Composition of ang. mom.

$$r_{cm} = \sum \frac{m_i r_i}{\sum m_i}$$

$$\vec{J} = \sum r_i \times p_i = \sum (r_i - r_{cm}) \times p_i + r_{cm} \times \sum p_i$$

$$= \sum (r_i - r_{cm}) \times m_i v_i + r_{cm} \times P_{cm}$$

$$= \vec{L}_{cm} + \sum (r_i - r_{cm}) \times m_i (v_i - v_{cm}) + \sum (r_i - r_{cm}) \times m_i v_{cm}$$

$$= \vec{L}_{cm} + \vec{S} + \cancel{\underbrace{\sum r_i m_i}_{\sum m_i r_{cm}} v_{cm}} - \cancel{\sum m_i r_{cm} \times v_{cm}}$$

$$\vec{J} = \vec{L}_{cm} + \vec{S}$$

$$\boxed{J^2 = L^2 + S^2 + 2L \cdot S}$$

$J_1 = L_1 + S_1$

$J_2 = L_2 + S_2$

$\boxed{J_3 = L_3 + S_3}$ add up.

$[L_i, S_j] = 0$

electron in p orbital

$\ell \;\; \ell_3$

$|1, 1\rangle$

$|1, 0\rangle$

$|1, -1\rangle$

spin $(\overset{s}{\tfrac{1}{2}}, \overset{S_3}{\pm\tfrac{1}{2}})$

$\|1\ 1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$	$\|1\ 1\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$
$\|1\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$	$\|1\ 0\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$
$\|1\ -1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle$	$\|1\ -1\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle$

$$\qquad\qquad\qquad\qquad\qquad\qquad \ell \quad s \quad j \quad j_3 \leftarrow \text{add up.}$$

$$|1\ 1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle \to |1\ \tfrac{1}{2}\ 3/2\ \ 3/2\rangle$$

$$|1\ 1\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle \to |1\ \tfrac{1}{2}\qquad 1/2\rangle$$

$$|1\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle \to |1\ \tfrac{1}{2}\qquad 1/2\rangle$$

$$|1\ 0\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle \to |1\ \tfrac{1}{2}\qquad -1/2\rangle$$

$$|1\ -1\ \tfrac{1}{2}\ \tfrac{1}{2}\rangle \to |1\ \tfrac{1}{2}\qquad -1/2\rangle$$

$$|1\ -1\ \tfrac{1}{2}\ -\tfrac{1}{2}\rangle \to |1\ \tfrac{1}{2}\ 3/2\ \ -3/2\rangle$$

maximum value of $j_3 = 3/2 \to j = 3/2$

$\boxed{j = 3/2}$ or $\boxed{j = 1/2}$

$$j_-\,|1\ \tfrac{1}{2}\ 3/2\ 3/2\rangle = (L_- + S_-)\,|1\ \tfrac{1}{2}\ 1\ \tfrac{1}{2}\rangle =$$

$$= \sqrt{2 - 0}\ |1\ \tfrac{1}{2}\ 0\ \tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{4} + \tfrac{1}{4}}\ |1\ \tfrac{1}{2}\ 1\ -\tfrac{1}{2}\rangle$$

$$= \sqrt{2}\ |1\ \tfrac{1}{2}\ 0\ \tfrac{1}{2}\rangle + |1\ \tfrac{1}{2}\ 1\ -\tfrac{1}{2}\rangle = \sqrt{\tfrac{15}{4} - \tfrac{3}{4}}\ |1\ \tfrac{1}{2}\ 3/2\ 1/2\rangle = \sqrt{3}\,|1\ \tfrac{1}{2}\ \tfrac{3}{2}\ \tfrac{1}{2}\rangle$$

$$|1\,\tfrac{1}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,|1\,\tfrac{1}{2}\,0\,\tfrac{1}{2}\rangle + \frac{1}{\sqrt{3}}\,|1\,\tfrac{1}{2}\,1\,-\tfrac{1}{2}\rangle$$

normalized.

