -2iK \cos Ka \end{pmatrix} = -2i \begin{pmatrix} K \cos Ka & \sin Ka \\ -K^2 \sin Ka & K \cos Ka \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{1}{2kK} \begin{pmatrix} -k e^{-ka/2} & -e^{-ka/2} \\ -k e^{ka/2} & e^{ka/2} \end{pmatrix} \begin{pmatrix} K cKa & sKa \\ -K^2 sKa & K cKa \end{pmatrix} \begin{pmatrix} e^{-ka/2} \\ k e^{-ka/2} \end{pmatrix} A$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{A e^{-ka/2}}{2kK} \begin{pmatrix} -k e^{-ka/2} & -e^{-ka/2} \\ -k e^{ka/2} & e^{ka/2} \end{pmatrix} \begin{pmatrix} k\,cKa + k\,sKa \\ -k^2 sKa + kK\,cKa \end{pmatrix} \qquad (3)$$

$$= -\frac{A e^{-ka/2}}{2kK} \begin{pmatrix} -kK\,cKa\, e^{-ka/2} - k^2 e^{-ka/2} sKa + K^2 sKa\, e^{-ka/2} - kK e^{-ka/2} cKa \\ e^{ka/2}\left(-kK\,cKa - k^2 sKa - \cancel{K}^2 sKa + kK\,cKa\right) \end{pmatrix}$$

$$-kK\,cKa - k^2 sKa + K^2 sKa - kK\,cKa = 0$$

$$(K^2-k^2)\,sKa = 2kK\,cKa$$

$$\boxed{\tan Ka = \frac{2kK}{K^2-k^2}}$$

$$G = -\frac{A}{2kK}\left(-(k^2+K^2)\right) sKa \Rightarrow \frac{G}{A} = \frac{k^2+K^2}{2kK}\, sKa$$

$$1+\tan^2 Ka = \frac{1}{c^2Ka} = 1 + \frac{4k^2K^2}{(K^2-k^2)^2} = \frac{(k^2+K^2)^2}{(k^2-K^2)^2} \Rightarrow c^2Ka = \frac{(k^2-K^2)^2}{(k^2+K^2)^2}$$

$$s^2Ka = \frac{4k^2K^2}{(k^2+K^2)^2} \qquad |sKa| = \frac{2kK}{k^2+K^2}$$

$$\boxed{\left|\frac{G}{A}\right| = 1}$$

sign(sKa) = + even, symmetric

− odd, anti-symmetric.

$k - iK$; $k^2 + K^2 = \frac{2m}{\hbar^2}(\cancel{+E} + V_0 \cancel{+E}) = \frac{2mV_0}{\hbar^2} = k_0^2$ (9)

$$\boxed{k - iK = k_0 e^{i\varphi}} \qquad k = k_0 \cos\varphi \qquad K = -k_0 \sin\varphi$$

$$\frac{k-iK}{k+iK} = e^{2i\varphi} = \frac{(k-iK)^2}{k^2+K^2} = \frac{k^2-K^2}{k^2+K^2} - \frac{2ikK}{k^2+K^2}$$

$$\tan 2\varphi = -\frac{2kK}{k^2-K^2} = \tan ka$$

$$\boxed{2\varphi = ka + n\pi} \Rightarrow \varphi = \frac{ka}{2} + \frac{n\pi}{2}$$

φ differs from $ka/2$ by $n\pi/2$ (for some n).

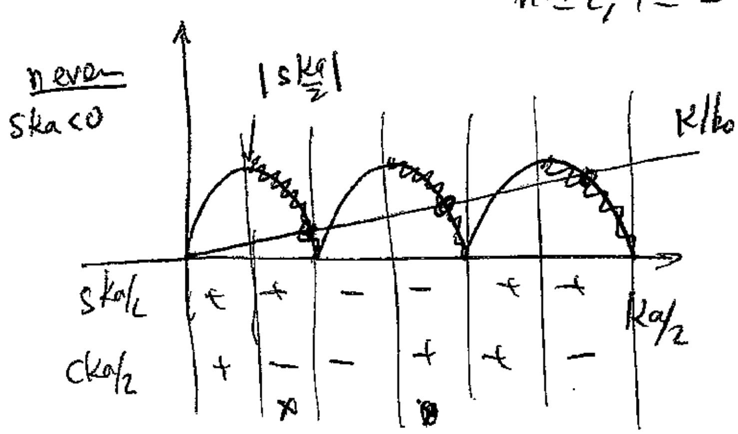
$$K = -k_0 \sin\left(\frac{ka}{2} + \frac{n\pi}{2}\right) \quad ; \left(\varphi - \frac{ka}{2}\right) = \frac{n\pi}{2}$$