$$|1\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,|1\,\tfrac{1}{2}\,0\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1\,\tfrac{1}{2}\,1\,-\tfrac{1}{2}\rangle$$

$$J_-\,|1\,\tfrac{1}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}\,(L_-+S_-)\,|1\,\tfrac{1}{2}\,0\,\tfrac{1}{2}\rangle + \frac{1}{\sqrt{3}}(L_-+S_-)\,|1\,\tfrac{1}{2}\,1\,-\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{2}{3}}\left(\sqrt{2-0}\,|1\,\tfrac{1}{2}\,-1\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{3}{4}+\tfrac{1}{4}}\,|1\,\tfrac{1}{2}\,0\,-\tfrac{1}{2}\rangle\right)$$

$$+\frac{1}{\sqrt{3}}\left(\sqrt{2}\,|1\,\tfrac{1}{2}\,0\,-\tfrac{1}{2}\rangle\right)$$

$$= \sqrt{\tfrac{4}{3}}\,|1\,\tfrac{1}{2}\,-1\,\tfrac{1}{2}\rangle + 2\sqrt{\tfrac{2}{3}}\,|1\,\tfrac{1}{2}\,0\,-\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{15}{4}+\tfrac{1}{4}}\,|1\,\tfrac{1}{2}\,\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = 2\,|1\,\tfrac{1}{2}\,\tfrac{3}{2}\,-\tfrac{1}{2}\rangle$$

$$\boxed{|1\,\tfrac{1}{2}\,\tfrac{3}{2}\,-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\,|1\,\tfrac{1}{2}\,-1\,\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1\,\tfrac{1}{2}\,0\,-\tfrac{1}{2}\rangle}$$

$$J_- |1 \tfrac{1}{2} \tfrac{1}{2} \tfrac{1}{2}\rangle = -\sqrt{\tfrac{1}{3}}\,(L_- + S_-)\,|1 \tfrac{1}{2} 0 \tfrac{1}{2}\rangle +$$

$$+ \sqrt{\tfrac{2}{3}}\,(L_- + S_-)\,|1 \tfrac{1}{2}\, 1\, -\tfrac{1}{2}\rangle$$

$$= -\sqrt{\tfrac{1}{3}}\left(\sqrt{2}\,|1 \tfrac{1}{2} -1 \tfrac{1}{2}\rangle + |1 \tfrac{1}{2} 0 -\tfrac{1}{2}\rangle\right)$$

$$+ \sqrt{\tfrac{2}{3}}\sqrt{2}\,|1 \tfrac{1}{2}\, 0 -\tfrac{1}{2}\rangle$$

$$= -\sqrt{\tfrac{2}{3}}\,|1 \tfrac{1}{2} -1 \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1 \tfrac{1}{2} 0 -\tfrac{1}{2}\rangle$$

$$= \sqrt{\tfrac{3}{4} + \tfrac{1}{4}}\,|1 \tfrac{1}{2}\, \tfrac{1}{2} -\tfrac{1}{2}\rangle = |1 \tfrac{1}{2}\, \tfrac{1}{2} -\tfrac{1}{2}\rangle$$

$$|3/2\ 3/2\rangle = |1\ 1\ 1/2\ 1/2\rangle$$

$$|3/2\ 1/2\rangle = \sqrt{\tfrac{2}{3}}\,|1\ 1/2\ 0\ 1/2\rangle + \tfrac{1}{\sqrt{3}}\,|1\ 1/2\ 1\ -1/2\rangle$$

$$|3/2\ -1/2\rangle = \tfrac{1}{\sqrt{3}}\,|1 \tfrac{1}{2} -1 \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1 \tfrac{1}{2}\ 0 -\tfrac{1}{2}\rangle$$

$$|3/2\ -3/2\rangle = |1\ -1\ 1/2\ -1/2\rangle$$

$$|1/2\ 1/2\rangle = -\tfrac{1}{\sqrt{3}}\,|1 \tfrac{1}{2} 0 \tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}\,|1 \tfrac{1}{2}\ 1 -\tfrac{1}{2}\rangle$$

$$|1/2\ -1/2\rangle = -\sqrt{\tfrac{2}{3}}\,|1 \tfrac{1}{2} -1 \tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}\,|1 \tfrac{1}{2}\ 0 -\tfrac{1}{2}\rangle$$

Clebsch – Gordan coefficients.

general case

$j_1 m_1 \quad j_2 m_2 \rightarrow j\, m \qquad |j m\rangle = \sum_{m_1 m_2} C^{jm}_{j_1 j_2 m_1 m_2} \, |j_1 j_2 m_1 m_2\rangle$

$|j_1 j_1 \; j_2 j_2\rangle \rightarrow |j_1 + j_2 \quad j_1 + j_2\rangle$

$j_{total}: \; j_1 + j_2 \,, \; j_1 + j_2 - 1 \; \ldots \; |j_1 - j_2|$

$j_1 > j_2$ e.g.

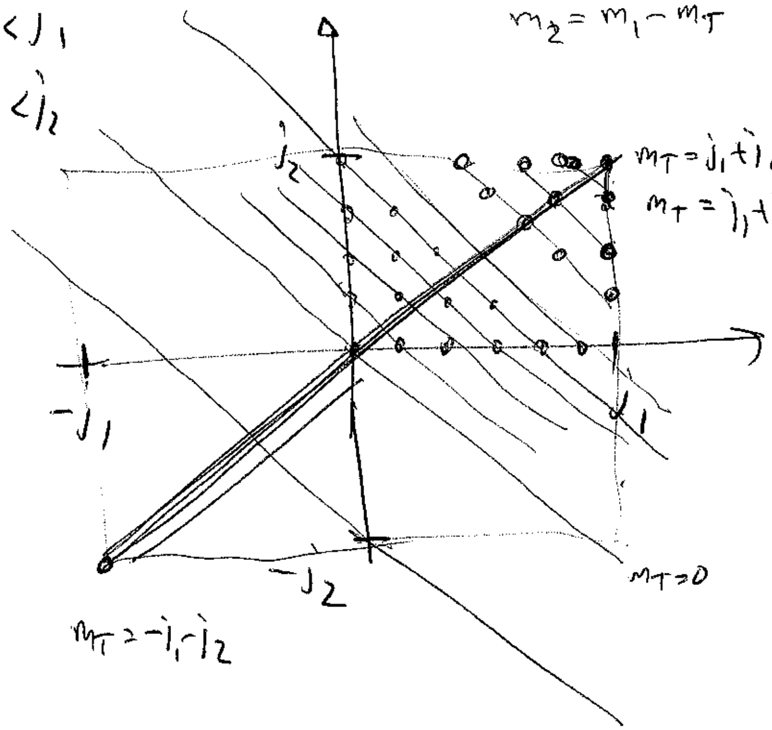
$\quad m_T = m_1 + m_2$

how many states $\#_{m_T} = \; j_1 + m_T - j_1$

$|m_1| < j_1$

$|m_2| < j_2$

$m_2 = m_1 - m_T$



$$(2l_1+1)(2l_2+1) = \sum_{l=l_{min}}^{l_1+l_2} (2l+1)$$

$\sum_1^2$

$$\sum_{n_0}^{n_1} l = \sum_1^{n_1} l - \sum_1^{n_0-1} l$$

$$= \frac{n_1(n_1+1)}{2} - \frac{(n_0-1)\,n_0}{2}$$

$$(2l_1+1)(2l_2+1) = \cancel{2}\left[\frac{(l_1+l_2)(l_1+l_2+1)}{\cancel{2}} - \frac{(l_m-1)\,l_m}{\cancel{2}}\right] +$$