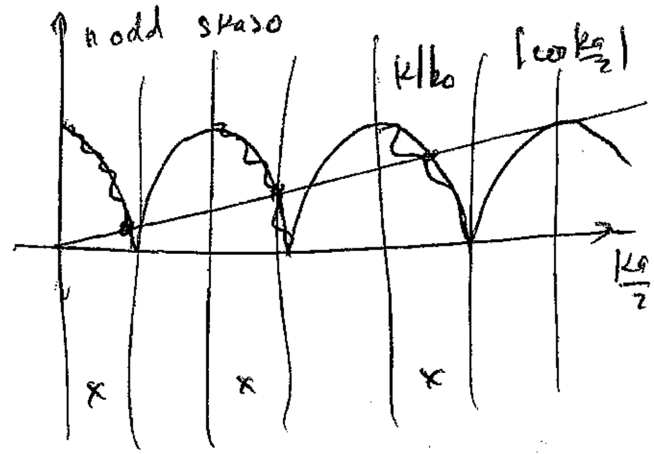
$\Rightarrow \quad \sin\frac{ka}{2} = \pm\frac{K}{k_0}$ (n even) or $\cos\frac{ka}{2} = \pm\frac{K}{k_0}$ (n odd)

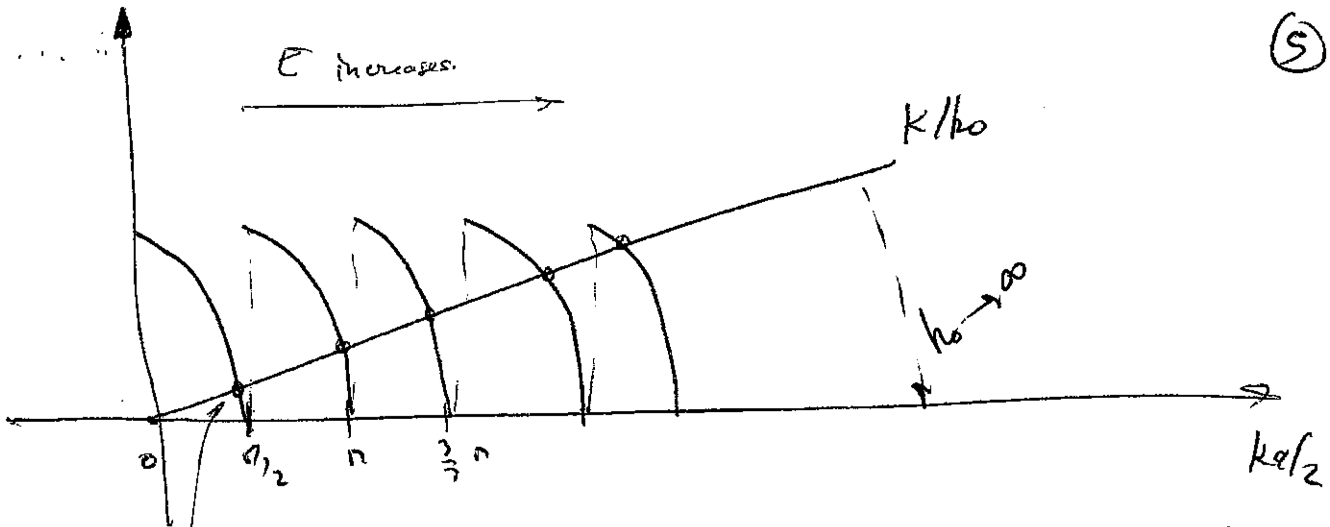
But $s\varphi < 0$ $(c\varphi > 0)$ $\quad s2\varphi < 0 \Rightarrow s(ka + n\pi) < 0 \Rightarrow (-)^n s(ka) < 0$

$\mathrm{sign}(ska) = (-)^{n+1}$ $\quad n = 1, 3, \ldots$ odd $\Rightarrow$ symmetric wave-function

$n = 2, 4, \ldots$ even $\Rightarrow$ antisymmetric wave-function.