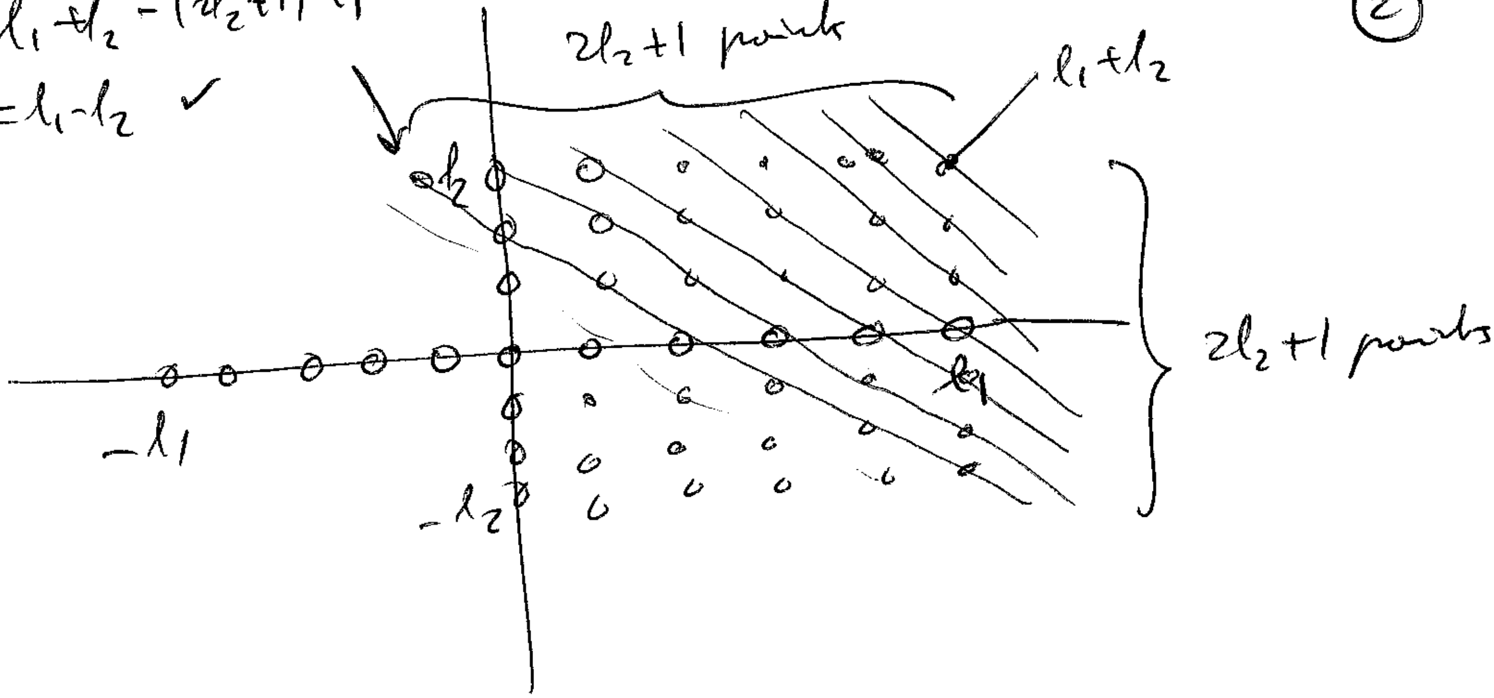
$$+ (l_1+l_2-l_{min}+1)$$

$$\cancel{4}\,4l_1l_2 + \cancel{2(l_1+l_2)+1} = l_1^2+l_2^2+2l_1l_2 - \cancel{+2l_1+2l_2} - l_m^2 + \cancel{l_m} - \cancel{l_m+1}$$

$$2l_1l_2 - l_1^2 - l_2^2 = -l_m^2 \Rightarrow l_m^2 = l_1^2+l_2^2-2l_1l_2 = (l_1-l_2)^2$$

$$\boxed{l_m = |l_1-l_2|} \checkmark$$

$l_1 + l_2 - (2l_2 + 1) + 1$
$= l_1 - l_2$ ✓



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				n	$\sim$
l_1+l_2	(l_1, l_2)			ϕ	0
l_1+l_2-1	(l_1-1, l_2)	(l_1, l_2-1)		2	1
l_1+l_2-2	(l_1-1, l_2-1)	(l_1-2, l_2)	(l_1, l_2-2)	3	2

$\vdots$

3

$\vdots$

$\underbrace{\times\times\times\times\times}_{n}$ $\qquad \times\times\times|\times\times\times \qquad \frac{(n+1)!}{n!} = \boxed{n+1}$

Tensor operators.

$\vec{x}, \vec{p}$.

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_{1\pm1} = \mp\sqrt{\frac{3}{4\pi}} \frac{x \pm iy}{\sqrt{2}r} = \sqrt{\frac{3}{4\pi}} \left(\mp \frac{x \pm iy}{\sqrt{2}r}\right)$$

$$x_0 = z \qquad x_+ = \frac{-x-iy}{\sqrt{2}} \qquad x_- = \frac{x-iy}{\sqrt{2}}.$$

rotate as spherical harmonics (Y_{1m})

What does it mean?

$$x|l\, l_z\rangle \;; \quad y|l\, l_z\rangle \;; \quad z|l\, l_z\rangle$$

$3 \times (2l+1)$ states

Ang. momentum?

$$L_i\, x_j |l l_z\rangle = [L_i, x_j] |l l_z\rangle + x_j L_i |l l_z\rangle$$

$$= i\epsilon_{ijk} x_k |l l_z\rangle + x_j L_i |l l_z\rangle$$