At least one eigenstate (for any k_0, i.e. V_0, a)

$$E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{\hbar^2}{2m}\left(k_0^2 - k^2\right) = -V_0 + \frac{\hbar^2 k^2}{2m}$$

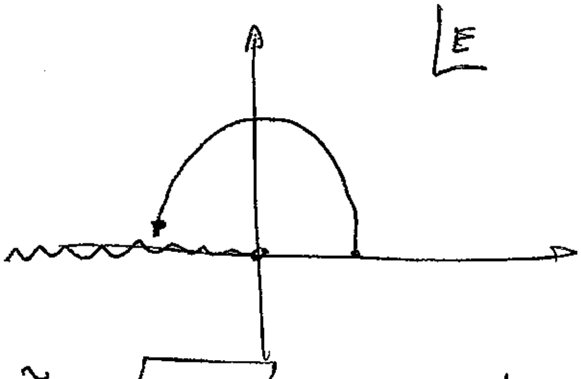
$k_0 \to \infty$ $\qquad \frac{ka}{2} = n\frac{\pi}{2}$ $\qquad \boxed{ka = n\pi}$ ∞ particle in a box.

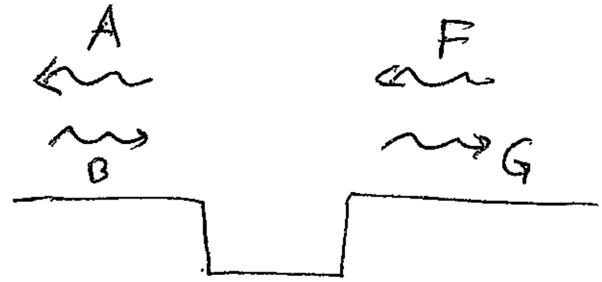
Continuum spectrum

$$\tilde{k} = \frac{\sqrt{2mE}}{\hbar} \qquad K = \frac{\sqrt{2m(V_0+E)}}{\hbar}$$

$$K^2 - \tilde{k}^2 = k_0^2$$

$E > 0$





$$\tilde{k} = \frac{\sqrt{2mE}}{\hbar} \longrightarrow ik$$

$$\tilde{k} = ik \qquad k = -i\tilde{k}$$

($B=0,\ F=0,\ A, G \neq 0$ bound state)

$$\psi_{I} = Ae^{-i\tilde{k}x} + Be^{i\tilde{k}x}$$

$$\psi_{II} = Ce^{iKx} + De^{-iKx}$$

$$\psi_{III} = Fe^{-i\tilde{k}x} + Ge^{i\tilde{k}x}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = \frac{-2i}{4\tilde{k}K} \begin{pmatrix} i\tilde{k}\, e^{i\tilde{k}a/2} & -e^{i\tilde{k}a/2} \\ i\tilde{k}\, e^{-i\tilde{k}a/2} & e^{-i\tilde{k}a/2} \end{pmatrix} \begin{pmatrix} KcKa & sKa \\ -K^2 sKa & KcKa \end{pmatrix} \begin{pmatrix} e^{i\frac{\tilde{k}a}{2}} & e^{-i\frac{\tilde{k}a}{2}} \\ -i\tilde{k}e^{i\frac{\tilde{k}a}{2}} & i\tilde{k}e^{-i\frac{\tilde{k}a}{2}} \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix}$$

$$= -\frac{i}{2\tilde{k}K} \begin{pmatrix} i\tilde{k}KcKa\, e^{i\frac{\tilde{k}a}{2}} + K^2 sKa\, e^{i\frac{\tilde{k}a}{2}} & e^{i\frac{\tilde{k}a}{2}}(i\tilde{k}sKa - KcKa) \\ e^{-i\frac{\tilde{k}a}{2}}(i\tilde{k}KcKa - K^2 sKa) & e^{-i\frac{\tilde{k}a}{2}}(i\tilde{k}sKa + KcKa) \end{pmatrix} \begin{pmatrix} e^{i\frac{\tilde{k}a}{2}} & e^{-i\frac{\tilde{k}a}{2}} \\ -i\tilde{k}e^{i\frac{\tilde{k}a}{2}} & i\tilde{k}e^{-i\frac{\tilde{k}a}{2}} \end{pmatrix}$$

7

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{2\tilde{k}K}\begin{pmatrix} e^{i\tilde{k}a}\left(i\tilde{k}KcKa + K^2 sKa + \tilde{k}^2 sKa + iK\tilde{k}cKa\right) & i\tilde{k}KcKa + K^2 sKa - \tilde{k}^2 sKa - i\tilde{k}KcKa \\ i\tilde{k}KcKa - K^2 sKa + \tilde{k}^2 sKa - i\tilde{k}KcKa & e^{-i\tilde{k}a}\left(i\tilde{k}KcKa - K^2 sKa - \tilde{k}^2 sKa + i\tilde{k}KcKa\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = -\frac{i}{2\tilde{k}K}\begin{pmatrix} e^{i\tilde{k}a}\left((K^2+\tilde{k}^2)sKa + 2iK\tilde{k}cKa\right) & k_0^2 sKa \\ -k_0^2 sKa & e^{-i\tilde{k}a}\left(-(K^2+\tilde{k}^2)sKa + 2iK\tilde{k}cKa\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\det T = -\frac{1}{4\tilde{k}^2K^2}\Big[-4\tilde{k}^2K^2c^2Ka \underbrace{- (K^2+\tilde{k}^2)^2 s^2Ka + \underset{(K^2-\tilde{k}^2)^2}{k_0^4}\, s^2Ka}_{-4K^2\tilde{k}^2 s^2Ka}$$

$$= \cancel{-}\frac{1}{\cancel{4\tilde{k}^2K^2}}\left(\cancel{-4\tilde{k}^2K^2}\right)\left(c^2Ka + s^2Ka\right) = 1.$$

$F = At_{11} + Bt_{12}$

$G = At_{21} + Bt_{22}$

Scattering matrix

$$\begin{pmatrix} G \\ A \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} B \\ F \end{pmatrix}$$

$$A = \frac{1}{t_{11}} F - \frac{t_{12}}{t_{11}} B \qquad ; \; G = \frac{t_{21}}{t_{11}} F - \frac{t_{21} t_{12}}{t_{11}} B + B t_{22}$$

$$G = \frac{\det T}{t_{11}} B + \frac{t_{21}}{t_{11}} F = \frac{1}{t_{11}} B + \frac{t_{21}}{t_{11}} F$$

$$\begin{pmatrix} G \\ A \end{pmatrix} = \frac{1}{t_{11}} \begin{pmatrix} 1 & t_{21} \\ -t_{12} & 1 \end{pmatrix} \begin{pmatrix} B \\ F \end{pmatrix}$$

$$S = \frac{i\, 2\tilde{k} K e^{-i\tilde{k}a}}{(k^2+\tilde{k}^2) s K a + 2i K \tilde{k}\, c K a} \begin{pmatrix} 1 & \frac{+ i k_0^2 s K a}{2\tilde{k} K} \\ \frac{+ i k_0^2 s K a}{2\tilde{k} K} & 1 \end{pmatrix}$$

$$S = \frac{2i\tilde{k} K \, e^{-i\tilde{k}a}}{(k^2+\tilde{k}^2) s K a + 2i\tilde{k} K c K a} \begin{pmatrix} 1 & + \frac{i k_0^2 s K a}{2\tilde{k} K} \\ \frac{i k_0^2 s K a}{2\tilde{k} K} & 1 \end{pmatrix}$$

$$\boxed{S = \frac{e^{-i\tilde{k}a}}{(k^2+\tilde{k}^2) s K a + 2i\tilde{k} K c K a} \begin{pmatrix} 2i\tilde{k} K & -k_0^2 s K a \\ -k_0^2 s K a & 2i\tilde{k} K \end{pmatrix}}$$

Spectrum vs double degenerate.

2 states $\boxed{\begin{matrix} B=1, & F=0 \\ G=S_{11} & A=S_{21} \end{matrix}}$; $\boxed{\begin{matrix} B=0; & F=1 \\ G=S_{12}, & A=S_{22} \end{matrix}}$

$$SS^{+} = \frac{1}{(k^2+\tilde{k}^2)^2 s^2Ka + 4\tilde{k}^2k^2 \underbrace{c^2Ka}_{1-s^2Ka}} \underbrace{\begin{pmatrix} 2i\tilde{k}K & -k_0^2 sKa \\ -k_0^2 sKa & 2i\tilde{k}K \end{pmatrix} \begin{pmatrix} -2i\tilde{k}K & -k_0^2 sKa \\ -k_0^2 sKa & -2i\tilde{k}K \end{pmatrix}}$$

$$\begin{pmatrix} 4\tilde{k}^2k^2 + k_0^4 s^2Ka & -2i\tilde{k}\cancel{k k_0^2} sKa + \cancel{2i\tilde{k}k k_0^2 sKa} \\ 0 & k_0^4 s^2Ka + 4\tilde{k}^2k^2 \end{pmatrix}$$

$$SS^{+} = \frac{\cancel{4\tilde{k}^2k^2 + k_0^2 s^2Ka}}{\cancel{4\tilde{k}^2k^2 + [(k^2+\tilde{k}^2)^2 - 4\tilde{k}^2k^2] s^2Ka}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(k^2-\tilde{k}^2)^2 = k_0^2$$

$$\boxed{SS^{+} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{unitary!}}$$

Other properties:

$k_0 \to 0$ (no potential) $\quad K^2 = \tilde{k}^2 \quad K = \tilde{k}$

$$S = \frac{e^{-i\tilde{k}a}}{(K^2+k^2)\, sKa + 2ik^2\, cKa} \begin{pmatrix} 2ik^2 & 0 \\ 0 & 2ik^2 \end{pmatrix}$$

$$= \frac{2ik^2\, e^{-i\tilde{k}a}}{2k^2\,(sKa + i\, cKa)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\cancel{i\, e^{-iKa}}}{\cancel{i\, e^{-iKa}}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$S \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ no potential.

Bound states. analytic continuation $\tilde{k} \to k$

$B, F = 0$, $A, G \neq 0$ $\quad S^{-1}\begin{pmatrix} G \\ A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$S \to \infty$ $\quad S$ has poles.

$(K^2 + \tilde{k}^2)\, sKa + 2i\tilde{k}K\, cKa = 0$ not possible but

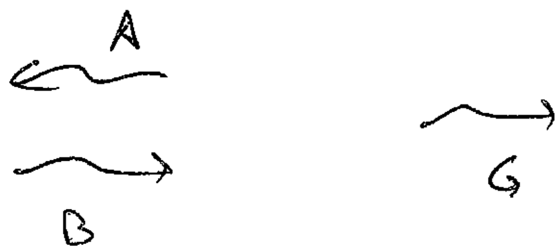
after $\tilde{k} \to ik$

$$(K^2 - k^2)\, sKa - 2kK\, cKa = 0 \Rightarrow \boxed{\tan Ka = \frac{2kK}{K^2 - k^2}}$$

✓

Transmission & reflection coefficients

$F=0$



$$T = \frac{|G|^2}{|B|^2} \qquad R = \frac{|A|^2}{|B|^2}$$

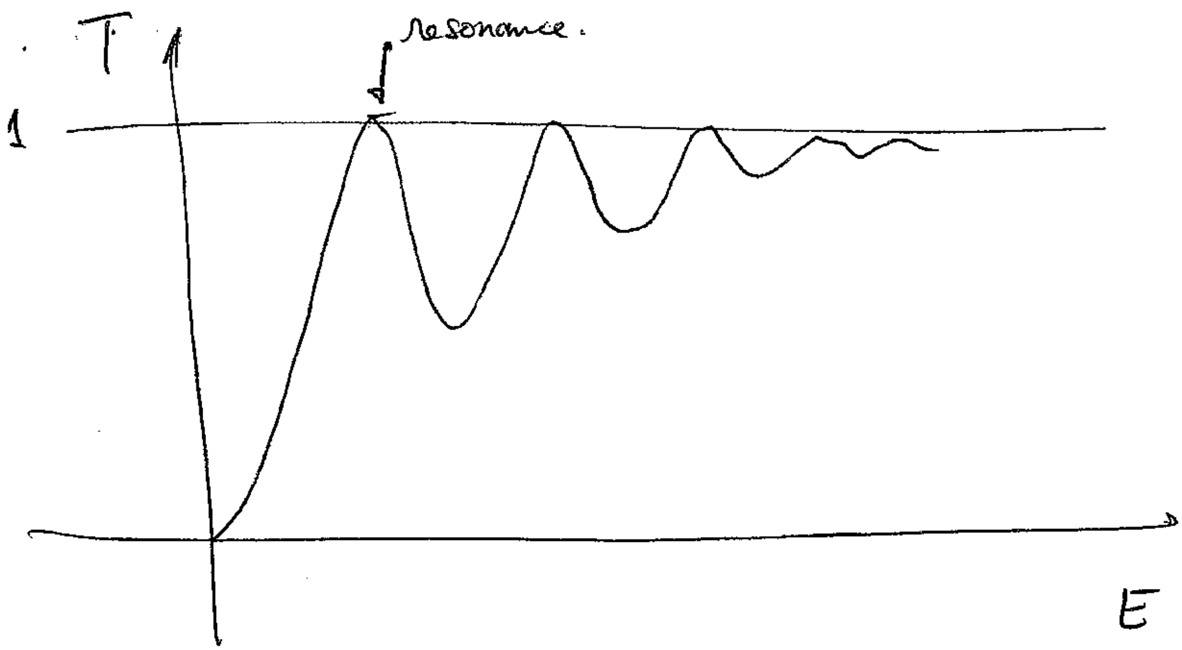
$$F=0 \Rightarrow \quad G = S_{11} B \qquad A = S_{21} B$$

$$T = |S_{11}|^2 \qquad R = |S_{21}|^2$$

$$T = \frac{4\tilde{k}^2 k^2}{(k^2+\tilde{k}^2)^2 s^2 Ka + 4\tilde{k}^2 k^2 \underbrace{c^2 Ka}_{1-s^2 Ka}} = \frac{4\tilde{k}^2 k^2}{4\tilde{k}^2 k^2 + \underbrace{[(k^2+\tilde{k}^2)^2 - 4\tilde{k}^2 k^2]}_{(k^2-\tilde{k}^2)^2 = k_0^4} s^2 Ka}$$

$$T = \frac{1}{1 + \frac{k_0^4}{4k^2\tilde{k}^2} s^2 Ka}$$

$$R = \frac{k_0^2 s^2 Ka}{4\tilde{k}^2 k^2 + k_0^4 s^2 Ka} = \frac{\frac{k_0^2 s^2 Ka}{4\tilde{k}^2 k^2}}{1 + \frac{k_0^4}{4k^2\tilde{k}^2} s^2 Ka}.$$



$SKa = 0$ $\quad \boxed{Ka = n\pi}$

$$\frac{\hbar^2 K^2}{2m} = V_0 + E \qquad \boxed{E = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 - V_0}$$

similar to bound states of square well. Fits wavelength

At resonance.

$$S = \frac{e^{-i\tilde{k}a}}{\cancel{2i\tilde{k}K}\, cKa}\begin{pmatrix} \cancel{2i\tilde{k}K} & 0 \\ 0 & \cancel{2i\tilde{k}K} \end{pmatrix} = (-)^n e^{-i\tilde{k}a}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

like no potential except for a <u>phase</u>

Finally:

Suppose $B = F$

$$\begin{pmatrix} G \\ A \end{pmatrix} = B \begin{pmatrix} S_{11}+S_{12} \\ S_{21}+S_{22} \end{pmatrix}$$

$$S_{11}+S_{12} = \frac{e^{-i\tilde{k}a}}{(K^2+\tilde{k}^2)sKa + 2i\tilde{k}KcKa}\left(2i\tilde{k}K - k_0^2 sKa\right)$$

$$S_{21}+S_{22} = S_{11}+S_{12} \Rightarrow G = A.$$

$$\frac{G}{B} = \frac{e^{-i\tilde{k}a}\left(2i\tilde{k}K - k_0^2 sKa\right)}{(K^2+\tilde{k}^2)sKa + 2i\tilde{k}KcKa}$$

$$\left|\frac{G}{B}\right| = \frac{4\tilde{k}^2K^2 + k_0^4 s^2Ka}{4\tilde{k}^2K^2 + k_0^4 s^2Ka} = 1 \;\; \underline{!}$$

$e^{i\delta} \leftarrow$ phase shift

$$(2i\tilde{k}K - k_0^2 sKa)^2 \cancel{(-2i\tilde{k}K - k_0^2 sKa)} = -4\tilde{k}^2K^2 + k_0^4 s^2Ka - 4i\tilde{k}Kk_0^2 sKa$$

$x \leftrightarrow -x$ symmetry.

$A \leftrightarrow G$

$B \leftrightarrow F$

$$\begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} F \\ B \end{pmatrix}$$

but also

$$\begin{pmatrix} A \\ G \end{pmatrix} = \begin{pmatrix} S_{22} & S_{21} \\ S_{12} & S_{11} \end{pmatrix} \begin{pmatrix} F \\ B \end{pmatrix}$$

$$\Rightarrow \quad \boxed{S_{11} = S_{22}} \quad \boxed{S_{12} = S_{21}} \quad \checkmark$$